Supplementary Notes for "Stable Matching in Large Economies"

(Not for Publication)

S.1 Analysis of the Example in Section 2

Let r be the number of workers with each of the two types who are matched to f. We consider the following cases:

1. Suppose $r > q/2$. For any such matching, at least one position is vacant at firm f' because f' has q positions, but strictly more than q workers are matched to f out of the total of $2q$ workers. Thus such a matching is blocked by f' and a type θ' worker who is currently matched to f.

2. Suppose $r < q/2$. Consider the following cases.

   (a) Suppose that there exists a type θ worker who is unmatched. Then such a matching is unstable because that worker and firm f' block it (note that f' prefers θ most).

   (b) Suppose that there exists no type θ worker who is unmatched. This implies that there exists a type θ' worker who is unmatched (because there are $2q$ workers in total, but firm f is matched to strictly fewer than q workers by assumption, and f' can be matched to at most q workers in any individually rational matching). Then, since f is the most preferred by all θ workers, a θ' worker prefers f to $\emptyset$, and there is some vacancy at f because $r < q/2$, the matching is blocked by a coalition of a type θ worker, a type θ' worker, and f.

S.2 Preliminaries for the Continuum Economy Model

S.2.1 Proof of Lemma 1

For any subset $\mathcal{Y} \subset \mathcal{X}$, define

$$\overline{Y}(E) := \sup\{\sum_i Y_i(E_i) \,|\, \{E_i\} \text{ is a finite partition of } E \text{ in } \Sigma \text{ and}$$
$$\{Y_i\} \text{ is a finite collection of measures in } \mathcal{Y}, \forall i\}, \forall E.$$

and $\underline{Y}$ analogously (by replacing "sup" with "inf"). We prove the lemma by showing that $\overline{Y} = \sup \mathcal{Y} \in \mathcal{Y}$ and $\underline{Y} = \inf \mathcal{Y} \in \mathcal{X}$.

First of all, note that $\overline{Y}$ and $\underline{Y}$ are monotonic, i.e. for any $E \subset D$, we have $\overline{Y}(D) \geq \overline{Y}(E)$ and $\underline{Y}(D) \geq \underline{Y}(E)$, whose proof is straightforward and thus omitted.

We next show that $\overline{Y}$ and $\underline{Y}$ are measures. We only prove the countable additivity of $\overline{Y}$, since the other properties are straightforward to prove and also since a similar argument applies to $\underline{Y}$. To this end, consider any countable collection $\{E_i\}$ of disjoint sets in Σ and let $D = \cup E_i$. We need to show that $\overline{Y}(D) = \sum_i \overline{Y}(E_i)$. For doing so, consider any finite partition $\{D_i\}$ of D and any finite collection of measures $\{Y_i\}$. Letting $E_{ij} = E_i \cap D_j$, for any i, the collection $\{E_{ij}\}_j$ is a finite partition of E_i in Σ. Thus, we have

$$\sum_i Y_i(D_i) = \sum_i \sum_j Y_i(E_{ij}) \leq \sum_i \overline{Y}(E_i).$$

Since this inequality holds for any finite partition $\{D_i\}$ of D and collection $\{Y_i\}$, we must have $\overline{Y}(D) \leq \sum_i \overline{Y}(E_i)$. To show that the reverse inequality also holds, for each E_i, we consider any finite partition $\{E_{ij}\}_j$ of E_i in Σ and collection of measures $\{Y_{ij}\}_j$ in $\mathcal{Y}$. We prove that $\overline{Y}(D) \geq \sum_i \sum_j Y_{ij}(E_{ij})$, which would imply $\overline{Y}(D) \geq \sum_i \overline{Y}(E_i)$ as desired since the partition $\{E_{ij}\}_j$ and collection $\{Y_{ij}\}_j$ are arbitrarily chosen for each i. Suppose not for contradiction. Then, we must have $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij})$ for some k. Letting $E := \cup_{i=1}^k (\cup_j E_{ij})$, this implies $\overline{Y}(D) < \sum_{i=1}^k \sum_j Y_{ij}(E_{ij}) \leq \overline{Y}(E)$, where the second inequality holds by the definitioin of $\overline{Y}$. This contradicts with the monotonicity of $\overline{Y}$ since $E \subset D$.

We now show that $\overline{Y}$ and $\underline{Y}$ are the supremum and infimum of $\mathcal{Y}$, respectively. It is straightforward to check that for any $Y \in \mathcal{Y}$, $Y \sqsubset \overline{Y}$ and $\underline{Y} \sqsubset Y$. Consider any $X, X' \in \mathcal{X}$ such that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$ and $X' \sqsubset Y$. We show that $\overline{Y} \sqsubset X$ and $X' \sqsubset \underline{Y}$. First, if $\overline{Y} \not\sqsubset X$ to the contrary, then there must be some $E \in \Sigma$ such that $\overline{Y}(E) > X(E)$. This means there are a finite partition $\{E_i\}$ of E and a collection of measures $\{Y_i\}$ in $\mathcal{Y}$ such that $\overline{Y}(E) \geq \sum Y_i(E_i) > X(E) = \sum X(E_i)$. Thus, for at least one i, we have $Y_i(E_i) > X(E_i)$, which contradicts the assumption that for all $Y \in \mathcal{Y}$, $Y \sqsubset X$. An analogous argument can be used to show $X' \sqsubset \underline{Y}$.

S.2.2 Proof of Proposition 1

Suppose that matching M is not Pareto efficient. Then by definition of Pareto efficiency, there exists $M' \neq M$ such that $M' \succeq_F M$ and $M' \succeq_\Theta M$. Let $f \in F$ be a firm such that $M'_f \neq M_f$. By assumption, $M' \succeq_f M$.

Next, since $M' \succeq_\Theta M$, for each $\tilde{f}$, $D^{\succeq \tilde{f}}(M') \sqsupset D^{\succeq \tilde{f}}(M)$, or

$$\sum_{f': f' \succeq_P \tilde{f}} M'_{f'}(\Theta_P \cap E) \geq \sum_{f': f' \succeq_P \tilde{f}} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

This implies that

$$\sum_{f': f' \succeq_P f^P_-} M'_{f'}(\Theta_P \cap E) \geq \sum_{f': f' \succeq_P f^P_-} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,$$

where f^P_- refers to the firm that is ranked immediately above f according to P (whenever it is well defined),[1] or equivalently

$$\sum_{f': f' \succ_P f} M'_{f'}(\Theta_P \cap E) \geq \sum_{f': f' \succ_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma.$$

This in turn implies that, for each P,

$$\sum_{f': f' \preceq_P f} M'_{f'}(\Theta_P \cap E) \leq \sum_{f': f' \preceq_P f} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,$$

or equivalently,

$$D^{\preceq f}(M') \sqsubset D^{\preceq f}(M).$$

By definition, $M'_f \sqsubset D^{\preceq f}(M')$, so we have $M'_f \sqsubset D^{\preceq f}(M)$.

Collecting the observations made so far, we conclude that f and M'_f block M, implying that M is not stable. Therefore, we have established that stability implies Pareto efficiency.

S.3 Substitutable Preferences

A class of preferences studied extensively in the matching theory literature are substitutable preferences. A well-known set of results, including existence of a stable matching,

[1] This is defined later as an immediate predecessor. Formally, $f^P_- \succ_P f$ and if $f' \succ_P f$, then $f' \succeq_P f^P_-$.

obtain under these preferences. We show that the same set of results follow in our continuum economy model with a suitable formulation of substitutable preferences. Since the arguments establishing these results are by now fairly standard, we shall be brief in our treatment of this case. One novel issue, though, is the question of uniqueness of a stable matching. Azevedo and Leshno (2014) show that multiplicity of stable matchings disappears in the large economy if firms have rich preferences over workers or if their quotas are generic. This striking result is obtained with the restricted preference domain of "responsive" preferences. We provide a condition for uniqueness of a stable matching under general substitutable preferences. We begin by defining the class of preferences:

Definition S1. Firm f's preference is **substitutable** if $R_f(X) \sqsubset R_f(X')$ whenever $X \sqsubset X'$.

In words, substitutability means that a firm rejects more of any given worker types when facing a bigger set of workers. Importantly, the assumption excludes the kind of complementary preferences studied in the main text of this paper. At the same time, the substitutable preferences are not a special case of the preferences considered in Section 4 either, since continuity of preferences need not be satisfied here.

Again, by Theorem 1, the fixed points of the map T characterize the stable matchings. Since we do not assume continuity of the choice mappings, however, Theorem 2 does not apply. Instead, as shown in the proof of the next theorem, substitutability of the firms' preferences implies that the map T is monotone increasing with respect to the partial order $\sqsubset_{\tilde{F}}$. Next, recall from Lemma 1 that a partially ordered set $(\mathcal{X}, \sqsubset)$, and thus the partially ordered set $(\mathcal{X}^{n+1}, \sqsubset_{\tilde{F}})$, is a complete lattice, where $X_{\tilde{F}} \sqsubset_{\tilde{F}} X'_{\tilde{F}}$ if $X_f \sqsubset X'_f$ for all $f \in \tilde{F}$. Hence, Tarski's fixed point theorem yields existence as well as the lattice structure of stable matchings.

To describe the lattice structure, it is also worth describing the extreme points based on the preference orders defined earlier. We say that a stable matching $\overline{M}$ is **firm-optimal** (resp., **firm-pessimal**) if $\overline{M} \succeq_F M$ (resp., $\overline{M} \preceq_F M$) for every stable matching M. A matching $\underline{M}$ is **worker-optimal** (resp., **worker-pessimal**) if $\underline{M} \succeq_\Theta M$ (resp., $\underline{M} \preceq_\Theta M$) for every stable matching M. The result is then stated as follows:

Theorem S1. *When the firms' preferences are substitutable, (i) the set $\mathcal{X}^*$ of fixed points of T is nonempty, and $(\mathcal{X}^*, \sqsubset_{\tilde{F}})$ is a complete lattice; and (ii) there exists a firm-optimal (and worker-pessimal) stable matching $\overline{M} = (C_f(\overline{X}_f))_{f \in \tilde{F}}$, where $\overline{X} = \sup_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$, and a firm-pessimal (and worker-optimal) stable matching $\underline{M} = (C_f(\underline{X}_f))_{f \in \tilde{F}}$, where $\underline{X} = \inf_{\sqsubset_{\tilde{F}}} \mathcal{X}^*$.*

Proof. The part (i) immediately follows from Tarski's fixed point theorem and the fact that each R_f is monotonic in $\sqsubset_{\tilde{F}}$ due to the substitutability of f's preference and thus T is monotonic as well.

We next prove part (ii). To see that the stable matching $\overline{M}$ is firm-optimal, observe first that for any stable matching M, there is some $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. Thus, we have $M_f \sqsubset X_f \sqsubset \overline{X}_f$, which implies that $\overline{M}_f = C_f(M_f \vee \overline{M}_f)$ by revealed preference since $\overline{M}_f = C_f(\overline{X}_f)$ and $(M_f \vee \overline{M}_f) \sqsubset \overline{X}_f$. Thus, $\overline{M}_f \succeq_f M_f$ for each $f \in F$, as desired. To show that $\overline{M}$ is worker-pessimal, fix any stable matching M. Then, by Theorem 1, there is some $X \in \mathcal{X}^*$ (i.e., a fixed point of T) such that $M_f = C_f(X_f)$ and $X_f = D^{\preceq f}(M)$ for all $f \in \tilde{F}$. Thus, for each $P \in \mathcal{P}$ and $E \in \Sigma$,[2]

$$T_{f_+^P}(X)(\Theta_P \cap E) = X_{f_+^P}(\Theta_P \cap E) = D^{\preceq f_+^P}(M)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} M_{f'}(\Theta_P \cap E).$$

Similarly, for $\overline{X}$, we have $T_{f_+^P}(\overline{X})(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \prec_P f} \overline{M}_{f'}(\Theta_P \cap E)$. Since T_f is monotonic and $X \sqsubset_{\tilde{F}} \overline{X}$, we obtain

$$\begin{aligned}\sum_{f' \in \tilde{F}: f' \succeq_P f} \overline{M}_{f'}(\Theta_P \cap E) =& G(\Theta_P \cap E) - T_{f_+^P}(\overline{X})(\Theta_P \cap E) \\ \leq& G(\Theta_P \cap E) - T_{f_+^P}(X)(\Theta_P \cap E) = \sum_{f' \in \tilde{F}: f' \succeq_P f} M_{f'}(\Theta_P \cap E) \qquad \text{(S1)}\end{aligned}$$

for all $P \in \mathcal{P}$, $E \in \Sigma$, and $f \in \tilde{F}$, as desired. ▮

As has been noted by Hatfield and Milgrom (2005), the algorithm finding the fixed point corresponds to the Gale and Shapley's deferred acceptance algorithm, although the algorithm may not terminate in finite rounds in our continuum model.

Consider an additional restriction on the preferences.

Definition S2. Firm f's preference exhibits the **law of aggregate demand** if for any $X, X' \in \mathcal{X}$ with $X \sqsubset X'$, $C_f(X)(\Theta) \leq C_f(X')(\Theta)$.[3]

This property simply ensures that a firm demands more workers (in terms of cardinality) when more workers (in terms of set inclusion) become available. This property is needed to obtain the next two results.

[2]Recall that f_+^P denotes an immediate successor of $f \in \tilde{F}$ at $P \in \mathcal{P}$.

[3]This property is an adaptation of the same property to our continuum economy that appears in the literature such as Hatfield and Milgrom (2005), Alkan (2002), and Fleiner (2003).

Theorem S2 (Rural hospital theorem). *If firms' preferences exhibit substitutability and the law of aggregate demand, then for any stable matching M, we have $M_f(\Theta) = \overline{M}_f(\Theta)$ for each $f \in F$ and $M_\varnothing = \overline{M}_\varnothing$.*

Proof. Let M be a stable matching. Then, by Theorem 1, there exists $X \in \mathcal{X}^*$ such that $M_f = C_f(X_f)$ for each $f \in F$. Since $X \sqsubset_{\tilde{F}} \overline{X}$ by Theorem S1, by the law of aggregate demand, we have

$$\overline{M}_f(\Theta) = C_f(\overline{X}_f)(\Theta) \geq C_f(X_f)(\Theta) = M_f(\Theta), \forall f \in F. \tag{S2}$$

Next since $\overline{M}$ is worker pessimal, (S1) holds for any $f \in \tilde{F}$. Let $w_P := \varnothing_-^P$ be the immediate predecessor of $\varnothing$ (i.e., the worst individually rational firm) for types in Θ_P. Then, setting $f = w_P$ in (S1), we obtain

$$\begin{aligned}\sum_{f' \in F} \overline{M}_{f'}(\Theta_P \cap E) &= \sum_{f' \in \tilde{F}: f' \succeq_P w_P} \overline{M}_{f'}(\Theta_P \cap E) \\ &\leq \sum_{f' \in \tilde{F}: f' \succeq_P w_P} M_{f'}(\Theta_P \cap E) = \sum_{f' \in F} M_{f'}(\Theta_P \cap E), \forall E \in \Sigma,\end{aligned}$$

or equivalently

$$\sum_{f' \in F} \overline{M}_{f'}(E) \leq \sum_{f' \in F} M_{f'}(E), \forall E \in \Sigma. \tag{S3}$$

Since this inequality must hold with $E = \Theta$, combining it with (S2) implies that $M_f(\Theta) = \overline{M}_f(\Theta)$ for all $f \in F$, as desired.

Further, we must have $\sum_{f \in F} \overline{M}_f = \sum_{f \in F} M_f$, which means that $\overline{M}_\varnothing = M_\varnothing$. To prove this, suppose otherwise. Then, by (S3), we must have $\sum_{f' \in F} \overline{M}_{f'}(E) < \sum_{f' \in F} M_{f'}(E)$ for some $E \in \Sigma$. Also, by (S3), $\sum_{f' \in F} \overline{M}_{f'}(E^c) \leq \sum_{f' \in F} M_{f'}(E^c)$. Combining these two inequalities, we obtain $\sum_{f' \in F} \overline{M}_{f'}(\Theta) < \sum_{f' \in F} M_{f'}(\Theta)$, which contradicts with (S2). ∎

The result implies that the measure of workers matched with each firm $f \in F$ as well as the measure of unmatched workers is identical across all stable matchings.

We next introduce a condition that would ensure uniqueness of a stable matching. The condition refers to some new notation. For any matching M and subset F' of firms, let $M^f_{F'}$ be a subpopulation of workers defined by

$$M^f_{F'}(E) := \sum_{P \in \mathcal{P}} \sum_{f': f \succ_P f', f' \notin F'} M_{f'}(\Theta_P \cap E) \text{ for each } E \in \Sigma,$$

who are matched outside firms F' and available to firm f under M. Recall the worker-optimal stable matching $\underline{M}$. Then, our condition is stated as follows.

Definition S3 (Rich preferences)**.** The firms' preferences are **rich** if for any individually rational matching $\hat{M} \neq \underline{M}$ such that $\hat{M} \succeq_F \underline{M}$, there exists $f^* \in F$ such that $\underline{M}_{f^*} \neq C_{f^*}((\underline{M}_{f^*} + \hat{M}^{f^*}_{\bar{F}}) \wedge G)$, where $\bar{F} := \{f \in F | \hat{M}_f \succ_f \underline{M}_f\}$.

In words, the condition is explained as follows. Consider any (individually rational) matching $\hat{M}$ that is preferred to the worker-optimal matching $\underline{M}$ by all firms, strictly by firms in $\bar{F} \subset F$. Then, the richness condition requires that, at matching $\underline{M}$, there must exist a firm f^* that would be happy to match with some workers who are not hired by the firms in $\bar{F}$ but are willing to match with f^* under $\hat{M}$. Since firms are more selective at $\hat{M}$ than at $\underline{M}$, it is intuitive that a firm would demand at the latter matching some workers that the more selective firms would not demand at the former matching. The presence of such worker types requires richness of the preference palette of firms as well as workers—hence the name. This point will be seen more clearly when one considers (a general class of) responsive preferences, and we shall illustrate this in an example later.

Theorem S3. *If firms' preferences are rich and substitutable, and exhibit the law of aggregate demand, then a unique stable matching exists.*

Proof. Suppose otherwise. Then there exists a stable matching M that differs from the worker-optimal stable matching $\underline{M}$. Let X and $\underline{X}$ be respectively fixed points of T such that $M_f = C_f(X_f)$, $\underline{M}_f = C_f(\underline{X}_f)$ and $\underline{X}_f \sqsubset X_f$, for each $f \in F$.

First of all, by Theorem S2, $\sum_{f\in F} M_f = \sum_{f \in F} \underline{M}_f$. Next, since $\underline{X}_f \sqsubset X_f$ for each $f \in F$, we have $(\underline{M}_f \vee M_f) \sqsubset X_f$. Revealed preference then implies that, for each $f \in F$,

$$M_f = C_f(\underline{M}_f \vee M_f)$$

or $M \succeq_F \underline{M}$. Moreover, since $M \neq \underline{M}$, the set $\bar{F} := \{f \in F | M_f \succ_f \underline{M}_f\}$ is nonempty. But then by the rich preferences, there exists $f^* \in F$ such that

$$\underline{M}_{f^*} \neq C_{f^*}((\underline{M}_{f^*} + M^{f^*}_{\bar{F}}) \wedge G).$$

For each $f \in F \setminus \bar{F}$, $M_f = \underline{M}_f$, by definition of $\bar{F}$, and Theorem S2 guarantees that $M_\varnothing = \underline{M}_\varnothing$. Consequently, we have for each $E \in \Sigma$, that

$$M^{f^*}_{\bar{F}}(E) = \sum_{P \in \mathcal{P}} \sum_{f' : f^* \succ_P f', f' \notin \bar{F}} M_{f'}(\Theta_P \cap E) = \sum_{P \in \mathcal{P}} \sum_{f' : f^* \succ_P f', f' \notin \bar{F}} \underline{M}_{f'}(\Theta_P \cap E) = \underline{M}^{f^*}_{\bar{F}}(E).$$

It then follows that $(\underline{M}_{f^*} + M^{f^*}_{\bar{F}}) \wedge G = (\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}}) \wedge G = \underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}}$ (since $\underline{M}$ is a matching), so

$$\underline{M}_{f^*} \neq C_{f^*}(\underline{M}_{f^*} + \underline{M}^{f^*}_{\bar{F}}). \tag{S4}$$

Letting $\hat{M}_{f^*} := C_{f^*}(\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*})$, we have $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} \vee \hat{M}_{f^*}) \sqsubset (\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*})$. Revealed preference then implies that

$$\hat{M}_{f^*} = C_{f^*}(\underline{M}_{f^*} \vee \hat{M}_{f^*}).$$

By (S4), we then have $\hat{M}_{f^*} \succ \underline{M}_{f^*}$. Further, $\hat{M}_{f^*} \sqsubset (\underline{M}_{f^*} + \underline{M}_{\bar{F}}^{f^*}) \sqsubset D^{\preceq f^*}(\underline{M})$. We therefore have a contradiction to the stability of $\underline{M}$. ▮

Checking the rich preference condition requires identifying the worker-optimal matching $\underline{M}$, which can be done by adapting the worker-proposing DA to the continuum economy and running it in a given environment.[4] Once $\underline{M}$ is found, it is often straightforward to inspect the existence of $\hat{M}$ and f^* that satisfy the stated property, as Example S1 will illustrate later. The rich preference condition can also be useful for identifying (possibly stronger) sufficient conditions for uniqueness. Specifically, a full support condition that Azevedo and Leshno (2014) have shown to yield a unique stable matching when firms have responsive preferences will be shown to be sufficient for our rich preferences condition in a more general environment in which firms have responsive preferences but may face caps on the number of workers they can hire from different groups of workers. Such group specific quotas, typically based on socio-economic status or other characteristics, may arise from affirmative action or diversity considerations. As pointed out by Abdulkadiroğlu and Sönmez (2003), the resulting preferences (or choice functions) may violate responsiveness but they nonetheless satisfy substitutability.

Responsive preferences with affirmative action. Assume that there is a finite set T of "ethnic types" that describe characteristics of a worker such as ethnicity, gender, and socio-economic status, such that type θ is mapped to T via some measurable function $\tau : \Theta \to T$. For each $t \in T$, a (measurable) set $\Theta^t := \{\theta \in \Theta | \tau(\theta) = t\}$ of agents has an ethnic type t. Each firm f faces (maximum) quota q_f for the workers and q_f^t for workers in ethnic type t. We assume $q_f \leq \sum_{t \in T} q_f^t$, allowing for the possibility that the quota for some ethnic type may not bind. Aside from the quotas, a firm's preference is responsive and described by a continuous score function $s_f : \Theta \to [0, 1]$, with the interpretation that firm f prefers a type θ' worker to a type θ worker if and only if $s_f(\theta') > s_f(\theta)$. We assume that G is absolutely continuous and admits density g in the interior of Θ.[5] Such firms are

[4]Note that we may need to take $\underline{M}$ as the limit of the algorithm in case it does not finish in a finite time. See a leading example of Azevedo and Leshno (2014), for instance.

[5]This assumption is reasonable, and is implied by the firms' preferences to involve no ties over a positive measure of worker types. One can define a worker's type as $\theta = (P, t, (s_f)_{f \in F})$, where P, t, and s_f are respectively the worker's preference, her ethnic type, and the firm f's score of the worker. For firm f's

said to have **responsive preferences with affirmative action**. The corresponding choice functions are defined formally in Section S.3.1. As shown there, the choice function exhibits substitutability and satisfies the law of aggregate demand. Consistent with Azevedo and Leshno (2014), the optimal choice by a firm is characterized by the cutoffs, possibly different across different ethnic types. The full support condition defined by Azevedo and Leshno (2014) in the context of pure responsive preferences is easily generalized to the current environment:

Definition S4 (Full Support)**.** Firms' preferences have **full support** if for each preference $P \in \mathcal{P}$, any ethnic type $t \in T$, and for any non-empty open cube set $S \subset [0,1]^n$, the worker types

$$\Theta_P^t(S) := \{\theta \in \Theta_P \cap \Theta^t \,|\, (s_f(\theta))_{f \in F} \in S\}$$

have a positive measure; i.e., $G(\Theta_P^t(S)) > 0$.

Our full support condition boils down to the full support condition of Azevedo and Leshno (2014), if T is a singleton set so there is no affirmative action constraint.

Proposition S1. *If firms have responsive preferences with affirmative action that satisfy the full support condition, then the preferences are rich.*

Proof. To simplify notation, let $M = \underline{M}$, i.e., the worker-optimal matching. Fix any individually rational matching $\hat{M}$ such that $\hat{M} \succeq_F M$ and assume that $\bar{F} := \{f' \in F | \hat{M}_{f'} \succ_f M_{f'}\}$ is nonempty. For any f, t, let $M_f^t := M_f(\Theta^t \cap \cdot)$ and $\hat{M}_f^t := \hat{M}_f(\Theta^t \cap \cdot)$. Since G is absolutely continuous, for any f, t, both M_f^t and $\hat{M}_f^t$, being its subpopulations, admit densities, denoted respectively by m_f^t and $\hat{m}_f^t$. Let p_f^t and $\hat{p}_f^t$ respectively denote the optimal cutoffs associated with $M_f = C_f(M_f)$ and $\hat{M}_f = C_f(M_f \vee \hat{M}_f)$.

Because C_f satisfies the law of aggregate demand (as established in Section S.3.1), $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$ and $M_f = C_f(M_f)$ imply $M_f(\Theta) \leq \hat{M}_f(\Theta)$ for each $f \in F$. Then, Proposition S1 follows from proving a sequence of claims.

Claim S1. $M_\varnothing = \hat{M}_\varnothing$.

Proof. Suppose to the contrary that $M_\varnothing \neq \hat{M}_\varnothing$. Then, with their densities denoted by $m_\varnothing$ and $\hat{m}_\varnothing$, $E_\varnothing = \{\theta \in \Theta \,|\, m_\varnothing(\theta) > \hat{m}_\varnothing(\theta)\}$ must be a non-empty set of positive (Lebesgue) measure, due to the fact that $M_\varnothing(\Theta) = G(\Theta) - \sum_{f \in F} M_f(\Theta) \geq G(\Theta) - \sum_{f \in F} \hat{M}_f(\Theta) = \hat{M}_\varnothing(\Theta)$. Also, letting $\hat{E}_f = \{\theta \in \Theta \,|\, \hat{m}_f(\theta) > m_f(\theta)\}$, there must be at least one firm f

preference to involve no ties among a positive measure of worker types, the marginal distribution of its scores, s_f, must not involve a mass point. This requires the distribution of θ to be absolutely continuous.

for which $E_\varnothing \cap \hat{E}_f$ is a non-empty set of positive measure, since otherwise we would have $\sum_{f' \in \tilde{F}} m_{f'}(\theta) > \sum_{f' \in \tilde{F}} \hat{m}_{f'}(\theta)$ for all $\theta \in E_\varnothing$, a contradiction. Now fixing such a firm f and letting $\tilde{E} = E_\varnothing \cap \hat{E}_f$, define

$$\tilde{m}_f(\theta) = \begin{cases} \min\{m_f(\theta) + m_\varnothing(\theta), \hat{m}_f(\theta)\} & \text{if } \theta \in \tilde{E} \\ m_f(\theta) & \text{otherwise.} \end{cases}$$

and let $\tilde{M}_f$ denote the corresponding measure. Note that $\tilde{m}_f(\theta) > m_f(\theta)$ for all $\theta \in \tilde{E}$, and also that $(M_f \vee \tilde{M}_f) = \tilde{M}_f \neq M_f$ and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$. Letting $M'_f = C_f(\tilde{M}_f)$, we show below that f and M'_f are a blocking coalition for M, contradicting the stability of M.

First of all, it follows from revealed preference that $C_f(M_f \vee M'_f) = M'_f$. To show that $M'_f \neq M_f$, note first that $\hat{m}_f(\theta) > m_f(\theta), \forall \theta \in \tilde{E}$ means $(\hat{M}_f \vee M_f)(\tilde{E}) = \hat{M}_f(\tilde{E})$, so

$$R_f(M_f \vee \hat{M}_f)(\tilde{E}) = (M_f \vee \hat{M}_f)(\tilde{E}) - C_f(M_f \vee \hat{M}_f)(\tilde{E}) = \hat{M}_f(\tilde{E}) - \hat{M}_f(\tilde{E}) = 0.$$

Then, since f has a substitutable preference and $\tilde{M}_f \sqsubset (M_f \vee \hat{M}_f)$, we have $R_f(\tilde{M}_f)(\tilde{E}) = 0$, which means $M'_f(\tilde{E}) = C_f(\tilde{M}_f)(\tilde{E}) = \tilde{M}_f(\tilde{E}) \neq M_f(\tilde{E})$. It only remains to show that $M'_f \sqsubset D^{\preceq f}(M)$. For this, note that since $\hat{M}$ is individually rational and $\hat{m}_f(\theta) > 0, \forall \theta \in \tilde{E}$, we have $f \succ_\theta \varnothing, \forall \theta \in \tilde{E}$. Given the definition of $\tilde{M}_f$, this implies that $\tilde{M}_f \sqsubset D^{\preceq f}(M)$ and thus $M'_f \sqsubset \tilde{M}_f \sqsubset D^{\preceq f}(M)$. ▌

Meanwhile, $\sum_{f \in F} M_f = \sum_{f \in F} \hat{M}_f$ since $M_\varnothing = \hat{M}_\varnothing$ as shown in the above claim. Hence, we conclude that $M_f(\Theta) = \hat{M}_f(\Theta)$ for each $f \in F$.

We then prove the next claim.

Claim S2. *For each $f \in \bar{F}$, there must be some t such that $p^t_f < \hat{p}^t_f$.*

Proof. Suppose to the contrary that $\hat{p}^t_f \leq p^t_f (< 1)$ for all $t \in T$. Since $\sum_{t \in T} M^t_f(\Theta) = M_f(\Theta) = \hat{M}_f(\Theta) = \sum_{t \in T} \hat{M}^t_f(\Theta)$ and $M_f \neq \hat{M}_f$, there must exist $t \in T$ such that the set $\{\theta \in \Theta^t | s_f(\theta) > p^t_f \geq \hat{p}^t_f \text{ and } m^t_f(\theta) > \hat{m}^t_f(\theta)\}$ has a positive measure. A contradiction then arises since, due to the fact that C_f selects all workers of type t whose scores are above the optimal cutoff $\hat{p}^t_f$ and that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$, the measure of type $\theta \in \Theta^t$ workers selected when $\hat{M}_f \vee M_f$ is available is equal to $\hat{m}^t_f(\theta) = \max\{\hat{m}^t_f(\theta), m^t_f(\theta)\}$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{p}^t_f$, which cannot be smaller than $m^t_f(\theta)$. ▌

Claim S3. *For any $f \in \bar{F}$ and $t \in T$, if $\hat{p}^t_f = 0$, then $\hat{M}_f(\Theta^t \cap \cdot) = M_f(\Theta^t \cap \cdot)$.*

Proof. Let us first observe that for any $f \in \bar{F}$ and t, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$, then we have $\hat{p}^t_f > p^t_f$ since, as we argued in the proof of Claim S2, the fact that $\hat{M}_f = C_f(\hat{M}_f \vee M_f)$

implies that $\hat{m}_f^t(\theta) = \max\{\hat{m}_f^t(\theta), m_f^t(\theta)\} \geq m_f^t(\theta)$ for all $\theta \in \Theta^t$ with $s_f(\theta) \geq \hat{p}_f^t$. We also know that if $\hat{M}_f(\Theta^t) < q_f^t$, then $\hat{p}_f^t = \hat{p}_f \leq \hat{p}_f^{t'}$, for all $t' \in T$. Hence, if $\hat{M}_f(\Theta^t) < M_f(\Theta^t)$ for $f \in \bar{F}$ and t, then $p_f^t < \hat{p}_f^t \leq \hat{p}_f^{t'}$, for all $t' \in T$.

Fix now any $f \in \bar{F}$ and $t \in T$ for which $\hat{p}_f^t = 0$. Since it means $\hat{p}_f^t \leq p_f^t$, we must have $\hat{M}_f(\Theta^t) \geq M_f(\Theta^t)$ according to the above argument. If, in addition, $\hat{M}_f(\Theta^t) > M_f(\Theta^t)$, then the fact that $\hat{M}_f(\Theta) = M_f(\Theta)$ implies that there must exist t' such that $\hat{M}_f(\Theta^{t'}) < M_f(\Theta^{t'})$. This means that $p_f^{t'} < \hat{p}_f^{t'} \leq \hat{p}_f^t = 0$, a contradiction. Hence, $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$.

Given $\hat{p}_f^t = 0$ (i.e. the lowest possible score), we must have $\max\{\hat{m}_f^t(\theta), m_f^t(\theta)\} = \hat{m}_f^t(\theta)$ for all $\theta \in \Theta^t$. In order that $\hat{M}_f(\Theta^t) = M_f(\Theta^t)$, we must then have $\hat{m}_f^t(\theta) = m_f^t(\theta)$ for (almost) all $\theta \in \Theta^t$, which leads to the desired conclusion. ▮

Claim S4. *For any $t \in T$, if there is some $f \in \bar{F}$ such that $\hat{p}_f^t > p_f^t$, then we must have $\hat{p}_{f'}^t > 0, \forall f' \in \bar{F}$.*

Proof. Fix a firm $f \in \bar{F}$ with $\hat{p}_f^t > p_f^t$. Suppose to the contrary that the set $\bar{F}_0 = \{f' \in \bar{F} | \hat{p}_{f'}^t = 0\}$ is nonempty, and note that $f \notin \bar{F}_0$. Then, let us define $\bar{F}_+ = \bar{F} \backslash \bar{F}_0$ and consider the set

$$\{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (p_f^t, \hat{p}_f^t), \text{ and } s_{f'}(\theta) < \hat{p}_{f'}^t \, \forall f' \in \bar{F}_+ \setminus \{f\}\}.$$

Since M is stable, all worker types in this set must be matched with f under M, which implies that they cannot be matched with any firm in $\tilde{F} \backslash \bar{F}$ under $\hat{M}$ since $\hat{M}_f = M_f$ for each $f \in F \backslash \bar{F}$ by assumption and also since $\hat{M}_\varnothing = M_\varnothing$ by Claim S1. Moreover, these workers cannot be matched with any firm $f' \in \bar{F}_+$ under $\hat{M}$ since their scores are below $\hat{p}_{f'}^t$. It thus follows that they must be matched with firms in $\bar{F}_0$ under $\hat{M}$ while being matched with $f \notin \bar{F}_0$ under M, which contradicts Claim S3. ▮

Claim S5. *Rich preferences hold.*

Proof. Fix any $f \in \bar{F}$ and $t \in T$ (given by Claim S2) such that $p_f^t < \hat{p}_f^t$, and let

$$\tilde{\Theta}_f^t := \{\theta \in \Theta | f \succ_\theta f'', \forall f'' \neq f, s_f(\theta) \in (p_f^t, \hat{p}_f^t), \text{ and } s_{f'}(\theta) < \hat{p}_{f'}^t \, \forall f' \in \bar{F} \setminus \{f\}\}$$

be a set of ethnic type t workers who prefer f to all other firms and have scores that will make them employable at f under M but not under $\hat{M}$ and not employable at all other firms in $\bar{F}$ under $\hat{M}$. Let $M' := \sum_{t \in T} G(\tilde{\Theta}_f^t \cap \cdot)$ denote the measure of these workers. The full support assumption and the fact (given by Claim S4) that $\hat{p}_{f'}^t > 0, \forall f' \in \bar{F}$ implies that $M'(\Theta) > 0$.

We show that these workers are not employed by any firm in $\bar{F}$ under either $\hat{M}$ or M. It is easy to see that these workers are not employed by any firm in $\bar{F}$ under $\hat{M}$ since their scores are below the cutoffs of these firms at $\hat{M}$. Since $\sum_{f\in F} M_f = \sum_{f\in F} \hat{M}_f$, and since $M_f = \hat{M}_f$ for each $f \in F \setminus \bar{F}$, we must have $\sum_{f\in\bar{F}} M_f = \sum_{f\in\bar{F}} \hat{M}_f$. It thus follows that these workers are not employed by firms in $\bar{F}$ under matching M either.

Next, note that the above argument implies $M' \sqsubset \hat{M}^f_{\bar{F}}$. Since $\hat{p}^t_f > p^t_f$, firm f will wish to replace some of its workers with these workers under M. Hence, $M_f \neq C_f((M_f + \hat{M}^f_{\bar{F}}) \wedge G)$, so the rich preferences property follows. ▌

The above claims complete the proof of the proposition. ▌

Proposition S1 and Theorem S3 then imply the following:

Corollary S1. *Suppose the firms' preferences are responsive with affirmative action, and satisfy the law of aggregate demand. If the full support condition holds, then a unique stable matching exists.*

Lastly, the next example demonstrates that the law of aggregate demand is also crucial for the uniqueness result: if the firm preferences violate the law of aggregate demand, then uniqueness of a stable matching does not necessarily hold even if the firm preferences are rich.

Example S1 (Necessity of LoAD for uniqueness)**.** Consider a continuum economy with worker types θ_1 and θ_2 (each with measure 1/2) and firms f_1 and f_2. Preferences are as follows:

1. Firm f_1 wants to hire as many workers of type θ_2 as possible if no worker of type θ_1 is available, but if any positive measure of type θ_1 workers is available, then f_1 wants to hire only type θ_1 workers and no type θ_2 workers at all, and f_1 wants to hire only up to measure 1/3 of type θ_1 workers.

2. The preference of firm f_2 is symmetric, changing the roles of worker types θ_1 and θ_2. More specifically, Firm f_2 wants to hire as many workers of type θ_1 as possible if no worker of type θ_2 is available, but if any positive measure of type θ_2 workers is available, then f_2 wants to hire only type θ_2 workers and no type θ_1 workers at all, and f_2 wants to hire only up to measure 1/3 of type θ_2 workers.

3. Worker preferences are as follows:

$$\theta_1 : f_2 \succ f_1 \succ \emptyset,$$
$$\theta_2 : f_1 \succ f_2 \succ \emptyset.$$

Clearly, the firm preferences are substitutable. To check the rich preference, note first that

$$\underline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{2}\theta_2 & \frac{1}{2}\theta_1 \end{pmatrix},$$

where the notation is such that measure 1/2 of type θ_1 workers are matched to f_2 and measure 1/2 of type θ_2 workers are matched to f_1.[6] Finally, firm preferences violate the law of aggregate demand because, for instance, the choice of f_1 from measure 1/2 of θ_2 is to hire all of them, but even adding a measure $\epsilon < 1/2$ of type θ_1 workers would cause f_1 to reject all θ_2 workers. As it turns out, there is a firm-optimal stable matching that is different from $\underline{M}$ and given as follow:

$$\overline{M} = \begin{pmatrix} f_1 & f_2 \\ \frac{1}{3}\theta_1 & \frac{1}{3}\theta_2 \end{pmatrix}.$$

S.3.1 Choice Functions Representing Responsive Preferences with Affirmative Action

For any firm f and subpopulation $X \sqsubset G$ available to f, we can define firm f's optimal choice from X as a solution to the following problem:

$$[C] \qquad \max_{Y \sqsubset X} \int s_f(\theta) dY$$

subject to

$$Y(\Theta) \leq q_f, \text{ and } Y(\Theta^t) \leq q_f^t, \forall t \in T.$$

As in Theorem 2, one can show that the feasible set is compact. Since its objective function is continuous in X (by the definition of weak convergence, given continuity of $s(\cdot)$), the maximum is well defined. Further, the set of optimal choices is closed, so it is compact.

[6] That this is a worker-optimal matching follows from the fact that the worker-proposing DA ends after the first round where each worker applies to her preferred firm while the firm accepts her. Then, under any matching $\hat{M} \neq \underline{M}$ that satisfies $\hat{M}_f = C_f(\hat{M}_f \vee \underline{M}_f)$ for all f, some firm, say f_1, must be matched with a positive measure of θ_1 workers. Given that $\hat{M}$ is individually rational, this implies that f_1 is not matched with any θ_2 workers. Also, since f_2 is matched with no more than measure 1/3 workers of θ_2 under any individual rational matching, at least measure 1/6 of θ_2 workers are unemployed under $\hat{M}$, which means that these workers belong to $\hat{M}_{\bar{F}}^{f_2}$ since they prefer f_2 to $\emptyset$ and $\emptyset \notin \bar{F}$. If they are available to f_2 in addition to $\underline{M}_{f_2}$, then f_2 would choose not to be matched with any θ_1 workers, to whom it is matched under $\underline{M}_{f_2}$. Thus, the rich preference condition is satisfied.

We next show that an optimal choice can be found in a class of feasible subpopulations with a cutoff structure. For each $t \in T$, let $\tilde{p}_f^t := \inf\{s \in [0,1] | X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq s\}) \leq q_f^t\}$. Then, we say Y is the **optimal cutoff rule for firm f at X** if $Y(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq p_f^t\}) = X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \geq p_f^t\})$, and $Y(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) < p_f^t\}) = 0$, where $p_f^t := \max\{p_f, \tilde{p}_f^t\}$, and "the common cutoff" p_f is the supremum of the set of common cutoffs that maximize $Y(\Theta)$ subject to $Y(\Theta) \leq q_f$ and $Y(\Theta^t) \leq q_f^t, \forall t \in T$. Note that the optimal cutoff rule is uniquely determined.

Claim S6. *The optimal cutoff rule for firm f at X is its optimal choice from X.*

Proof. For any feasible solution Y to $[C]$, consider a cutoff rule $\hat{Y}$, given by $\hat{Y}(\Theta^t \cap \{\theta | s_f(\theta) \geq p_f^t\} \cap E) = X(\Theta^t \cap \{\theta | s_f(\theta) \geq p_f^t\} \cap E)$ for each $E \in \Sigma$, and $\hat{Y}(\Theta^t \cap \{\theta | s_f(\theta) < p_f^t\} \cap E) = 0$, for each $E \in \Sigma$, for some cutoff score p_f^t, for each $t \in T$. In words, a cutoff rule selects all workers above a certain cutoff score and rejects all workers below that score. As the cutoff score p_f^t rises, $\hat{Y}(\Theta^t)$ falls continuously (since $\hat{Y}$, being a subpopulation of G, is absolutely continuous), and it equals $X(\Theta^t)$ when $p_f^t = 0$ and zero when $p_f^t = 1$. Hence, there exists $p_f^t \in [0,1]$ such that $\hat{Y}(\Theta^t) = Y(\Theta^t)$.

Since both Y and $\hat{Y}$, being subpopulation of G which has density, have density functions say y and $\hat{y}$, respectively. In that case, $\hat{y}(\theta) = x(\theta) \geq y(\theta)$ if $s_f(\theta) \geq p_f^t$ and $\hat{y}(\theta) = 0 \leq y(\theta)$ if $s_f(\theta) < p_f^t$. Hence,

$$\int_{\Theta^t} s_f(\theta)\hat{y}(\theta)d\theta - \int_{\Theta^t} s_f(\theta)y(\theta)d\theta = \int_{\Theta^t} s_f(\theta)(\hat{y}(\theta) - y(\theta))d\theta$$
$$\geq \int_{\Theta^t} p_f^t(\hat{y}(\theta) - y(\theta))d\theta = p_f^t[\hat{Y}(\Theta^t) - Y(\Theta^t)] = 0.$$

In short, $\hat{Y}$ is feasible and yields a weakly higher value of objective than does Y. It follows that an optimal choice can be found in the class of cutoff rules. Moreover, if Y differs from $\hat{Y}$ for a positive measure, the inequality is strict. This implies that an optimal choice must coincide with a cutoff rule almost everywhere (i.e., for every positive measure set).

Fix any optimal choice Y that is a cutoff rule. If $Y(\Theta^t) < q_f^t$ for some t, then there exists an optimal cutoff rule in which $p_f^t \leq p_f^{t'}$ for all $t' \neq t$. To see this, suppose an optimal choice has $p_f^t > p_f^{t'}$, where t' is the ethnic type with the lowest cutoff at the optimal choice. We can assume without loss of generality that $p_f^{t'} = \inf\{s_f(\theta) | \theta \in \Theta^{t'}, y(\theta) > 0\}$, or else we can raise $p_f^{t'}$ slightly without any consequence. If $X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \in [p_f^{t'}, p_f^t)\}) = 0$, then we can lower p_f^t without consequence to $p_f^{t'}$, so the claim holds. If $X(\Theta^t \cap \{\theta \in \Theta | s_f(\theta) \in [p_f^{t'}, p_f^t)\}) > 0$, then we can slightly lower p_f^t and slightly raise $p_f^{t'}$ so as to keep all constraints satisfied, which increases the value of the objective, producing a contradiction.

This observation implies that there exists a common cutoff p_f that applies to all t whose quota is not binding, and the cutoffs for ethnic types with binding quotas are weakly higher than p_f. Hence, we can write $p_f^t := \max\{p_f, \tilde{p}_f^t\}$, where $\tilde{p}_f^t$ is defined above.

The common cutoff p_f should be chosen to maximize $Y(\Theta)$ subject to $Y(\Theta) \leq q_f$ and $Y(\Theta^t) \leq q_f^t, \forall t \in T$, or else it can be lowered to increase the employment (and thus increase the value of the objective). Let P_f^t be the set of maximizers,[7] and let $\bar{p}_f^t := \sup P_f^t$. Then, $\bar{p}_f^t \in P_f^t$ due to the compactness of the optimal choices: Any sequence of the optimal cutoff rules with common cutoff $p_f^t \in P_f^t$ converging to the cutoff rule with common cutoff $\bar{p}_f^t$ must be optimal as well, so its limit must be optimal given the compactness of the optimal choices. ∎

Based on Claim S6, we define a choice function C_f to be an optimal cutoff rule. The resulting choice function then satisfies the revealed preference property: If $X \sqsubset X'$ and $C_f(X') \sqsubset X$, then $C_f(X')$ is also an optimal cutoff rule at X.

Next, it is routine to see that C_f satisfies the law of aggregate demand. If $X \sqsubset \hat{X}$, then the optimal cutoff rule at the latter leads to the firm choosing a weakly higher mass of workers than the optimal cutoff rule at X.

It is also easy to see C_f exhibits substitutability. Again fix $X \sqsubset \hat{X}$. We show that $R_f(X) \sqsubset R_f(\hat{X})$, where R_f is defined before. For non-triviality, assume $R_f(X)(\Theta) > 0$. Let $(\hat{p}_f^t)_t$ be the cutoffs associated with $C_f(\hat{X})$ and let $(p_f^t)_t$ be the cutoffs associated with $C_f(X)$. Note first if the quota for t is binding at the optimal choice from X, we can only have $\hat{p}_f^t \geq p_f^t$, or else the quota for t will be violated at $\hat{X}$. There are two cases. First, suppose first $C_f(X)(\Theta) < q_f$. In this case, no mass of agents from X is rejected at $C_f(X)$ except for violating quotas for ethnic types, and those who are rejected for violating the ethnic type quotas must be rejected as well at $C_f(\hat{X})$ since their cutoffs are weakly higher. Hence, $R_f(X) \sqsubset R_f(\hat{X})$. Suppose next $C_f(X)(\Theta) = q_f$. In this case, the common cutoff $\hat{p}_f$ at $C_f(\hat{X})$ must be weakly higher than the common cutoff p_f at $C_f(X)$. If not, then feasibility of $C_f(\hat{X})$ implies that there exists t such that its quota is binding at $C_f(X)$ but not at $C_f(\hat{X})$. But then $\hat{p}_f^t > p_f^t \geq p_f > \hat{p}_f$, which implies that $C_f(\hat{X})$ violates the property of the optimal cutoff rule at $\hat{X}$. Since all cutoffs are uniformly higher at $C_f(\hat{X})$, we conclude that $R_f(X) \sqsubset R_f(\hat{X})$.

[7] The set P_f^t may not be a singleton. Suppose for instance that the measure of available workers is strictly smaller than the capacity of a firm, and say the firm has no affirmative action constraint and the infimum score of the available workers is say $s_m > 0$. Then any $p_f^t \in [0, s_m]$ will be an optimal cutoff, since selecting all available workers is optimal for the firm.

S.4 Matching with Contracts

Our paper has assumed that the terms of employment contracts are exogenously given. In many applications, however, they are decided endogenously. To study such a situation, we generalize our basic model by introducing a continuum-population version of the "matching with contracts" model due to Hatfield and Milgrom (2005).

Let Ω denote a finite set of all available contracts with its typical element denoted as ω. Assume that Ω is partitioned into subsets, $\{\Omega_f\}_{f\in\tilde{F}}$, where Ω_f is the set of contacts for $f \in \tilde{F}$ and $\Omega_\emptyset = \{\omega_\emptyset\}$ (where $\omega_\emptyset$ denotes the option of not contracting with any firm). Each contract ω specifies contract terms a firm f may offer to a worker.[8] Let $f(\omega) \in \tilde{F}$ denote the firm associated with contract ω (or the outside option if $\omega = \omega_\emptyset$). Thus, $f(\omega) = f$ if and only if $\omega \in \Omega_f$. We use $P \in \mathcal{P}$ to denote workers' preference defined over Ω. Let $\omega_-^P \in \Omega$ denote a contract that is an immediate predecessor of ω according to preference P, that is, ω_-^P is the contract with the property $\omega_-^P \succ_P \omega$ and $\omega' \succeq_P \omega_-^P$ for all $\omega' \succ_P \omega$. As before, Θ_P denotes the subset of types in Θ whose preference is given by P.

In the current framework, the relevant unit of analysis is the measure of workers assigned to a particular contract. We let $X_\omega \in \mathcal{X}$ denote the subpopulation assigned to contract $\omega \in \Omega$ and $X_f = (X_\omega)_{\omega\in\Omega_f}$ denote a profile of subpopulations contracting with firm f. For any profiles $X, X' \in \mathcal{X}^{|\Omega_f|}$, we denote $X \sqsubset_f X'$ if $X_\omega \sqsubset X'_\omega$ for all $\omega \in \Omega_f$. Given a profile $X_f = (X_\omega)_{\omega\in\Omega_f}$, we use

$$X_f^{\preceq\omega}(\cdot) := \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega_f:\omega'\preceq_P\omega} X_{\omega'}(\Theta_P \cap \cdot), \tag{S5}$$

to denote the measure of workers hired by f under contract ω or worse; these are the workers who are willing to work for f under ω given their current contracts. We then let $X_f^{\preceq} = (X_f^{\preceq\omega})_{\omega\in\Omega_f}$.

For any $\omega \in \Omega_f$, let $X_\omega \in \mathcal{X}$ denote the subpopulation of workers who are available to firm f under the contract ω. Given any profile $X_f = (X_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|}$, each firm f's choice is described by a map $X_f \mapsto C_f(X_f) = (C_\omega(X_f))_{\omega\in\Omega_f} \in \mathcal{Y}_f(X_f)$, where

$$\mathcal{Y}_f(X_f) := \{Y_f \in \mathcal{X}^{|\Omega_f|} \mid Y_f^{\preceq\omega} \sqsubset X_\omega, \forall \omega \in \Omega_f\}.$$

For any profile of subpoulations in $\mathcal{Y}_f(X_f)$, the measure of workers who are hired by f under any contract $\omega \in \Omega_f$ or worse cannot exceed the measure of workers, X_ω, who are available under ω. The requirement that the output of C_f should belong to $\mathcal{Y}_f(X_f)$ is

[8] Note that the contract itself does not contain information about the associated worker type, and that each firm's preference is determined by what worker types it is matched with under what contracts.

based on the premise that each firm f is aware of workers' preferences and also believes (correctly) that only those workers who are available under $\omega \in \Omega_f$ can be hired under the contracts that are weakly inferior to ω, and thus put an upper bound on the measure of workers that can be hired under the latter contracts. As before, we let $C_{\omega_\varnothing}(X_{\omega_\varnothing}) = X_{\omega_\varnothing}$. We then assume the revealed preference property that for any $X, X' \in \mathcal{X}^{|\Omega_f|}$ with $X' \sqsubset_f X$ and for $M_f = C_f(X)$, if $M_f \in \mathcal{Y}_f(X')$, then $M_f = C_f(X')$.

An **allocation** is $M = (M_\omega)_{\omega\in\Omega}$ such that $M_\omega \in \mathcal{X}$ for all $\omega \in \Omega$ and $\sum_{\omega\in\Omega} M_\omega = G$. Let $M_f = (M_\omega)_{\omega\in\Omega_f} \in \mathcal{X}^{|\Omega_f|}$ denote a profile of subpopulations who are matched with f. Given $M_f = (M_\omega)_{\omega\in\Omega_f}$, define $M_f^{\preceq\omega}$ by (S5) and let $M_f^{\preceq} = (M_f^{\preceq\omega})_{\omega\in\Omega_f}$. Note that $M_f^{\preceq\omega}$ corresponds to a subpopulation of workers already hired by firm f who are willing to work for f under ω given their current contracts. In other words, $M_f^{\preceq}$ does *not* include the workers available to firm f who are currently matched with firms other than f. A subpopulation of *all* workers—not only those hired by firm f—who are available to $f \in \tilde{F}$ under contract $\omega \in \Omega_f$ is denoted as before by

$$D^{\preceq\omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap \cdot).$$

Let $D^{\preceq f}(M) = (D^{\preceq\omega}(M))_{\omega\in\Omega_f}$.

Definition S5. An allocation $M = (M_\omega)_{\omega\in\Omega}$ is **stable** if

1. (Individual Rationality) $M_\omega(\Theta_P) = 0$ for all $P \in \mathcal{P}$ and $\omega \in \Omega$ satisfying $\omega \prec_P \omega_\varnothing$; and for each $f \in F$, $M_f = C_f(M_f^{\preceq})$, and
2. (No Blocking Coalition) There exist no $f \in F$ and $\widetilde{M}_f \in \mathcal{X}^{|\Omega_f|}, \widetilde{M}_f \neq M_f$ such that

$$\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \text{ and } \widetilde{M}_f^{\preceq} \sqsubset_f D^{\preceq f}(M).$$

Note that this definition reduces to the notion of stability in Definition 2 if each firm is associated with exactly one contract.

Let us now define a map $T = (T_\omega)_{\omega\in\Omega} : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$ by specifying, for each $\omega \in \Omega$ and $E \in \Sigma$,

$$T_\omega(X)(E) := \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E). \tag{S6}$$

Theorem S4. *$M = (M_\omega)_{\omega\in\Omega}$ is a stable allocation if and only if $M_f = C_f(X_f), \forall f \in \tilde{F}$, where $X = (X_\omega)_{\omega\in\Omega}$ is a fixed point of mapping T.*

Proof. **("Only if" part)** Suppose M is a stable allocation in $\mathcal{X}^{|\Omega|}$. We prove that $X = (D^{\preceq \omega}(M))_{\omega\in\Omega}$ is a fixed point of T. Let us first show that for each $\omega \in \Omega$, $X_\omega \in \mathcal{X}$. It is clear that as each M_ω is countably additive, so is $M_\omega(\Theta_P \cap \cdot)$, which implies that $X_\omega(\cdot) = D^{\preceq \omega}(M)(\cdot) = \sum_{P\in\mathcal{P}} \sum_{\omega'\in\Omega:\omega'\preceq_P \omega} M_{\omega'}(\Theta_P \cap \cdot)$ is also countably additive. It is also clear that since $(M_\omega)_{\omega\in\Omega}$ is an allocation, $X_\omega \sqsubset G$. Thus, we have $X_\omega \in \mathcal{X}$.

We next claim that $M_f = C_f(X_f)$ for all $f \in \tilde{F}$. This is immediate for $f = \emptyset$ since $M_\emptyset = X_\emptyset = C_\emptyset(X_\emptyset)$. To prove the claim for $f \neq \emptyset$, suppose for a contradiction that $M_f \neq C_f(X_f)$, and let us denote $\widetilde{M}_f = C_f(X_f)$. Since $C_f(X_f) \in \mathcal{Y}_f(X_f)$ by definition, we have $\widetilde{M}_f^{\preceq} \sqsubset_f X_f$ and thus $(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \sqsubset_f X_f$. Given this and $\widetilde{M}_f \in \mathcal{Y}_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, we have $\widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$ by revealed preference, which means that M is not stable since $\widetilde{M}_f^{\preceq} \sqsubset_f X_f = D^{\preceq f}(M)$, yielding the desired contradiction.

We next prove $X = T(X)$. The fact that $M_\omega = C_\omega(X_{f(\omega)}), \forall \omega \in \Omega$ means that $X_\omega - M_\omega = R_\omega(X_{f(\omega)}), \forall \omega \in \Omega$. Then, for each $\omega \in \Omega$ and $E \in \Sigma$, we obtain

$$\begin{aligned}
&\sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap E) \\
&= \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \Big(X_{\omega_-^P}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E)\Big) \\
&= \sum_{P:P(1)=\omega} G(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \left(\sum_{\omega'\in\Omega:\omega'\preceq_P \omega_-^P} M_{\omega'}(\Theta_P \cap E) - M_{\omega_-^P}(\Theta_P \cap E)\right) \\
&= \sum_{P:P(1)=\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) + \sum_{P:P(1)\neq\omega} \sum_{\omega'\in\Omega:\omega'\preceq_P\omega} M_{\omega'}(\Theta_P \cap E) = X_\omega(E),
\end{aligned}$$

where the second and fourth equalities follow from the definition of $X_{\omega_-^P}$ and X_ω, respectively, while the third from the fact that ω_-^P is an immediate predecessor of ω and $\sum_{\omega'\in\Omega:\omega'\preceq_P P(1)} M_{\omega'}(\Theta_P \cap E) = G(\Theta_P \cap E)$. The above equation holds for every contract $\omega \in \Omega$, so we conclude that $X = T(X)$, i.e. X is a fixed point of T.

("If" part) Let us first introduce some notations. Let ω_+^P denote an **immediate successor** of $\omega \in \Omega$ at $P \in \mathcal{P}$: that is, $\omega_+^P \prec_P \omega$, and for any $\omega' \prec_P \omega$, $\omega' \preceq_P \omega_+^P$. Note that for any $\omega, \tilde{\omega} \in \Omega$, $\omega = \tilde{\omega}_-^P$ if and only if $\tilde{\omega} = \omega_+^P$.

Suppose now that $X = (X_\omega)_{\omega\in\Omega} \in \mathcal{X}^{|\Omega|}$ is a fixed point of T. For each contract $\omega \in \Omega$ and $E \in \Sigma$, define

$$M_\omega(E) = X_\omega(E) - \sum_{P:P(|\Omega|)\neq\omega} X_{\omega_+^P}(\Theta_P \cap E), \tag{S7}$$

where $P(|\Omega|) \neq \omega$ means that ω is not ranked lowest at P.

We first verify that for each $\omega \in \Omega$, $M_\omega \in \mathcal{X}$. First, it is clear that for each $\omega \in \Omega$, as both $X_\omega(\cdot)$ and $X_{\omega_+^P}(\Theta_P \cap \cdot)$ are countably additive, so is M_ω. It is also clear that for each $\omega \in \Omega$, $M_\omega \sqsubset X_\omega$.

Let us next show that for all $\omega \in \Omega$, $P \in \mathcal{P}$, and $E \in \Sigma$,

$$X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \tag{S8}$$

which means that $X_\omega = D^{\preceq\omega}(M)$. To do so, consider first a contract ω that is ranked lowest at P. By (S7) and the fact that $X_{\omega_+^P}(\Theta_P \cap \cdot) \equiv 0$, we have $M_\omega(\Theta_P \cap E) = X_\omega(\Theta_P \cap E)$. Hence, (S8) holds for such ω. Consider now any $\omega \in \Omega$ which is not ranked last, and assume for an inductive argument that (S8) holds true for ω_+^P, so $X_{\omega_+^P}(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E)$. Then, by (S7), we have

$$\begin{aligned} X_\omega(\Theta_P \cap E) = M_\omega(\Theta_P \cap E) + X_{\omega_+^P}(\Theta_P \cap E) &= M_\omega(\Theta_P \cap E) + \sum_{\omega' \in \Omega : \omega' \preceq_P \omega_+^P} M_{\omega'}(\Theta_P \cap E) \\ &= \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E), \end{aligned}$$

as desired.

To show that $M = (M_\omega)_{\omega \in \Omega}$ is an allocation, let $\omega = P(1)$. Then, the definition of T and the fact that X is a fixed point of T imply that for any $E \in \Sigma$,

$$G(\Theta_P \cap E) = X_\omega(\Theta_P \cap E) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P \cap E) = \sum_{\omega' \in \Omega} M_{\omega'}(\Theta_P \cap E),$$

where the second equality follows from (S8). Since the above equation holds for every $P \in \mathcal{P}$, M is an allocation.

We now prove that $(M_\omega)_{\omega \in \Omega}$ is stable. To prove the first part of Condition 1 of Definition S5, note first that $C_{\omega_\varnothing}(X_{\omega_\varnothing}) = \{X_{\omega_\varnothing}\}$ and thus $R_{\omega_\varnothing} = 0$. Fix any $P \in \mathcal{P}$ and assume $\varnothing \neq P(|\Omega|)$, since there is nothing to prove if $\varnothing$ is ranked lowest at P. Consider a contract ω such that $\omega_-^P = \omega_\varnothing$. Then, X being a fixed point of T means $X_\omega(\Theta_P) = R_{\omega_-^P}(\Theta_P) = R_{\omega_\varnothing}(\Theta_P) = 0$, which implies by (S8) that $0 = X_\omega(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \preceq_P \omega} M_{\omega'}(\Theta_P) = \sum_{\omega' \in \Omega : \omega' \prec_P \omega_\varnothing} M_{\omega'}(\Theta_P)$, as desired.

To prove the second part of Condition 1 of Definition S5, we first show that $M_\omega = C_\omega(X_{f(\omega)})$, which is equivalent to showing $X_\omega - M_\omega = R_\omega(X_{f(\omega)})$. Since $X = T(X)$, we have $X_\omega(\Theta_P \cap \cdot) = R_{\omega_-^P}(X_{f(\omega_-^P)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(1)$, or $X_{\omega_+^P}(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot)$ for all $\omega \neq P(|\Omega|)$. Then, (S7) implies that for any $\omega \in \Omega$,

$$X_\omega(\cdot) - M_\omega(\cdot) = \sum_{P : P(|\Omega|) \neq \omega} X_{\omega_+^P}(\Theta_P \cap \cdot) = \sum_{P : P(|\Omega|) \neq \omega} R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = R_\omega(X_{f(\omega)})(\cdot),$$

as desired. The last equality here follows from the fact that $R_\omega(\Theta_P \cap \cdot) = 0$ if $\omega = P(|\Omega|)$. To see this, note that if $\omega = P(|\Omega|) = \omega_\emptyset$, then $R_\omega(X_{f(\omega)}) = R_{\omega_\emptyset}(X_\emptyset) = 0$ by definition of $R_{\omega_\emptyset}$, and that if $\omega = P(|\Omega|) \prec_P \omega_\emptyset$, then the individual rationality of M for workers implies that $X_\omega(\Theta_P \cap \cdot) = M_\omega(\Theta_P \cap \cdot) = 0$, which in turn implies $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) = 0$ since $R_\omega(X_{f(\omega)})(\Theta_P \cap \cdot) \sqsubset X_\omega(\Theta_P \cap \cdot)$. Given that $M_\omega = C_\omega(X_{f(\omega)})$ for all $\omega \in \Omega$ or $M_f = C_f(X_f)$ for all $f \in F$, $M_f = C_f(M_f^{\preceq})$ follows from the revealed preference and the fact that $M_f^{\preceq} \sqsubset_f X_f$.

It only remains to check Condition 2 of Definition S5. Suppose for a contradiction that it fails. Then, there exist f and $\widetilde{M}_f$ such that

$$M_f \neq \widetilde{M}_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \text{ and } \widetilde{M}_f^{\preceq} \sqsubset_f D^{\preceq f}(M). \tag{S9}$$

Then, we have $M_f \in \mathcal{Y}_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, $(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq}) \sqsubset_f D^{\preceq f}(M) = X_f$, and $M_f = C_f(X_f)$, which, by revealed preference, implies $M_f = C_f(\widetilde{M}_f^{\preceq} \vee M_f^{\preceq})$, contradicting (S9). We have thus proven that M is stable. ▌

Given this characterization result, the existence of stable allocation follows from assuming that for each $f \in F$, $C_f : \mathcal{X}^{|\Omega_f|} \to \mathcal{X}^{|\Omega_f|}$ is continuous, since it guarantees that $T : \mathcal{X}^{|\Omega|} \to \mathcal{X}^{|\Omega|}$ is also continuous:

Theorem S5. *If each firm's preference is continuous, then there a stable allocation exists.*

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