Optimal Monetary Policy with Heterogeneous Agents*

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Abstract

Incomplete markets models with heterogeneous agents are increasingly used for policy analysis. We propose a novel methodology for solving fully dynamic optimal policy problems in models of this kind, both under discretion and commitment. We illustrate our methodology by studying optimal monetary policy in an incomplete-markets model with non-contingent nominal assets and costly inflation. Under discretion, an inflationary bias arises from the central bank's attempt to redistribute wealth towards debtor households, which have a higher marginal utility of net wealth. Under commitment, this inflationary force is counteracted over time by the incentive to prevent expectations of future inflation from being priced into new bond issuances; under certain conditions, long run inflation is zero as both effects cancel out asymptotically. For a plausible calibration, we find that the optimal commitment features first-order initial inflation followed by a gradual decline towards its (near zero) long-run value. Welfare losses from discretionary policy are first-order in magnitude, affecting both debtors and creditors.

Keywords: optimal monetary policy, commitment and discretion, incomplete markets, nominal debt, inflation, redistributive effects, continuous time

JEL codes: E5, E62,F34.

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1 Introduction

Ever since the seminal work of Bewley (1983), Hugget (1993) and Aiyagari (1994), incomplete markets models with uninsurable idiosyncratic risk have become a workhorse for policy analysis in macro models with heterogeneous agents.[1] Among the different areas spawned by this literature, the analysis of the dynamic aggregate effects of fiscal and monetary policy has begun to receive considerable attention in recent years.[2]

As is well known, one difficulty when working with incomplete markets models is that the state of the economy at each point in time includes the cross-household wealth distribution, which is an infinite-dimensional object.[3] The development of numerical methods for computing equilibrium in these models has made it possible to study the effects of aggregate shocks and of particular policy rules. However, the infinite-dimensional nature of the endogenously-evolving wealth distribution has made it difficult to make progress in the analysis of *optimal* fiscal or monetary policy problems in this class of models.

In this paper, we propose a novel methodology for solving fully dynamic optimal policy problems in incomplete-markets models with uninsurable idiosyncratic risk, both under discretion and commitment. Key to our approach is that we cast the model in continuous time. This allows us to exploit the fact that the dynamics of the cross-sectional distribution are then characterized by a partial differential equation known as the *Kolmogorov forward* (KF) or *Fokker-Planck* equation, and therefore the problem can be solved by using calculus techniques in infinite-dimensional Hilbert spaces. To this end, we employ a generalized version of the classical differential known as Gateaux differential.

We illustrate our methodology by analyzing optimal monetary policy in an incomplete-markets economy. Our framework is close to Huggett's (1993) standard formulation. As in the latter, households trade non-contingent claims, subject to an exogenous borrowing limit, in order to smooth consumption in the face of idiosyncratic income shocks. Aside from casting the model in continuous time, we depart from Huggett's real framework by considering *nominally* non-contingent bonds with an arbitrarily long maturity, which allows monetary policy to have an effect on equilibrium allocations. In particular, our model features a classic Fisherian channel (Fisher, 1933), by which unanticipated inflation redistributes wealth from lending to borrowing households.[4] In order to have a meaningful trade-off in the choice of the inflation path, we also assume that inflation is

[1] For a survey of this literature, see e.g. Heathcote, Storesletten & Violante (2009).

[2] See our discussion of the related literature below.

[3] See e.g. Ríos-Rull (1995).

[4] See Doepke and Schneider (2006a) for an influential study documenting net nominal asset positions across US household groups and estimating the potential for inflation-led redistribution. See Auclert (2016) for a recent analysis of the Fisherian redistributive channel in a more general incomplete-markets model that allows for additional redistributive mechanisms.

costly, which can be rationalized on the basis of price adjustment costs; in addition, expected future inflation raises the nominal cost of new debt issuances through inflation premia. Finally, we depart from the standard closed-economy setup by considering a small open economy, with the aforementioned (domestic currency-denominated) bonds being also held by risk-neutral foreign investors. This, aside from making the framework somewhat more tractable,[5] also makes the policy analysis richer, by making the redistributive Fisherian channel operate not only between domestic lenders and borrowers, but also between the latter and foreign bond holders.[6]

On the analytical front, we show that discretionary optimal policy features an 'inflationary bias', whereby the central bank tries to use inflation so as to redistribute wealth and hence consumption. In particular, we show that at each point in time optimal discretionary inflation increases with the average cross-household net liability position weighted by each household's marginal utility of net wealth. This reflects the two redistributive motives mentioned before. On the one hand, inflation redistributes from foreign investors to domestic borrowers (*cross-border redistribution*). On the other hand, and somewhat more subtly, under market incompleteness and standard concave preferences for consumption, borrowing households have a higher marginal utility of net wealth than lending ones. As a result, they receive a higher effective weight in the optimal inflation decision, giving the central bank an incentive to redistribute wealth from creditor to debtor households (*domestic redistribution*).

Under commitment, the same redistributive motives to inflate exist, but they are counteracted by an opposing force: the central bank internalizes how investors' expectations of future inflation affect their pricing of the long-term nominal bonds from the time the optimal commitment plan is formulated ('time zero') onwards. At time zero, inflation is close to that under discretion, as no prior commitments about inflation exist. But from then on, the fact that the price of newly issued bonds incorporates promises about the future inflation path gives the central bank an incentive to commit to reducing inflation over time. Importantly, we show that under certain conditions on preferences and parameter values, the steady state inflation rate under the optimal commitment is zero;[7] that is, in the long run the redistributive motive to inflate exactly cancels out with the incentive to reduce inflation expectations and nominal yields for an economy that is a net debtor.

[5] We restrict our attention to equilibria in which the domestic economy remains a net debtor *vis-à-vis* the rest of the World, such that domestic bonds are always in positive net supply. As a result, the usual bond market clearing condition in closed-economy models is replaced by a no arbitrage condition for foreign investors that effectively prices the nominal bond. This allows us to reduce the number of constraints in the policy-maker's problem featuring the infinite-dimensional wealth distribution.

[6] As explained by Doepke and Schneider (2006a), large net holdings of nominal (domestic currency-denominated) assets by foreign investors increase the potential for a large inflation-induced wealth transfer from foreigners to domestic borrowers.

[7] In particular, assuming separable preferences, then in the limiting case in which the central bank's discount rate is arbitrarily close to that of foreign investors, optimal steady-state inflation under commitment is arbitrarily close to zero.

We then solve numerically for the full transition path under commitment and discretion. We calibrate our model to match a number of features of a prototypical European small open economy, such as the size of gross household debt or their net international position.[8] We find that optimal time-zero inflation, which as mentioned before is very similar under commitment and discretion, is first-order in magnitude. We also show that both the cross-border and the domestic redistributive motives are quantitatively relevant for initial inflation. Under discretion, inflation remains high due to the inflationary bias discussed before. Under commitment, by contrast, inflation falls gradually towards its long-run level (essentially zero, under our calibration), reflecting the central bank's efforts to prevent expectations of future inflation from being priced into new bond issuances. In summary, under commitment the central bank front-loads inflation so as to transitorily redistribute existing wealth from lenders to borrowing households, but commits to gradually undo such initial inflation.

In welfare terms, the discretionary policy implies sizable (first-order) losses relative to the optimal commitment. Such losses are suffered by creditor households, but also by debtor ones. The reason is that, under discretion, expectations of permanent future positive inflation are fully priced into current nominal yields. This impairs the very redistributive effects of inflation that the central bank is trying to bring about, and leaves only the direct welfare costs of permanent inflation, which are born by creditor and debtor households alike.

Overall, our findings shed some light on current policy and academic debates regarding the appropriate conduct of monetary policy once household heterogeneity is taken into account. In particular, our results suggest that an optimal plan that includes a commitment to price stability in the medium/long-run may also justify a relatively large (first-order) positive initial inflation rate, with a view to shifting resources to households that have a relatively high marginal utility of net wealth.

Related literature. Our main contribution is methodological. To the best of our knowledge, ours is the first paper to solve for a fully dynamic optimal policy problem, both under commitment and discretion, in a general equilibrium model with uninsurable idiosyncratic risk in which the cross-sectional net wealth distribution (an infinite-dimensional, endogenously evolving object) is a state in the planner's optimization problem. Different papers have analyzed Ramsey problems in similar setups. Dyrda and Pedroni (2014) study the optimal dynamic Ramsey taxation in a discrete-time Aiyagari economy. They assume that the paths for the optimal taxes follow splines with nodes set at a few exogenously selected periods, and perform a numerical search of the optimal node values. Acikgoz (2014), instead, follows the work of Davila et al. (2012) in employing calculus of variations to characterize the optimal Ramsey taxation in a similar setting. However,

[8] These targets are used to inform the calibration of the gap between the central bank's and foreign investors' discount rates, which as explained before is a key determinant of long-run inflation under commitment.

after having shown that the optimal long-run solution is independent of the initial conditions, he analyzes quantitatively the steady state but does not solve the full dynamic optimal path.[9] Other papers, such as Gottardi, Kajii, and Nakajima (2011) or Itskhoki and Moll (2015), are able to find the optimal Ramsey policies in incomplete-market models under particular assumptions that allow for closed-form solutions. In contrast to these papers, we introduce a methodology for computing the full dynamics under commitment in a general incomplete-markets setting.[10] Regarding discretion, we are not aware of any previous paper that has quantitatively analyzed the Markov Perfect Equilibrium (MPE) in models with uninsurable idiosyncratic risk.

The use of infinite-dimensional calculus in continuous-time problems with non-degenerate distributions was first introduced in Lucas and Moll (2014) and Nuño and Moll (2015) to find the first-best and the constrained-efficient allocation in heterogeneous-agents models. In the latter models, a social planner directly decides on individual policies in order to control a distribution of states subject to idiosyncratic shocks. Here, by contrast, we show how these techniques may be extended to a game-theoretical setting involving several agents, who are moreover forward-looking. Under commitment, as is well known, this requires the policy-maker to internalize how her promised future decisions affect private agents' expectations; the problem is then augmented by introducing costates that reflect the value of deviating from the promises made at time zero.[11] If commitment is not possible, the value of these costates is zero at all times.[12]

Aside from the methodological contribution, our paper relates to several strands of the literature. As explained before, our analysis assigns an important role to the Fisherian redistributive channel of monetary policy, a long-standing topic that has experienced a revival in recent years. Doepke and Schneider (2006a) document net nominal asset positions across US sectors and

[9] Werning (2007) studies optimal fiscal policy in a heterogeneous-agents economy in which agent types are permanently fixed. Park (2014) extends this approach to a setting of complete markets with limited commitment in which agent types are stochastically evolving. Both papers provide a theoretical characterization of the optimal policies based on the primal approach introduced by Lucas and Stokey (1983). Park (2014) analyzes numerically the steady state but not the transitional dynamics, due to the complexity of solving the latter problem with that methodology.

[10] In addition, the numerical solution of the model is greatly improved in continuous-time, as discussed in Achdou, Lasry, Lions and Moll (2015) or Nuño and Thomas (2015). This is due to two properties of continuous-time models. First, the HJB equation is a deterministic partial differential equation which can be solved using efficient finite difference methods. Second, the dynamics of the distribution can be computed relatively quickly as they amount to calculating a matrix adjoint: the KF operator is the adjoint of the infinitesimal generator of the underlying stochastic process. This computational speed is essential as the computation of the optimal policies requires several iterations along the complete time-path of the distribution. In a home PC, the Ramsey problem presented here can be solved in less than five minutes.

[11] In the commitment case, we construct a Lagrangian in a suitable function space and obtain the corresponding first order conditions. The resulting optimal policy is time inconsistent (reflecting the effect of investors' inflation expectations on bond pricing), depending only on time and the initial wealth distribution.

[12] Under discretion, we work with a generalization of the Bellman principle of optimality and the Riesz representation theorem to obtain the time-consistent optimal policies depending on the distribution at any moment in time.

household groups and estimate empirically the redistributive effects of different inflation scenarios. Adam and Zhu (2014) perform a similar analysis for Euro Area countries, adding the cross-country redistributive dimension to the picture.

A recent literature addresses the Fisherian and other channels of monetary policy transmission in the context of general equilibrium models with incomplete markets and household heterogeneity. In terms of modelling, our paper is closest to Auclert (2016), Kaplan, Moll and Violante (2016), Gornemann, Kuester and Nakajima (2012), McKay, Nakamura and Steinsson (2015) or Luetticke (2015), who also employ different versions of the incomplete-markets, uninsurable idiosyncratic risk framework.[13] Other contributions, such as Doepke and Schneider (2006b), Meh, Ríos-Rull and Terajima (2010), Sheedy (2014), Challe et al. (2015) or Sterk and Tenreyro (2015), analyze the redistributive effects of monetary policy in environments where heterogeneity is kept finite-dimensional. We contribute to this literature by analyzing fully dynamic optimal monetary policy, both under commitment and discretion, in a standard incomplete markets model with uninsurable idiosyncratic risk.

Although this paper focuses on monetary policy, the techniques developed here lend themselves naturally to the analysis of other policy problems, e.g. optimal fiscal policy, in this class of models. Recent work analyzing fiscal policy issues in incomplete-markets, heterogeneous-agent models includes Heathcote (2005), Oh and Reis (2012), Kaplan and Violante (2014) and McKay and Reis (2016).

Finally, our paper is related to the literature on *mean-field games* in Mathematics. The name, introduced by Lasry and Lions (2006a,b), is borrowed from the mean-field approximation in statistical physics, in which the effect on any given individual of all the other individuals is approximated by a single averaged effect. In particular, our paper is related to Bensoussan, Chau and Yam (2015), who analyze a model of a major player and a distribution of atomistic agents that shares some similarities with the Ramsey problem discussed here.[14]

2 Model

We extend the basic Huggett framework to an open-economy setting with nominal, non-contingent, long-term debt contracts and disutility costs of inflation. Let $(\Sigma, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ be a filtered probability space. Time is continuous: $t \in [0, \infty)$. The domestic economy is composed of a measure-one continuum of households that are heterogeneous in their net financial wealth. There is a single,

[13] For work studying the effects of different aggregate shocks in related environments, see e.g. Guerrieri and Lorenzoni (2016), Ravn and Sterk (2013), and Bayer et al. (2015).

[14] Other papers analyzing mean-field games with a large non-atomistic player are Huang (2010), Nguyen and Huang (2012a,b) and Nourian and Caines (2013). A survey of mean-field games can be found in Bensoussan, Frehse and Yam (2013).

freely traded consumption good, the World price of which is normalized to 1. The domestic price (equivalently, the nominal exchange rate) at time t is denoted by P_t and evolves according to

$$dP_t = \pi_t P_t dt, \tag{1}$$

where π_t is the domestic inflation rate (equivalently, the rate of nominal exchange rate depreciation).

2.1 Households

2.1.1 Output and net assets

Household $k \in [0, 1]$ is endowed with an income y_{kt} units of the good at time t, where y_{kt} follows a two-state Poisson process: $y_{kt} \in \{y_1, y_2\}$, with $y_1 < y_2$. The process jumps from state 1 to state 2 with intensity λ_1 and vice versa with intensity λ_2.

Households trade a nominal, non-contingent, long-term, domestic-currency-denominated bond with one another and with foreign investors. Let A_{kt} denote the net holdings of such bond by household k at time t; assuming that each bond has a nominal value of one unit of domestic currency, A_{kt} also represents the total nominal (face) value of net assets. For households with a negative net position, $(-)\, A_{kt}$ represents the total nominal (face) value of outstanding net liabilities ('debt' for short). We assume that outstanding bonds are amortized at rate $\delta > 0$ per unit of time.[15] The nominal value of the household's net asset position thus evolves as follows,

$$dA_{kt} = \left(A_{kt}^{new} - \delta A_{kt}\right) dt,$$

where A_{kt}^{new} is the flow of new assets purchased at time t. The nominal market price of bonds at time t is Q_t. Let c_{kt} denote the household's consumption. The budget constraint of household k is then

$$Q_t A_{kt}^{new} = P_t \left(y_{kt} - c_{kt}\right) + \delta A_{kt}.$$

Combining the last two equations, we obtain the following dynamics for net nominal wealth,

$$dA_{kt} = \left(\frac{\delta}{Q_t} - \delta\right) A_{kt} dt + \frac{P_t \left(y_{kt} - c_{kt}\right)}{Q_t} dt. \tag{2}$$

We define real net wealth as $a_{kt} \equiv A_{kt}/P_t$. Its dynamics are obtained by applying Itô's lemma to equations (1) and (2),

$$da_{kt} = \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t\right) a_{kt} + \frac{y_{kt} - c_{kt}}{Q_t}\right] dt. \tag{3}$$

[15] This tractable form of long-term bonds was first introduced by Leland and Toft (1986).

We assume that each household faces the following exogenous borrowing limit,

$$a_{kt} \geq \phi. \tag{4}$$

where $\phi \leq 0$.

For future reference, we define the *nominal bond yield* r_t implicit in a nominal bond price Q_t as the discount rate for which the discounted future promised cash flows equal the bond price. The discounted future promised payments are $\int_0^\infty e^{-(r_t+\delta)s}\delta ds = \delta/(r_t+\delta)$. Therefore, the nominal bond yield is

$$r_t = \frac{\delta}{Q_t} - \delta. \tag{5}$$

2.1.2 Preferences

Household have preferences over paths for consumption c_{kt} and domestic inflation π_t discounted at rate $\rho > 0$,

$$U_{k0} \equiv \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_{kt}, \pi_t) dt \right], \tag{6}$$

with $u_c > 0$, $u_\pi > 0$, $u_{cc} < 0$ and $u_{\pi\pi} < 0$.[16] From now onwards we drop subscripts k for ease of exposition. The household chooses consumption at each point in time in order to maximize its welfare. The *value function* of the household at time t can be expressed as

$$v(t,a,y) = \max_{\{c_s\}_{s=t}^\infty} \mathbb{E}_t \left[\int_t^\infty e^{-\rho(s-t)} u(c_s, \pi_s) ds \right], \tag{7}$$

subject to the law of motion of net wealth (3) and the borrowing limit (4). We use the short-hand notation $v_i(t,a) \equiv v(t,a,y_i)$ for the value function when household income is low ($i=1$) and high ($i=2$). The *Hamilton-Jacobi-Bellman* (HJB) equation corresponding to the problem above is

$$\rho v_i(t,a) = \frac{\partial v_i}{\partial t} + \max_c \left\{ u(c, \pi(t)) + s_i(t,a,c) \frac{\partial v_i}{\partial a} \right\} + \lambda_i \left[v_j(t,a) - v_i(t,a) \right], \tag{8}$$

for $i, j = 1, 2$, and $j \neq i$, where $s_i(t,a,c)$ is the *drift* function, given by

$$s_i(t,a,c) = \left(\frac{\delta}{Q(t)} - \delta - \pi(t) \right) a + \frac{y_i - c}{Q(t)}, \ i = 1, 2. \tag{9}$$

[16] The general specification of disutility costs of inflation nests the case of costly price adjustments *à la* Rotemberg. See Section 4.1 for further discussion.

The first order condition for consumption is

$$u_c(c_i\,(t,a)\,,\pi\,(t)) = \frac{1}{Q\,(t)}\frac{\partial v_i(t,a)}{\partial a}. \tag{10}$$

Therefore, household consumption increases with nominal bond prices and falls with the slope of the value function. Intuitively, a higher bond price (equivalently, a lower yield) gives the household an incentive to save less and consume more. A steeper value function, on the contrary, makes it more attractive to save so as to increase net asset holdings.

2.2 Foreign investors

Households trade bonds with competitive risk-neutral foreign investors that can invest elsewhere at the risk-free real rate $\bar{r}$. As explained before, domestic bonds are amortized at rate δ. Foreign investors also discount future future nominal payoffs with the accumulated domestic inflation (i.e. exchange rate depreciation) between the time of the bond purchase and the time such payoffs accrue. Therefore, the nominal price of the bond at time t is given by

$$Q(t) = \int_t^\infty \delta e^{-(\bar{r}+\delta)(s-t)-\int_t^s \pi_u du} ds. \tag{11}$$

Taking the derivative with respect to time, we obtain

$$Q(t)\,(\bar{r}+\delta+\pi(t)) = \delta + Q'(t). \tag{12}$$

The partial differential equation (12) provides the risk-neutral pricing of the nominal bond. The boundary condition is

$$\lim_{t\to\infty} Q(t) = \frac{\delta}{\bar{r}+\delta+\pi\,(\infty)}, \tag{13}$$

where $\pi\,(\infty)$ is the inflation level in the steady state, which we assume exits.

2.3 Central Bank

There is a central bank that chooses monetary policy. We assume that there are no monetary frictions so that the only role of money is that of a unit of account. The monetary authority chooses the inflation rate π_t.[17] In Section 3, we will study in detail the optimal inflationary policy of the central bank.

[17] This could be done, for example, by setting the nominal interest rate on a lending (or deposit) short-term nominal facility with foreign investors.

2.4 Competitive equilibrium

The state of the economy at time t is the joint distribution of net wealth and output, $f(t,a,y_i) \equiv f_i(t,a)$, $i = 1,2$. The dynamics of this distribution are given by the *Kolmogorov Forward* (KF) equation,

$$\frac{\partial f_i(t,a)}{\partial t} = -\frac{\partial}{\partial a}\left[s_i\left(t,a\right)f_i(t,a)\right] - \lambda_i f_i(t,a) + \lambda_j f_j(t,a), \tag{14}$$

$\forall a \in [\phi, \infty)$, $i,j = 1,2$, $j \neq i$. The distribution satisfies the normalization

$$\sum_{i=1}^{2}\int_{\phi}^{\infty} f_i\left(t,a\right)da = 1. \tag{15}$$

We define a competitive equilibrium in this economy.

Definition 1 (Competitive equilibrium) *Given a sequence of inflation rates $\pi\left(t\right)$ and an initial wealth-output distribution $f(0,a,y)$, a competitive equilibrium is composed of a household value function $v(t,a,y)$, a consumption policy $c(t,a,y)$, a bond price function $Q\left(t\right)$ and a distribution $f(t,a,y)$ such that:*

1. *Given π, the price of bonds in (12) is Q.*
2. *Given Q and π, v is the solution of the households' problem (8) and c is the optimal consumption policy.*
3. *Given Q, π, and c, f is the solution of the KF equation (14).*

Notice that, given π, the problem of foreign investors can be solved independently of that of the household, which in turn only depends on π and Q but not on the aggregate distribution.

We can compute some aggregate variables of interest. The aggregate real net financial wealth in the economy is

$$\bar{a}_t \equiv \sum_{i=1}^{2}\int_{\phi}^{\infty} af_i\left(t,a\right)da. \tag{16}$$

We may similarly define gross real household debt as $\bar{b}_t \equiv \sum_{i=1}^{2}\int_{\phi}^{0}\left(-a\right)f_i\left(t,a\right)da$. Aggregate consumption is

$$\bar{c}_t \equiv \sum_{i=1}^{2}\int_{\phi}^{\infty} c_i\left(a,t\right)f_i\left(t,a\right)da,$$

where $c_i\left(a,t\right) \equiv c(t,a,y_i)$, $i = 1,2$, and aggregate output is

$$\bar{y}_t \equiv \sum_{i=1}^{2}\int_{\phi}^{\infty} y_i f_i\left(t,a\right)da.$$

These quantities are linked by the current account identity,

$$\begin{aligned}
\frac{d\bar{a}_t}{dt} &= \sum_{i=1}^{2}\int_{\phi}^{\infty} a\frac{\partial f_i(t,a)}{\partial t}da = \sum_{i=1}^{2}\int_{\phi}^{\infty} a\left[-\frac{\partial}{\partial a}\left(s_i f_i\right)da - \lambda_i f_i(t,a) + \lambda_j f_j(t,a)\right]da \\
&= \sum_{i=1}^{2}\int_{\phi}^{\infty} -a\frac{\partial}{\partial a}\left(s_i f_i\right)da = -\sum_{i=1}^{2} a s_i f_i|_{\phi}^{\infty} + \sum_{i=1}^{2}\int_{\phi}^{\infty} s_i f_i da \\
&= \left(\frac{\delta}{Q_t} - \delta - \pi_t\right)\bar{a}_t + \frac{\bar{y}_t - \bar{c}_t}{Q_t}, \qquad (17)
\end{aligned}$$

where we have used (14) in the second equality, and we have applied the boundary conditions $s_1(t,\phi) f_1(t,\phi) + s_2(t,\phi) f_2(t,\phi) = 0$ in the last equality.[18]

Finally, we make the following assumption.

Assumption 1 *The value of parameters is such that in equilibrium the economy is always a net debtor against the rest of the World:* $\bar{a}_t \leq 0\ \forall t$.

This condition is imposed for tractability. We have restricted households to save only in bonds issued by other households, and this would not be possible if the country was a net creditor *vis-à-vis* the rest of the World. In addition to this, we have assumed that the bonds issued by the households are priced by foreign investors, which requires that there should be a positive net supply of bonds to the rest of the World to be priced. In any case, this assumption is consistent with the experience of the small open economies that we target for calibration purposes, as we explain in Section 4.

3 Optimal monetary policy

We now turn to the design of the optimal monetary policy. Following standard practice, we assume that the central bank is utilitarian, i.e. it gives the same Pareto weight to each household. In order to illustrate the role of commitment vs. discretion in our framework, we will consider both the case in which the central bank can credibly commit to a future inflation path (the Ramsey problem), and the time-consistent case in which the central bank decides optimal current inflation given the current state of the economy (the Markov Perfect equilibrium).

[18] This condition is related to the fact that the KF operator is the adjoint of the infinitesimal generator of the stochastic process (3). See Appendix A for more information. See also Appendix B.6 in Achdou et al. (2015).

3.1 Central bank preferences

The central bank is assumed to be benevolent and hence maximizes economy-wide aggregate welfare,

$$U_0^{CB} = \int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} v_i\left(0,a\right) f_i(0,a)da. \tag{18}$$

It will turn out to be useful to express the above welfare criterion as follows.

Lemma 1 *The welfare criterion (18) can alternatively be expressed as*

$$U_0^{CB} \equiv \int_0^{\infty} e^{-\rho s}\left[\int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} u\left(c_i\left(a,s\right),\pi\left(s\right)\right) f_i(s,a)da\right] ds. \tag{19}$$

3.2 Discretion (Markov Perfect Equilibrium)

Consider first the case in which the central bank cannot commit to any future policy. The inflation rate π then depends only on the current value of the aggregate state variable, the net wealth distribution $\{f_i\left(t,a\right)\}_{i=1,2} \equiv f\left(t,a\right)$; that is, $\pi\left(t\right) \equiv \pi^{MPE}\left[f\left(t,a\right)\right]$. This is a Markovian problem in a space of distributions. The value functional of the central bank is given by

$$J^{MPE}\left[f\left(t,\cdot\right)\right] = \max_{\{\pi_s\}_{s=t}^{\infty}} \int_t^{\infty} e^{-\rho(s-t)}\left[\sum_{i=1}^{2}\int_{\phi}^{\infty} u\left(c_{is}\left(a\right),\pi_s\right) f_i(s,a)da\right] ds, \tag{20}$$

subject to the law of motion of the distribution (14). Notice that the optimal value J^{MPE} and the optimal policy π^{MPE} are not ordinary functions, but *functionals*, as they map the infinite-dimensional state variable $f\left(t,a\right)$ into $\mathbb{R}$.

Let $f_0\left(\cdot\right) \equiv \{f_i\left(0,a\right)\}_{i=1,2}$ denote the initial distribution. We can define the equilibrium in this case.

Definition 2 (Markov Perfect Equilibrium) *Given an initial distribution* f_0, *a* symmetric Markov Perfect Equilibrium *is composed of a sequence of inflation rates* $\pi\left(t\right)$, *a household value function* $v(t,a,y)$, *a consumption policy* $c(t,a,y)$, *a bond price function* $Q\left(t\right)$ *and a distribution* $f(t,a,y)$ *such that:*

1. *Given* π, *then* v, c, Q *and* f *are a competitive equilibrium.*

2. *Given* c, Q *and* f, π *is the solution to the central bank problem (20).*

The fact that v, c, Q and f are part of a competitive equilibrium needs to be imposed in the definition of Markov Perfect Equilibrium, as it is not implicit in the central bank's problem (20).

Using standard dynamic programming arguments, the problem (20) can be expressed recursively as

$$J^{MPE}\left[f\left(t,\cdot\right)\right]=\max_{\{\pi_s\}_{s=t}^{\tau}}\int_t^{\tau}e^{-\rho(s-t)}\left[\int_{\phi}^{\infty}\sum\nolimits_{i=1}^{2}u\left(c_{is},\pi_s\right)f_i(s,a)da\right]ds+e^{-\rho(\tau-t)}J^{MPE}\left[f\left(\tau,\cdot\right)\right], \tag{21}$$

for any $\tau>t$ and subject to the law of motion of the distribution (14).

The following proposition characterizes the solution to the central bank's problem under discretion.

Proposition 1 (Optimal inflation - MPE) *In addition to equations (14), (12), (8) and (10), if a solution to the MPE problem (20) exists, the inflation rate function $\pi\left(t\right)$ must satisfy*

$$\sum_{i=1}^{2}\int_{\phi}^{\infty}\left[a\frac{\partial v_i}{\partial a}-u_{\pi}\left(c_i\left(t,a\right),\pi\left(t\right)\right)\right]f_i\left(t,a\right)da=0. \tag{22}$$

In addition, the value functional must satisfy

$$J^{MPE}\left[f\left(t,\cdot\right)\right]=\sum_{i=1}^{2}\int_{\phi}^{\infty}v_i\left(t,a\right)f_i(t,a)da, \tag{23}$$

The proof is in Appendix A. Our approach combines the dynamic programming representation (21) with the *Riesz Representation Theorem*, which allows decomposing the central bank value functional J^{MPE} as an aggregation of individual values $v_i\left(t,a\right)$ across agents.

Equation (22) captures the basic static trade-off that the central bank faces when choosing inflation under discretion. The central bank balances the marginal utility cost of higher inflation across the economy (u_{π}) against the marginal welfare effects due to the impact of inflation on the real value of households' nominal net positions ($a\frac{\partial v_i}{\partial a}$). For borrowing households ($a<0$), the latter effect is positive as inflation erodes the real value of their debt burden, whereas the opposite is true for creditor ones ($a>0$). Moreover, provided that the value function is *concave* in net wealth $\left(\frac{\partial^2 v_i}{\partial a^2}<0, i=1,2,\right)$, and given Assumption 1 (the country as a whole is a net debtor), the central bank has a double motive to use inflation for redistributive purposes.[19] On the one hand, it will try to redistribute wealth from foreign investors to domestic borrowers (*cross-border redistribution*). On the other hand, and somewhat more subtly, since borrowing households have a higher marginal utility of net wealth than creditor ones, the central bank will be led to redistribute from the latter to the former, as such course of action is understood to raise welfare in the domestic economy as a whole (*domestic redistribution*).

[19] The concavity of the value function is guaranteed for the separable utility function presented in Assumption 2 below.

3.3 Commitment

Assume now that the central bank can credibly commit at time zero to an inflation path $\{\pi(t)\}_{t=0}^{\infty}$. The optimal inflation path is now a function of the initial distribution $f_0(a)$ and of time: $\pi(t) \equiv \pi^R[t, f_0(a)]$. The value functional of the central bank is now given by

$$J^R[f_0(\cdot)] = \max_{\{\pi_s, Q_s, v(s,\cdot), c(s,\cdot), f(s,\cdot)\}_{s=0}^{\infty}} \int_0^{\infty} e^{-\rho s} \left[\int_{\phi}^{\infty} \sum\nolimits_{i=1}^{2} u(c_{is}, \pi_s) f_i(s,a) da\right] ds, \tag{24}$$

subject to the law of motion of the distribution (14), the bond pricing equation (12), and household's HJB equation (8) and optimal consumption choice (10). The optimal value J^R and the optimal policy π^R are again functionals, as in the discretionary case, only now they map the *initial* distribution $f_0(\cdot)$ into $\mathbb{R}$, as opposed to the distribution at each point in time. Notice that the central bank maximizes welfare taking into account not only the state dynamics (14), but *also* the HJB equation (8) and the bond pricing condition (12). That is, the central bank understands how it can steer households' and foreign investors' expectations by committing to an inflation path. This is unlike in the discretionary case, where the central bank takes the expectations of other agents as given.

Definition 3 (Ramsey problem) *Given an initial distribution f_0, a* Ramsey problem *is composed of a sequence of inflation rates $\pi(t)$, a household value function $v(t,a,y)$, a consumption policy $c(t,a,y)$, a bond price function $Q(t)$ and a distribution $f(t,a,y)$ such that they solve the central bank problem (24).*

If v, f, c and Q are a solution to the problem (24), given π, they constitute a competitive equilibrium, as they satisfy equations (14), (12), (8) and (10). Therefore the Ramsey problem could be redefined as that of finding the π such that v, f, c and Q are a competitive equilibrium and the central bank's welfare criterion is maximized.

The Ramsey problem is an optimal control problem in a suitable function space. The following proposition characterizes the solution to the central bank's problem under commitment.

Proposition 2 (Optimal inflation - Ramsey) *In addition to equations (14), (12), (8) and (10), if a solution to the Ramsey problem (24) exists, the inflation path $\pi(t)$ must satisfy*

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \left[a \frac{\partial v_{it}}{\partial a} - u_{\pi}(c_i(t,a), \pi(t))\right] f_i(t,a) da - \mu(t) Q(t) = 0, \tag{25}$$

where $\mu(t)$ is a costate with law of motion

$$\frac{d\mu(t)}{dt} = (\rho - \bar{r} - \pi(t) - \delta) \mu(t) + \sum_{i=1}^{2} \int_{\phi}^{\infty} \frac{\partial v_{it}}{\partial a} \frac{\delta a + y_i - c_i(t,a)}{Q(t)^2} f_i(t,a) da \tag{26}$$

and initial condition $\mu(0) = 0$.

The proof can also be found in Appendix A. Our approach is to solve the constrained optimization problem (24) in an infinite-dimensional Hilbert space. To this end, we need to employ a generalized version of the classical differential known as 'Gateaux differential'.[20]

The equation determining optimal inflation under commitment (25), is identical to that in the discretionary case (22), *except* for the presence of the costate $\mu(t)$, which is the Lagrange multiplier associated to the bond pricing equation (12). Intuitively, $\mu(t)$ captures the value to the central bank of promises about time-t inflation made to foreign investors at time 0. Such value is zero only at the time of announcing the Ramsey plan ($t = 0$), because the central bank is not bound by previous commitments, but it will generally be different from zero at any time $t > 0$. By contrast, in the MPE case no promises are made at any point in time, hence the absence of such costate. Therefore, the static trade-off between the welfare cost of inflation and the welfare gains from inflating away net liabilities, explained above in the context of the MPE solution, is now modified by the central bank's need to respect past promises to investors about current inflation. If $\mu(t) < 0$, then the central bank's incentive to create inflation at time $t > 0$ so as to redistribute wealth will be tempered by the fact that it internalizes how expectations of higher inflation affect investors' bond pricing prior to time t.

Notice that the Ramsey problem is not time-consistent, due precisely to the presence of the (forward-looking) bond pricing condition in that problem.[21] If at some future time $\tilde{t} > 0$ the central bank decided to re-optimize given the current state $f\left(\tilde{t}, a, y\right)$, the new path for optimal inflation $\tilde{\pi}(t) \equiv \pi^R\left[t, f\left(\tilde{t}, \cdot\right)\right]$ would not need to coincide with the original path $\pi(t) \equiv \pi^R\left[t, f(0, \cdot)\right]$, as the value of the costate at that point would be $\tilde{\mu}\left(\tilde{t}\right) = 0$ (corresponding to a new commitment formulated at time t), whereas under the original commitment it is $\mu\left(\tilde{t}\right) \neq 0$.[22]

3.4 Some analytical results

In order to provide some additional analytical insights on optimal policy, we make the following assumption on preferences.

[20] The system composed of equations (8), (12), (14), (10), (25) and (26) is technically known as *forward-backward*, as both households and investors proceed backwards in order to compute their optimal values, policies and bond prices, whereas the distributional dynamics proceed forwards.

[21] As is well known, the MPE solution is time consistent, as it only depends on the current state.

[22] We also note that in the Ramsey equilibrium the Lagrange multipliers associated to households' HJB equation (8) and optimal consumption decision (10) are zero in all states (see the Appendix). That is, households' forward-looking optimizing behavior does not represent a source of time-inconsistency, as the monetary authority would choose at all times the same individual consumption and saving policies as the households themselves.

Assumption 2 *Consider the class of separable utility functions*

$$u(c, \pi) = u^c(c) - u^\pi(\pi).$$

The consumption utility function u^c *is bounded, concave and continuous with* $u_c^c > 0$, $u_{cc}^c < 0$ *for* $c > 0$. *The inflation disutility function* u^π *satisfies* $u_\pi^\pi > 0$ *for* $\pi > 0$, $u_\pi^\pi < 0$ *for* $\pi < 0$, $u_{\pi\pi}^\pi > 0$ *for all* π, *and* $u^\pi(0) = u_\pi^\pi(0) = 0$.

We first obtain the following result.

Lemma 2 *Let preferences satisfy Assumption 2. The optimal value function is concave.*

The following result establishes the existence of a positive *inflationary bias* under discretionary optimal monetary policy.

Proposition 3 (Inflation bias under discretion) *Let preferences satisfy Assumption 2. Optimal inflation under discretion is then positive at all times:* $\pi(t) > 0$ *for all* $t \geq 0$.

The proof can be found in Appendix A. To gain intuition, we can use the above separable preferences in order to express the optimal inflation decision under discretion (equation 22) as

$$u_\pi^\pi(\pi(t)) = \sum_{i=1}^{2} \int_\phi^\infty (-a) \frac{\partial v_i}{\partial a} f_i(t, a)\, da. \tag{27}$$

That is, under discretion inflation increases with the average net liabilities *weighted by each household's marginal utility of wealth*, $\partial v_i / \partial a$. Notice first that, from Assumption 1, the country as a whole is a net debtor: $\sum_{i=1}^{2} \int_\phi^\infty (-a) f_i(t, a)\, da = (-)\, \bar{a}_t \geq 0$. This, combined with the strict concavity of the value function (such that debtors effectively receive more weight than creditors), makes the right-hand side of (27) strictly positive. Since $u_\pi^\pi(\pi) > 0$ only for $\pi > 0$, it follows that inflation must be positive. Notice that, even if the economy as a whole is neither a creditor or a debtor ($\bar{a}_t = 0$), the concavity of the value function implies that, as long as there is wealth dispersion, the central bank will have a reason to inflate.

The result in Proposition 3 is reminiscent of the classical inflationary bias of discretionary monetary policy originally emphasized by Kydland and Prescott (1977) and Barro and Gordon (1983). In those papers, the source of the inflation bias is a persistent attempt by the monetary authority to raise output above its natural level. Here, by contrast, it arises from the welfare gains that can be achieved for the country as a whole by redistributing wealth towards debtors.

We now turn to the commitment case. Under the above separable preferences, from equation (25) optimal inflation under commitment satisfies

$$u_{\pi}^{\pi}(\pi(t)) = \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(t, a)\, da + \mu(t) Q(t). \tag{28}$$

In this case, the inflationary pressure coming from the redistributive incentives is counterbalanced by the value of time-0 promises about time-t inflation, as captured by the costate $\mu(t)$. Thus, a negative value of such costate leads the central bank to choose a *lower* inflation rate than the one it would set *ceteris paribus* under discretion.

Unfortunately, we cannot solve analytically for the optimal path of inflation. However, we are able to establish the following important result regarding the long-run level of inflation under commitment.

Proposition 4 (Optimal long-run inflation under commitment) *Let preferences satisfy Assumption 2. In the limit as $\rho \to \bar{r}$, the optimal steady-state inflation rate under commitment tends to zero: $\lim_{\rho \to \bar{r}} \pi(\infty) = 0$.*

Provided households' (and the benevolent central bank's) discount factor is arbitrarily close to that of foreign investors, then optimal long-run inflation under commitment will be arbitrarily close to zero. The intuition is the following. The inflation path under commitment converges over time to a level that optimally balances the marginal welfare costs and benefits of trend inflation. On the one hand, the welfare costs include the direct utility costs, but also the increase in nominal bond yields that comes about with higher expected inflation; indeed, from the definition of the yield (5) and the expression for the long-run nominal bond price (13), the long-run nominal bond yield is given by the following long-run Fisher equation,

$$r(\infty) = \frac{\delta}{Q(\infty)} - \delta = \bar{r} + \pi(\infty), \tag{29}$$

such that nominal yields increase one-for-one with (expected) inflation in the long run. On the other hand, the welfare benefits of inflation are given by its redistributive effect (for given nominal yields). As $\rho \to \bar{r}$, these effects tend to exactly cancel out precisely at zero inflation.

Proposition 4 is reminiscent of a well-known result from the New Keynesian literature, namely that optimal long-run inflation in the standard New Keynesian framework is exactly zero (see e.g. Benigno and Woodford, 2005). In that framework, the optimality of zero long-run inflation arises from the fact that, at that level, the welfare gains from trying to exploit the short-run output-inflation trade-off (i.e. raising output towards its socially efficient level) exactly cancel out with the welfare losses from permanently worsening that trade-off (through higher inflation expectations).

Key to that result is the fact that, in that model, price-setters and the (benevolent) central bank have the same (steady-state) discount factor. Here, the optimality of zero long-run inflation reflects instead the fact that, at zero trend inflation, the welfare gains from trying to redistribute wealth from creditors to debtors becomes arbitrarily close to the welfare losses from lower nominal bond prices when the discount rate of the investors pricing such bonds is arbitrarily close to that of the central bank.

Assumption 1 restricts us to have $\rho > \bar{r}$, as otherwise households would we able to accumulate enough wealth so that the country would stop being a net debtor to the rest of the World. However, Proposition 4 provides a useful benchmark to understand the long-run properties of optimal policy in our model when ρ is very close to $\bar{r}$. This will indeed be the case in our subsequent numerical analysis.

4 Numerical analysis

In the previous section we have characterized the optimal monetary policy in our model. In this section we solve numerically for the dynamic equilibrium under optimal policy, using numerical methods to solve continuous-time models with heterogeneous agents, as in Achdou et al. (2015) or Nuño and Moll (2015). Before analyzing the dynamic path of this economy under the optimal policy, we first analyze the steady state towards which such path converges asymptotically. The numerical algorithms that we use are described in Appendices B (steady-state) and C (transitional dynamics).

4.1 Calibration

The calibration is intended to be mainly illustrative, given the model's simplicity and parsimoniousness. We calibrate the model to replicate some relevant features of a prototypical European small open economy.[23] Let the time unit be one year. For the calibration, we consider that the economy rests at the steady state implied by a zero inflation policy.[24] When integrating across

[23]We will focus for illustration on the UK, Sweden, and the Baltic countries (Estonia, Latvia, Lithuania). We choose these countries because they (separately) feature desirable properties for the purpose at hand. On the one hand, UK and Sweden are two prominent examples of relatively small open economies that retain an independent monetary policy, like the economy in our framework. This is unlike the Baltic states, who recently joined the euro. However, historically the latter states have been relatively large debtors against the rest of the World, which make them square better with our theoretical restriction that the economy remains a net debtor at all times (UK and Sweden have also remained net debtors in basically each quarter for the last 20 years, but on average their net balance has been much closer to zero).

[24]This squares reasonably well with the experience of our target economies, which have displayed low and stable inflation for most of the recent past.

households, we therefore use the stationary wealth distribution associated to such steady state.[25]

We assume the following specification for preferences,

$$u(c,\pi) = \log(c) - \frac{\psi}{2}\pi^2. \tag{30}$$

As discussed in Appendix D, our quadratic specification for the inflation utility cost, $\frac{\psi}{2}\pi^2$, can be micro-founded by modelling firms explicitly and allowing them to set prices subject to standard quadratic price adjustment costs *à la* Rotemberg (1982). We set the scale parameter ψ such that the slope of the inflation equation in a Rotemberg pricing setup replicates that in a Calvo pricing setup for reasonable calibrations of price adjustment frequencies and demand curve elasticities.[26]

We jointly set households' discount rate ρ and borrowing limit ϕ such that the steady-state net international investment position (NIIP) over GDP $(\bar{a}/\bar{y})$ and gross household debt to GDP $(\bar{b}/\bar{y})$ replicate those in our target economies.[27]

We target an average bond duration of 4.5 years, as in Auclert (2016). In our model, the Macaulay bond duration equals $1/(\delta+\bar{r})$. We set the world real interest rate $\bar{r}$ to 3 percent. Our duration target then implies an amortization rate of $\delta = 0.19$.

The idiosyncratic income process parameters are calibrated as follows. We follow Huggett (1993) in interpreting states 1 and 2 as 'unemployment' and 'employment', respectively. The transition rates between unemployment and employment (λ_1, λ_2) are chosen such that (i) the unemployment rate $\lambda_2/(\lambda_1+\lambda_2)$ is 10 percent and (ii) the job finding rate is 0.1 at monthly frequency or $\lambda_1 = 0.72$ at annual frequency.[28] These numbers describe the 'European' labor market

[25] The wealth dimension is discretized by using 1000 equally-spaced grid points from $a = \phi$ to $a = 10$. The upper bound is needed only for operational purposes but is fully innocuous, because the stationary distribution places essentially zero mass for wealth levels above $a = 8$.

[26] The slope of the continuous-time New Keynesian Phillips curve in the Calvo model can be shown to be given by $\chi(\chi+\rho)$, where χ is the price adjustment rate (the proof is available upon request). As shown in Appendix D, in the Rotemberg model the slope is given by $\frac{\varepsilon-1}{\psi}$, where ε is the elasticity of firms' demand curves and ψ is the scale parameter in the quadratic price adjustment cost function in that model. It follows that, for the slope to be the same in both models, we need

$$\psi = \frac{\varepsilon-1}{\chi(\chi+\rho)}.$$

Setting ε to 11 (such that the gross markup $\varepsilon/(\varepsilon-1)$ equals 1.10) and χ to 4/3 (such that price last on average for 3 quarters), and given our calibration for ρ, we obtain $\psi = 5.5$.

[27] According to Eurostat, the NIIP/GDP ratio averaged minus 48.6% across the Baltic states in 2016:Q1, and only minus 3,8% across UK-Sweden. We thus target a NIIP/GDP ratio of minus 25%, which is about the midpoint of both values. Regarding gross household debt, we use BIS data on 'total credit to households', which averaged 85.9% of GDP across Sweden-UK in 2015:Q4 (data for the Baltic countries are not available). We thus target a 90% household debt to GDP ratio.

[28] Analogously to Blanchard and Galí (2010; see their footnote 20), we compute the equivalent annual rate λ_1 as

$$\lambda_1 = \sum_{i=1}^{12} \left(1-\lambda_1^m\right)^{i-1} \lambda_1^m,$$

calibration in Blanchard and Galí (2010). We normalize average income $\bar{y} = \frac{\lambda_2}{\lambda_1+\lambda_2} y_1 + \frac{\lambda_1}{\lambda_1+\lambda_2} y_2$ to 1. We also set y_1 equal to 71 percent of y_2, as in Hall and Milgrom (2008). Both targets allow us to solve for y_1 and y_2. Table 1 summarizes our baseline calibration.

Table 1. Baseline calibration

Parameter	Value	Description	Source/Target
$\bar{r}$	0.03	world real interest rate	standard
ψ	5.5	scale inflation disutility	slope NKPC in Calvo model
δ	0.19	bond amortization rate	Macaulay duration = 4.5 years
λ_1	0.72	transition rate unemployment-to-employment	monthly job finding rate of 0.1
λ_2	0.08	transition rate employment-to-unemployment	unemployment rate 10 percent
y_1	0.73	income in unemployment state	Hall & Milgrom (2008)
y_2	1.03	income in employment state	$E(y) = 1$
ρ	0.0302	subjective discount rate	NIIP -25% of GDP
ϕ	-3.6	borrowing limit	HH debt/GDP ratio 90%

Figure 1 displays a number of objects in the zero-inflation steady state, including the value functions $v_i(a, \infty) \equiv v_i(a)$ and the consumption policies $c_i(a)$, for $i = 1, 2$. Importantly, while the figure displays the steady-state value functions, it should be noted by their concavity is preserved in the time-varying value functions implied by the optimal policy paths, .

4.2 Steady state under optimal policy

We start our numerical analysis of optimal policy by computing the steady state equilibrium to which each monetary regime (commitment and discretion) converges. Table 2 displays a number of steady-state objects. Under commitment, the optimal long-run inflation is close to zero (-0.05 percent), consistently with Proposition 4 and the fact ρ and $\bar{r}$ are very closed to each other in our calibration.[29] As a result, long-run gross household debt and net total assets (as % of GDP) are very similar to those under zero inflation. From now on, we use $x \equiv x(\infty)$ to denote the steady state value of any variable x. As shown in the previous section, the long-run nominal yield is $r = \bar{r} + \pi$, where the World real interest rate $\bar{r}$ equals 3 percent in our calibration.

where λ_1^m is the monthly job finding rate.

[29]As explained in section 3, in our baseline calibration we have $\bar{r} = 0.03$ and $\rho = 0.0302$.

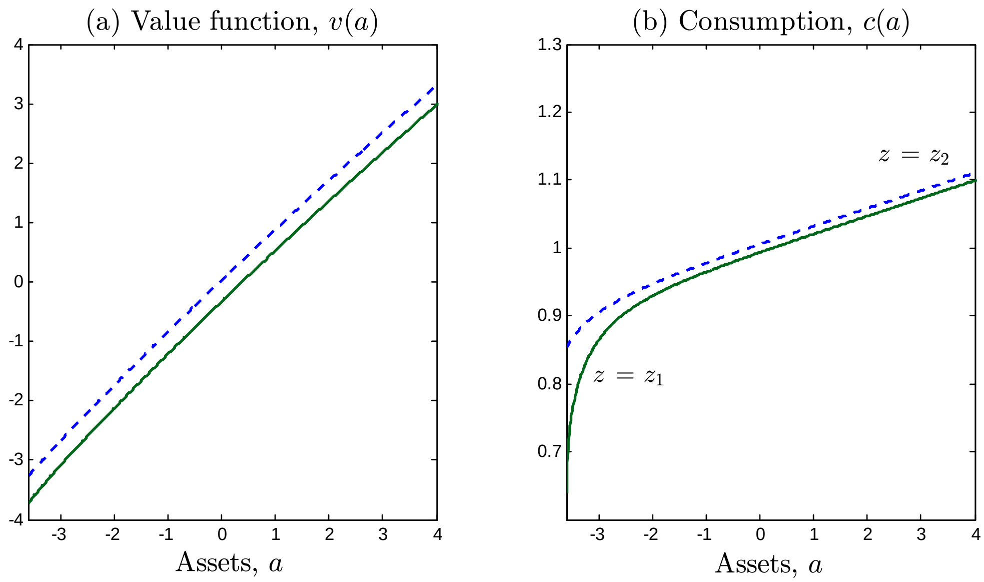

Figure 1: Steady state with zero inflation.

Table 2. Steady-state values under optimal policy

	units	Ramsey	MPE
Inflation, π	%	-0.05	1.68
Nominal yield, r	%	2.95	4.68
Net assets, $\bar{a}$	% GDP	-24.1	-0.6
Gross assets (creditors)	% GDP	65.6	80.0
Gross debt (debtors), $\bar{b}$	% GDP	89.8	80.6
Current acc. deficit, $\bar{c} - \bar{y}$	% GDP	-0.63	-0.01

Under discretion, by contrast, long run inflation is 1.68 percent, which reflects the inflationary bias discussed in the previous section. The presence of an inflationary bias makes nominal interest rates higher through the Fisher equation (29). The economy's aggregate net liabilities fall substantially relative to the commitment case (0.6% vs 24.1%), mostly reflecting larger asset accumulation by creditor households.

4.3 Optimal transitional dynamics

As explained in Section 3, the optimal policy paths depend on the initial (time-0) net wealth distribution across households, $\{f_i(0, a)\}_{i=1,2}$, which is an (infinite-dimensional) primitive in our model.[30] In the interest of isolating the effect of the policy regime (commitment vs discretion) on the equilibrium allocations, we choose a common initial distribution in both cases. For the purpose of illustration, we consider the stationary distribution under zero inflation as the initial distribution. In section 4.5 we will analyze the robustness of our results to a wide range of alternative initial distributions.

Consider first the case under commitment (Ramsey policy). The optimal paths are shown by the green solid lines in Figure 2.[31] Under our assumed functional form for preferences in (30), we have from equation (28) that initial optimal inflation is given by

$$\pi(0) = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i(0,a)}{\partial a} f_i(0,a)\, da,$$

where we have used the fact that $\mu(0) = 0$, as there are no pre-commitments at time zero. Therefore, the time-0 inflation rate, of about 4.6 percent, reflects exclusively the redistributive motive (both cross-border and domestic) discussed in Section 3. This domestic redistribution can be clearly seen in panels (h) and (i) of Figure 2: the transitory inflation created under commitment gradually reduces both the assets of creditor households and the liabilities of debtor ones. The cross-border redistribution is apparent from panel (g): the country as a whole temporarily reduces its net liabilities *vis-à-vis* the rest of the World.

As time goes by, optimal inflation under commitment gradually declines towards its (near) zero long-run level. The intuition is straightforward. At the time of formulating its commitment, the central bank exploits the existence of a stock of nominal bonds issued in the past. This means that the inflation created by the central bank has no effect on the prices at which those bonds were issued. However, the price of nominal bonds issued from time 0 onwards *does* incorporate the expected future inflation path. Under commitment, the central bank internalizes the fact that higher future inflation reduces nominal bond prices, i.e. it raises nominal bond yields, which hurts net bond issuers. This effect becomes stronger and stronger over time, as the fraction of total nominal bonds that were issued before the time-0 commitment becomes smaller and smaller. This gives the central bank the right incentive to gradually reduce inflation over time. Formally, in the

[30] As explained in section 3.1, in our numerical exercises we assume that the income distribution starts at its ergodic limit: $f_y(y_i) = \lambda_{j \neq i} / (\lambda_1 + \lambda_2), i = 1, 2$. Also, in all our subsequent exercises we assume that the time-0 net wealth distribution conditional on being in state 1 (unemployment) is identical to that conditional on state 2 (employment): $f_{a|y}(0, a \mid y_2) = f_{a|y}(0, a \mid y_1) \equiv f_0(a)$. Therefore, the initial joint density is simply $f(0, a, y_i) = f_0(a) \lambda_{j \neq i} / (\lambda_1 + \lambda_2)$, $i = 1, 2$.

[31] We have simulated 800 years of data at monthly frequency.

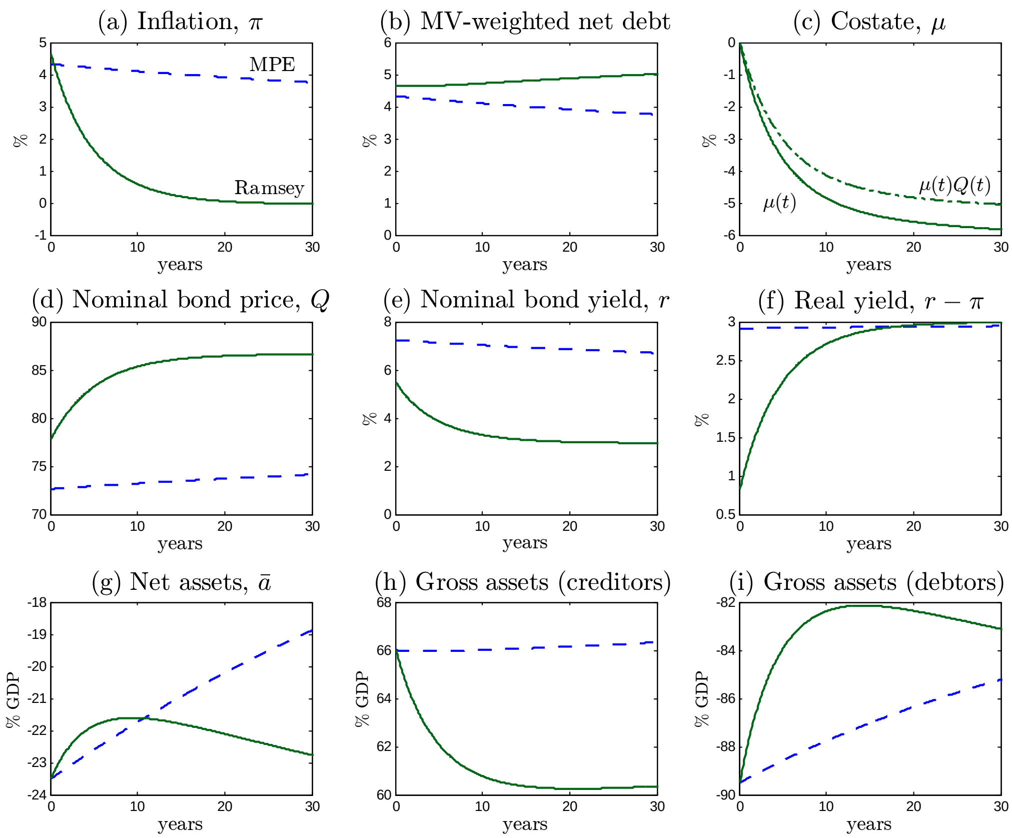

Figure 2: Dynamics under optimal monetary policy and zero inflation.

equation that determines optimal inflation at $t \geq 0$,

$$\pi(t) = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(t,a)\, da + \frac{1}{\psi} \mu(t) Q(t), \tag{31}$$

the (absolute) value of the costate $\mu(t)$, which captures the effect of time-t inflation on the price of bonds issued during the period $[0, t)$, becomes larger and larger over time. As shown in panels (c) and (b) of Figure 2, the increase in $|\mu(t) Q(t)|$ dominates that of the marginal-value-weighted average net liabilities, $\sum_i \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i(t,a)\, da$, which from equation (31) produces the gradual fall in inflation.[32] Importantly, the fact that investors anticipate the relatively short-lived nature of the initial inflation explains why nominal yields (panel e) increase much less than instantaneous inflation itself. This allows the ex-post real yield $r_t - \pi_t$ (panel f) to fall sharply at time zero, thus giving rise to the aforementioned redistribution.

In summary, under the optimal commitment the central bank *front-loads* inflation in order to redistribute net wealth towards domestic borrowers but also commits to gradually reducing inflation in order to prevent inflation expectations from permanently raising nominal yields.

Under discretion (dashed blue lines in Figure 2), time-zero inflation is 4.3 percent, close to the value under commitment.[33] In contrast to the commitment case, however, from time zero onwards optimal discretionary inflation remains relatively high, declining very slowly to its asymptotic value of 1.68 percent. The reason is the inflationary bias that stems from the central bank's attempt to redistribute wealth to borrowing households. This inflationary bias is not counteracted by any concern about the effect of inflation expectations on nominal bond yields; that is, the costate $\mu(t)$ in equation (31) is zero at all times under discretion. This inflationary bias produces permanently lower nominal bond prices (higher nominal yields) than under commitment. Contrary also to the Ramsey equilibrium, the discretionary policy largely *fails* to deliver the very redistribution it tries to achieve. The reason is that investors anticipate high future inflation and price the new bonds accordingly. The resulting jump in nominal yields (panel e) undoes most of the instantaneous inflation, such that the ex-post real yield (panel f) barely falls.

[32] Panels (b) and (c) in Figure 2 display the two terms on the right-hand side of (31), i.e. the marginal-value-weighted average net liabilities and $\mu(t) Q(t)$ both *rescaled* by the inflation disutility parameter ψ. Therefore, the sum of both terms equals optimal inflation under commitment.

[33] Since $\mu(0) = 0$, and given a common initial wealth distribution, time-0 inflation under commitment and discretion differ only insofar as time-0 value functions in both regimes do. Numerically, the latter functions are similar enough that $\pi(0)$ is very similar in both regimes.

4.4 Welfare analysis

We now turn to the welfare analysis of alternative policy regimes. Aggregate welfare is defined as

$$\int_{\phi}^{\infty}\sum_{i=1}^{2} v_i(0,a)\, f_i(t,a)da = \int_0^{\infty} e^{-\rho t}\int_{\phi}^{\infty}\sum_{i=1}^{2} u\left(c_i(t,a),\pi(t)\right) f_i(t,a)dadt \equiv W[c],$$

Table 3 displays the welfare losses of suboptimal policies *vis-à-vis* the Ramsey optimal equilibrium. We express welfare losses as a permanent consumption equivalent, i.e. the number Θ (in %) that satisfies in each case $W^R\left[c^R\right] = W\left[(1+\Theta)\,c\right]$, where R denotes the Ramsey equilibrium.[34] The table also displays the welfare losses incurred respectively by creditors and debtors.[35] The welfare losses from discretionary policy versus commitment are of first order: 0.31% of permanent consumption. This welfare loss is suffered not only by creditors (0.23%), *but also by debtors* (0.08%), despite the fact that the discretionary policy is aimed precisely at redistributing wealth towards debtors. As explained in the previous subsection, under the discretionary policy the increase in nominal yields undoes most of the impact of inflation on ex post real yields and hence on net asset accumulation. As a result, discretionary policy largely fails at producing the very redistribution towards debtor households that it intends to achieve in the first place, while leaving both creditor and debtor households to bear the direct welfare costs of permanent positive inflation.

Table 3. Welfare losses relative to the optimal commitment

	Economy-wide	Creditors	Debtors
Discretion	0.31	0.23	0.08
Zero inflation	0.05	-0.17	0.22

Note: welfare losses are expressed as a % of permanent consumption

We also compute the welfare losses from a policy of zero inflation, $\pi(t) = 0$ for all $t \geq 0$. As the table shows, the latter policy approximates the welfare outcome under commitment very closely, for two reasons. First, the welfare gains losses suffered by debtor households due to the absence of initial transitory inflation are largely compensated by the corresponding gains for creditor households. Second, zero inflation avoids too the welfare costs from the inflationary bias.

[34] Under our assumed separable preferences with log consumption utility, it is possible to show that $\Theta = \exp\left\{\rho\left(W^R\left[c^R\right] - W[c]\right)\right\} - 1$.

[35] That is, we report $\Theta^{a>0}$ and $\Theta^{a<0}$, where

$$\Theta^{a>0} = \exp\left[\rho\left(W^{R,a>0} - W^{MPE,a>0}\right)\right] - 1,$$

with $\Theta^{a<0}$ defined analogously, and where for each policy regime we have defined $W^{a>0} \equiv \int_0^{\infty}\sum_{i=1}^{2} v_i(0,a)\, f_i(t,a)da$, $W^{a<0} \equiv \int_{\phi}^{0}\sum_{i=1}^{2} v_i(0,a)\, f_i(t,a)da$. Notice that $\Theta^{a>0}$ and $\Theta^{a>0}$ do not exactly add up to Θ, as the expontential function is not a linear operator. However, Θ is sufficiently small that $\Theta \approx \Theta^{a>0} + \Theta^{a>0}$.

4.5 Robustness

Steady state inflation. In Proposition 4, we established that the Ramsey optimal long-run inflation rate converges to zero as the central bank's discount rate ρ converges to that of foreign investors, $\bar{r}$. In our baseline calibration, both discount rates are indeed very close to each other, implying that Ramsey optimal long-run inflation is essentially zero. We now evaluate the sensitivity of Ramsey optimal steady state inflation to the difference between both discount rates. From equation (31), Ramsey optimal steady state inflation is

$$\pi = \frac{1}{\psi} \sum_{i=1}^{2} \int_{\phi}^{\infty} (-a) \frac{\partial v_i}{\partial a} f_i (a) \, da + \frac{1}{\psi} \mu Q, \tag{32}$$

where the first term on the right hand side captures the redistributive motive to inflate in the long run, and the second one reflects the effect of central bank's commitments about long-run inflation. Figure 3 displays π (left axis), as well as its two determinants (right axis) on the right-hand side of equation (32). Optimal inflation decreases approximately linearly with the gap $\rho - \bar{r}$. As the latter increases, two counteracting effects take place. On the one hand, it can be shown that as the households become more impatient relative to foreign investors, the net asset distribution shifts towards the left, i.e. more and more households become net borrowers and come close to the borrowing limit, where the marginal utility of wealth is highest.[36] As shown in the figure, this increases the central bank's incentive to inflate for the purpose of redistributing wealth towards debtors. On the other hand, the more impatient households become relative to foreign investors, the more the central bank internalizes in present-discounted value terms the welfare consequences of creating expectations of higher inflation in the long run. This provides the central bank an incentive to committing to *lower* long run inflation. As shown by Figure 3, this second 'commitment' effect dominates the 'redistributive' effect, such that in net terms optimal long-run inflation becomes more negative as the discount rate gap widens.

Initial inflation. As explained before, time-0 optimal inflation and its subsequent path depend on the initial net wealth distribution across households, which is an infinite-dimensional object. In our baseline numerical analysis, we set it equal to the stationary distribution in the case of zero inflation. We now investigate how initial inflation depends on such initial distribution. To make the analysis operational, we restrict our attention to the class of Normal distributions truncated at the borrowing limit ϕ. That is,

$$f(0, a) = \begin{cases} \phi(a; \mu, \sigma) / [1 - \Phi(\phi; \mu, \sigma)], & a \geq \phi \\ 0, & a < \phi \end{cases}, \tag{33}$$

[36] The evolution of the long-run wealth distribution as $\rho - \bar{r}$ increases is available upon request.

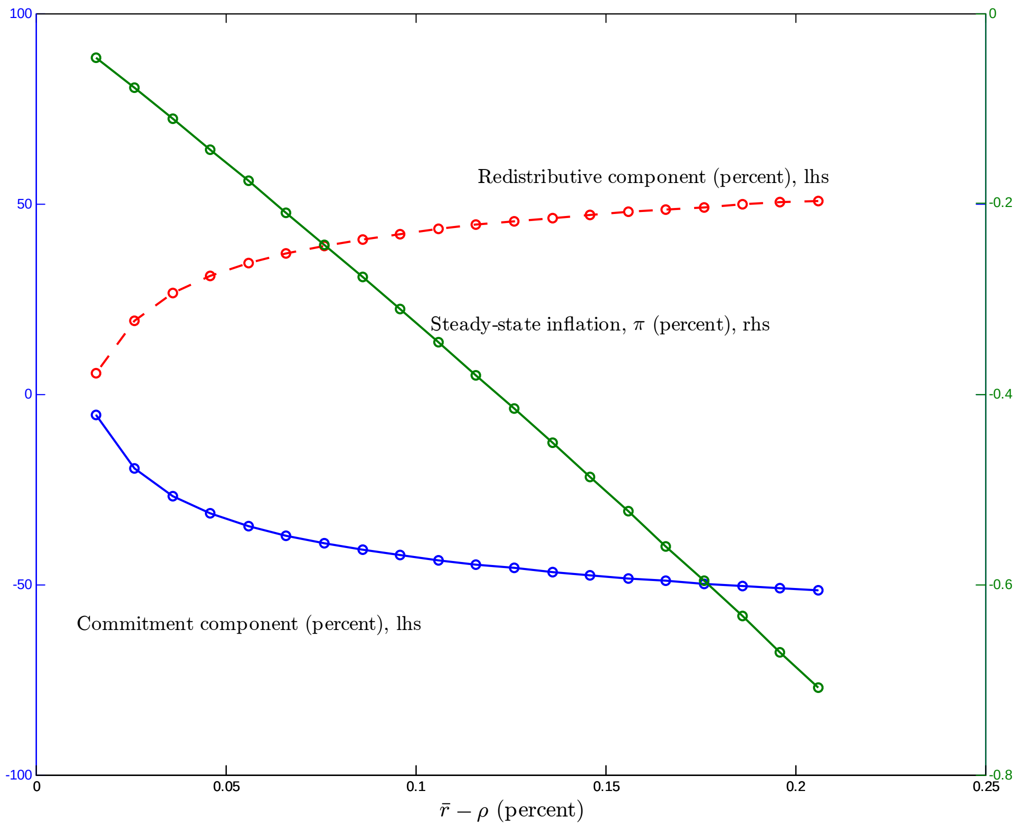

Figure 3: Sensitivity analysis to changes in $\rho - \bar{r}$.

where $\phi(\cdot;\mu,\sigma)$ and $\Phi(\cdot;\mu,\sigma)$ are the Normal pdf and cdf, respectively.[37] The parameters μ and σ allow us to control both (i) the initial net foreign asset position and (ii) the domestic dispersion in household wealth, and hence to isolate the effect of each factor on the optimal inflation path. Notice also that optimal long-run inflation rates do *not* depend on $f(0,a)$ and are therefore exactly the same as in our baseline numerical analysis regardless of μ and σ.[38] This allows us to focus here on inflation at time 0, while noting that the transition paths towards the respective long-run levels are isomorphic to those displayed in Figure 2.[39] Moreover, we report results for the commitment case, both for brevity and because results for discretion are very similar.[40]

Figure 4 displays optimal initial inflation rates for alternative initial net wealth distributions. In the first row of panels, we show the effect of increasing wealth dispersion while restricting the country to have a zero net position *vis-à-vis* the rest of the World, i.e. we increase σ and simultaneously adjust μ to ensure that $\bar{a}(0) = 0$.[41] In the extreme case of a (quasi) degenerate initial distribution at zero net assets (solid blue line in the upper left panel), the central bank has no incentive to create inflation, and thus optimal initial inflation is zero. As the degree of initial wealth dispersion increases, so does optimal initial inflation.

The bottom row of panels in Figure 4 isolates instead the effect of increasing the liabilities with the rest of the World, while assuming at the same time $\sigma \simeq 0$, i.e. eliminating any wealth dispersion.[42] As shown by the lower right panel, optimal inflation increases fairly quickly with the external indebtedness; for instance, an external debt-to-GDP ratio of 50 percent justifies an initial inflation of over 6 percent.

We can finally use Figure 4 to shed some light on the contribution of each redistributive motive (cross-border and domestic) to the initial optimal inflation rate, $\pi(0) = 4.6\%$, found in our baseline analysis. We may do so in two different ways. First, we note that the initial wealth distribution used in our baseline analysis implies a consolidated net foreign asset position of $\bar{a}(0) = -25\%$ of GDP ($\bar{y} = 1$). Using as initial condition a *degenerate* distribution at exactly that level (i.e. $\mu = -0.205$ and $\sigma \simeq 0$) delivers $\pi(0) = 3.1\%$ (see panel d). Therefore, the pure *cross-border*

[37] As explained in Section 5.2, in all our simulations we assume that the initial net asset distribution conditional on being in a boom or in a recession is the same: $f_{a|y}(0,a \mid y_2) = f_{a|y}(0,a \mid y_1) \equiv f_0(a)$. This implies that the marginal asset density coincides with its conditional density: $f(0,a) = \sum_{i=1,2} f_{a|y}(0,a \mid y_i)\, f_y(y_i) = f_0(a)$.

[38] As shown in Table 2, long-run inflation is -0.05% under commitment, and 1.68% under discretion.

[39] The full dynamic optimal paths under any of the alternative calibrations considered in this section are available upon request.

[40] As explained before, time-0 inflation in both policy regimes differ only insofar as the respective time-0 value functions do, but numerically we found the latter to be always very similar to each other. Results for the discretion case are available upon request.

[41] We verify that for all the calibrations considered here, the path of $\bar{a}_t$ after time 0 satisfies Assumption 1. In particular, the redistributive effect from foreign lenders to the domestic economy due to the initial positive increase in inflation is more than compensated by the increase in debtors' consumption.

[42] That is, we approximate 'Dirac delta' distributions centered at different values of μ. Since such distributions are not affected by the truncation at $a = \phi$, we have $\bar{a}(0) = \mu$, i.e. the net foreign asset position coincides with μ.

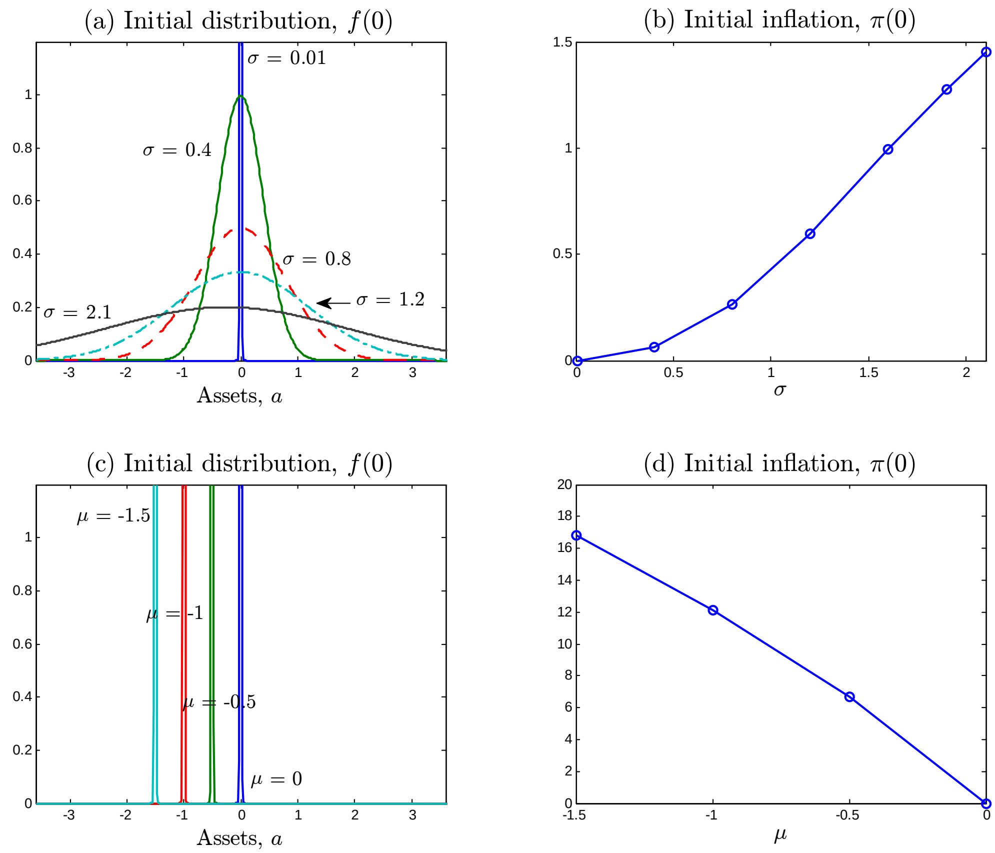

Figure 4: Ramsey optimal initial inflation for different initial net asset distributions.

redistributive motive explains a significant part (about two thirds) but not all of the optimal time-0 inflation under the Ramsey policy. Alternatively, we may note that our baseline initial distribution has a standard deviation of 1.95. We then find the (σ, μ) pair such that the (truncated) normal distribution has the same standard deviation and is centered at $\bar{a}(0) = 0$ (thus switching off the cross-border redistributive motive); this requires $\sigma = 2.1$, which delivers $\pi(0) = 1.5\%$ (panel b). We thus find again that the pure *domestic* redistributive motive explains about a third of the baseline optimal initial inflation. We conclude that both the cross-border and the domestic redistributive motives are quantitatively important for explaining the optimal inflation chosen by the monetary authority.

4.6 Aggregate ("MIT") shocks

So far we have restricted our analysis to the transitional dynamics, given the economy's initial state, while abstracting from aggregate shocks. We now extend our analysis to allow for aggregate shocks. We will consider an "MIT" shock to the World real interest rate, i.e. a one-time, temporary, unanticipated change in one of the structural parameters. In particular, we allow the World real interest rate $\bar{r}$ to vary over time and simulate a one-off, unanticipated increase at time 0 followed by a gradual return to its baseline value of 3%.[43] We will restrict our attention to optimal commitment policy. An issue that arises here is how long after 'time zero' (the implementation date of the Ramsey optimal commitment) the aggregate shock is assumed to take place. Since we do not want to take a stand on this dimension, we consider the limiting case in which the Ramsey optimal commitment has been going on for a sufficiently long time that the economy is essentially at its stationary equilibrium by the time the shock arrives. This can be viewed as an example of optimal policy 'from a timeless perspective', in the sense of Woodford (2003). In practical terms, it requires solving the optimal commitment problem analyzed in Section 3.3 with two modifications: (i) the initial wealth distribution is the stationary distribution implied by the optimal commitment itself, and (ii) the initial condition $\mu(0) = 0$ (absence of precommitments) is replaced by $\mu(0) = \mu(\infty)$, where the latter object is the stationary value of the costate in the commitment case. Both modifications guarantee that the central bank behaves as if it had been following the time-0 optimal commitment for an arbitrarily long time.

Figure 5 displays the results for two shock sizes (1pp and 2pp) and their impact on a number of variables in each case. The shock raises nominal (and real) bond yields, which leads households to reduce their consumption. Inflation rises slightly on impact, as the central bank tries to partially counteract the negative effect of the shock on household consumption. However, the inflation reaction is an order of magnitude smaller than that of the shocks themselves. Intuitively, the value of sticking to past commitments to keep inflation near zero weighs more in the central bank's decision than the value of using inflation transitorily so as to stabilize consumption in response to an unforeseen event. Therefore, we conclude that aggregate shocks such as those considered here barely affect the Ramsey optimal inflation path.

5 Conclusion

We have analyzed optimal monetary policy, under commitment and discretion, in a continuous-time, small-open-economy version of the standard incomplete-markets model extended to allow for nominal noncontingent claims and costly inflation. Our analysis sheds light on a recent policy and academic debate on the consequences that wealth heterogeneity across households should have for the appropriate conduct of monetary policy. Our main contribution is methodological: to the

[43] The dynamics of $\bar{r}_t$ are given by

$$d\bar{r}_t = \eta_r (\bar{r} - \bar{r}_t)\, dt,$$

with $\bar{r} = 0.03$ and $\eta_r = 0.5$.

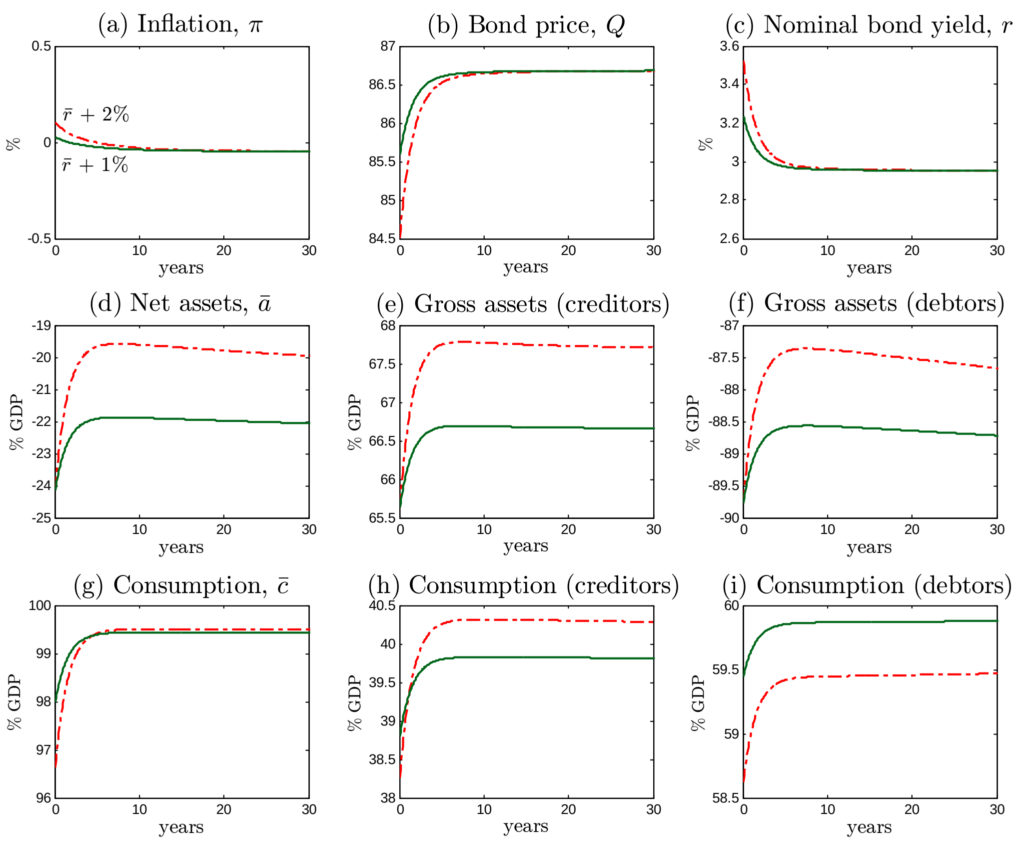

Figure 5: Impact of an interest rate shock under commitment (from a *timeless* perspective).

best of our knowledge, our paper is the first to solve for a fully dynamic optimal policy problem, both under commitment and discretion, in a standard incomplete-markets model with uninsurable idiosyncratic risk. While models of this kind have been established as a workhorse for policy analysis in macro models with heterogeneous agents, the fact that in such models the infinite-dimensional, endogenously-evolving wealth distribution is a state in the policy-maker's problem has made it difficult to make progress in the analysis of fully optimal policy problems. Our analysis proposes a novel methodology for dealing with this kind of problems in a continuous-time environment.

We show analytically that, whether under discretion or commitment, the central bank has an incentive to create inflation in order to redistribute wealth from lending to borrowing households, because the latter have a higher marginal utility of net wealth under incomplete markets. It also aims at redistributing wealth away from foreign investors, to the extent that these are net creditors *vis-à-vis* the domestic economy as a whole. Under commitment, however, these redistributive motives to inflate are counteracted by the central bank's understanding of how expectations of future inflation affect current nominal bond prices. We show moreover that, in the limiting case in which the central bank's discount factor is arbitrarily close to that of foreign investors, the long-run inflation rate under commitment is also arbitrarily close to zero.

We calibrate the model to replicate relevant features of a subset of prominent European small open economies, including their net foreign asset positions and gross household debt ratios. We show that the optimal policy under commitment features first-order positive initial inflation, followed by a gradual decline towards its (near zero) long-run level. That is, the central bank front-loads inflation so as to transitorily redistribute existing wealth both within the country and away from international investors, while committing to gradually abandon such redistributive stance. By contrast, discretionary monetary policy keeps inflation permanently high; such a policy is shown to reduce welfare substantially, both for creditor and for debtor households, as both groups suffer the consequences of the redistribution-led inflationary bias.

Our analysis thus suggest that, in an economy with heterogenous net nominal positions across households, inflationary redistribution should only be used temporarily, avoiding any temptation to prolong positive inflation rates over time.

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Online appendix (not for publication)

A. Proofs

Mathematical preliminaries

First we need to introduce some mathematical concepts. An *operator* T is a mapping from one vector space to another. Given the stochastic process a_t in (3), dfine an operator $\mathcal{A}$

$$\mathcal{A}v \equiv \begin{pmatrix} s_1(t,a)\frac{\partial v_1(t,a)}{\partial a} + \lambda_1 \left[v_2(t,a) - v_1(t,a)\right] \\ s_2(t,a)\frac{\partial v_2(t,a)}{\partial a} + \lambda_2 \left[v_1(t,a) - v_2(t,a)\right] \end{pmatrix}, \tag{34}$$

so that the HJB equation (8) can be expressed as

$$\rho v = \frac{\partial v}{\partial t} + \max_c \left\{ u\left(c,\pi\right) + \mathcal{A}v \right\},$$

where $v \equiv \binom{v_1(t,a)}{v_2(t,a)}$ and $u\left(c,\pi\right) \equiv \binom{u(c_1,\pi)}{u(c_2,\pi)}$.[44]

From now on, we assume that there is an upper bound arbitrarily large $\varkappa$ such that $f(t,a,y) = 0$ for all $a > \varkappa$. In steady state this can be proved in general following the same reasoning as in Proposition 2 of Achdou et al. (2015). Alternatively, we may assume that there is a maximum constraint in asset holding such that $a \leq \varkappa$, and that this constraint is so large that it does not affect to the results. In any case, let $\Phi \equiv [\phi, \varkappa]$ be the valid domain. The space of Lebesgue-integrable functions $L^2\left(\Phi\right)$ with the inner product

$$\left\langle v, f \right\rangle_\Phi = \sum_{i=1}^{2} \int_\Phi v_i f_i da = \int_\Phi v^{\mathbf{T}} f da, \ \forall v, f \in L^2\left(\Phi\right),$$

is a Hilbert space.[45]

Given an operator $\mathcal{A}$, its *adjoint* is an operator $\mathcal{A}^*$ such that $\left\langle f, \mathcal{A}v \right\rangle_\Phi = \left\langle \mathcal{A}^* f, v \right\rangle_\Phi$. In the case of the operator defined by (34) its adjoint is the operator

$$\mathcal{A}^* f \equiv \begin{pmatrix} -\frac{\partial(s_1 f_1)}{\partial a} - \lambda_1 f_1 + \lambda_2 j_2 \\ -\frac{\partial(s_2 f_2)}{\partial a} - \lambda_i f_i + \lambda_1 j_1 \end{pmatrix},$$

with boundary conditions

$$s_i\left(t,\phi\right) f_i\left(t,\phi\right) = s_i\left(t,\varkappa\right) f_i\left(t,\varkappa\right) = 0, \quad i = 1, 2, \tag{35}$$

[44] The *infinitesimal generator* of the process is thus $\frac{\partial v}{\partial t} + \mathcal{A}v$.

[45] See Luenberger (1969) or Brezis (2011) for references.

such that the KF equation (14) results in

$$\frac{\partial f}{\partial t} = \mathcal{A}^* f, \tag{36}$$

for $f \equiv \binom{f_1(t,a)}{f_2(t,a)}$. We can see that $\mathcal{A}$ and $\mathcal{A}^*$ are adjoints as

$$\begin{aligned} \langle \mathcal{A}v, f \rangle_\Phi &= \int_\Phi (\mathcal{A}v)^\mathbf{T} f da = \sum_{i=1}^{2} \int_\Phi \left[s_i \frac{\partial v_i}{\partial a} + \lambda_i \left[v_j - v_i \right] \right] f_i da \\ &= \sum_{i=1}^{2} v_i s_i f_i |_\phi^\varkappa + \sum_{i=1}^{2} \int_\Phi v_i \left[-\frac{\partial}{\partial a} \left(s_i f_i \right) - \lambda_i f_i + \lambda_j j_j \right] da \\ &= \int_\Phi v^\mathbf{T} \mathcal{A}^* f da = \langle v, \mathcal{A}^* f \rangle_\Phi . \end{aligned}$$

We introduce the concept of Gateaux and Frechet differentials as generalizations of the standard concept of derivative to infinite-dimensional spaces.[46]

Definition 4 (Gateaux differential) *Let $J[f]$ be a functional and let h be arbitrary in $L^2(\Omega)$, where $\Omega \subset \mathbb{R}^n$. If the limit*

$$\delta J[f;h] = \lim_{\alpha \to 0} \frac{J[f + \alpha h] - J[f]}{\alpha} \tag{37}$$

exists, it is called the Gateaux differential *of J at f with increment h. If the limit (37) exists for each $h \in L^2(\Omega)$, the functional J is said to be* Gateaux differentiable *at f.*

If the limit exists, it can be expressed as $\delta J[f;h] = \frac{d}{d\alpha} J[f + \alpha h]|_{\alpha=0}$. A more restricted concept is that of the Fréchet differential.

Definition 5 (Fréchet differential) *Let h be arbitrary in $L^2(\Omega)$. If for fixed $f \in L^2(\Omega)$ there exists $\delta J[f;h]$ which is linear and continuous with respect to h such that*

$$\lim_{\|h\|_{L^2(\Omega)} \to 0} \frac{|J[f+h] - J[f] - \delta J[f;h]|}{\|h\|_{L^2(\Omega)}} = 0,$$

then J is said to be Fréchet differentiable *at f and $\delta[Jf;h]$ is the* Fréchet differential *of J at f with increment h.*

The following proposition links both concepts.

Theorem 1 *If the Fréchet differential of J exists at f, then the Gateaux differential exists at f and they are equal.*

[46]See Luenberger (1969), Gelfand and Fomin (1991) or Sagan (1992).

Proof. See Luenberger (1969, p. 173). ■

The familiar technique of maximizing a function of a single variable by ordinary calculus can be extended in infinite dimensional spaces to a similar technique based on more general differentials. We use the term *extremum* to refer to a maximum or a minimum over any set. A a function $f \in L^2(\Omega)$ is a maximum of $J[f]$if for all functions h, $\|h - f\|_{L^2(\Omega)} < \varepsilon$ then $J[f] \geq J[h]$. The following theorem is the Fundamental Theorem of Calculus.

Theorem 2 *Let J have a Gateaux differential, a necessary condition for J to have an extremum at f is that $\delta J[f; h] = 0$ for all $h \in L^2(\Omega)$.*

Proof. Luenberger (1969, p. 173), Gelfand and Fomin (1991, pp. 13-14) or Sagan (1992, p. 34). ■

In the case of constrained optimization in an infinite-dimensional Hilbert space, we have the following Theorem.

Theorem 3 (Lagrange multipliers) *Let H be a mapping from $L^2(\Omega)$ into $\mathbb{R}^p$. If J has a continuous Fréchet differential, a necessary condition for J to have an extremum at f under the constraint $H[f] = 0$ at the function f is that there exists a function $\eta \in L^2(\Omega)$ such that the Lagrangian functional*

$$\mathcal{L}[f] = J[f] + \langle \eta, H[f] \rangle_\Omega \tag{38}$$

is stationary in f, i.e., $\delta\mathcal{L}[f; h] = 0$.

Proof. Luenberger (1969, p. 243). ■

Proof of Lemma 1

Given the welfare criterion defined as in (18), we have

$$\begin{aligned} U_0^{CB} &= \int_\phi^\infty \sum_{i=1}^2 v_i(0,a) f_i(0,a) da = \int_\phi^\infty \sum_{i=1}^2 \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} u(c_t, \pi_t) dt | a(0) = a, \ y(0) = y_i \right] f_i(0,a) da \\ &= \int_\phi^\infty \sum_{i=1}^2 \left[\sum_{j=1}^2 \int_\phi^\infty \int_0^\infty e^{-\rho t} u(c,\pi) f(t, \tilde{a}, \tilde{y}_j; a, y_i) dt d\tilde{a} \right] f_i(0,a) da \\ &= \int_0^\infty \sum_{j=1}^2 e^{-\rho t} \int_\phi^\infty u(c,\pi) \left[\sum_{i=1}^2 \int_\phi^\infty f(t, \tilde{a}, \tilde{y}_j; a, y_i) f_i(0,a) da \right] d\tilde{a} dt \\ &= \int_0^\infty \sum_{i=1}^2 e^{-\rho t} \int_\phi^\infty u(c,\pi) f_j(t, \tilde{a}) d\tilde{a} dt, \end{aligned}$$

where $f(t, \tilde{a}, \tilde{y}_j; a, y)$ is the transition probability from $a_0 = a$, $y_0 = y_i$ to $a_t = \tilde{a}$, $y_t = \tilde{y}_j$ and

$$f_j(t, \tilde{a}) = \sum_{j=1}^{2} \int_{\phi}^{\infty} f(t, \tilde{a}, \tilde{y}_j; a, y_i) f_i(0, a) da,$$

is the Chapman–Kolmogorov equation.

Proof of Proposition 1. Solution to the MPE

The idea of the proof is to employ dynamic programming in order to transform the problem of the central bank in a family -indexed by time- of static calculus of variations problems. Then we solve each of these problems using differenciation techniques in infinite-dimensional Hilbert spaces.

Step 1: Dynamic programming Given an initial condition $f(t, \cdot)$ we have an optimal control path $\{\pi(s)\}_{s=t}^{\infty}$. The first step is to apply the *Bellman's Principle of Optimality*

$$\begin{aligned} J[f(t,\cdot)] &= \sum_{i=1}^{2} \int_t^{\infty} e^{-\rho(s-t)} \int_{\Phi} u(c_s, \pi_s) f_i(s,a) da ds \\ &= \sum_{i=1}^{2} \int_t^{t'} e^{-\rho(s-t)} \int_{\Phi} u(c_s, \pi_s) f_i(s,a) da ds + e^{-\rho(t'-t)} J[f(t',\cdot)], \end{aligned} \tag{39}$$

where $J[f]$ is the central bank's optimal value functional.

Let $\Xi[f]$ be defined as

$$\Xi[f] \equiv \sum_{i=1}^{2} \int_{\Phi} u(c_t, \pi_t) f_i(t,a) da = \langle u, f \rangle_{\Phi}.$$

According to Definition 5 above, if the Fréchet differential $\delta J[f(t,\cdot)]$ of $J[f(t,\cdot)]$ exists then it is linear and continuous. We may apply the Riesz representation theorem to express it as an inner product.

Theorem 4 (Riesz representation theorem) *Let $\delta J[f;h] : L^2(\Phi) \to \mathbb{R}$ be a linear continuous functional. Then there exists a unique function known as the* functional derivative *of J with respect to f, $j[f] = \frac{\delta J}{\delta f}[f] \in L^2(\Phi)$, such that*

$$\delta J[f;h] = \langle j, h \rangle_{\Phi} = \sum_{i=1}^{2} \int_{\Phi} j_i[f](a) h_i(a) da.$$

Proof. See Brezis (2011, pp. 97-98). ■

Notice that, given a distribution $f(t,\cdot)$, j is a function that maps Φ into $\mathbb{R}^2$. As there is no aggregate uncertainty the dynamics of the distribution only depend on time and thus we may consider that j is a function mapping both time and wealth: $j(t,a):[0,\infty)\times\Phi\rightarrow\mathbb{R}^2$. As it will be clear below $j(t,a)$ is the *central bank's value* at time t of a household with debt a.

Taking derivatives with respect to time in equation (39) and the limit as $t'\rightarrow t$:

$$0=\Xi[f]-\rho J[f(t,\cdot)]+\frac{\partial}{\partial t}J[f(t,\cdot)]=\Xi[f]-\rho J[f(t,\cdot)]+\sum_{i=1}^{2}\int_{\Phi}j_i(t,a)\frac{\partial f_i}{\partial t}da, \tag{40}$$

where we have applied the chain rule.

Equation (40) is the Hamilton-Jacobi-Bellman equation of the problem (39) in the infinite-dimensional space $L^2(\Phi)$. The last term of the right-hand side equation is

$$\sum_{i=1}^{2}\int_{\Phi}j_i(t,a)\frac{\partial f_i}{\partial t}da=\left\langle j,\frac{\partial f}{\partial t}\right\rangle_{\Phi}=\langle j,\mathcal{A}^*f\rangle_{\Phi}=\langle\mathcal{A}j,f\rangle_{\Phi},$$

where in the last equality we have applied the KF equation (36).

We may express the Hamilton-Jacobi-Bellman equation as

$$\rho J[f]=\max_{\pi}\int_{\Phi}\left(u(c,\pi)+\frac{\partial j}{\partial t}+\mathcal{A}j\right)^{\mathbf{T}}f\,da. \tag{41}$$

Step 2: Optimal inflation The first order condition with respect to inflation in (41) is

$$\begin{aligned}&\frac{\partial}{\partial\pi}\int_{\Phi}\left(u(c,\pi)+\frac{\partial j}{\partial t}+\mathcal{A}j\right)^{\mathbf{T}}f\,da \\ =&\sum_{i=1}^{2}\int_{\Phi}\left(u_{\pi}f_i+\frac{\partial s_i}{\partial\pi}\frac{\partial j_i}{\partial a}\right)f_i\,da \\ =&\sum_{i=1}^{2}\int_{\Phi}\left(u_{\pi}f_i-af_i\frac{\partial j_i}{\partial a}\right)da=0,\end{aligned} \tag{42}$$

so that the optimal inflation should satisfy

$$\sum_{i=1}^{2}\int_{\Phi}\left(af_i\frac{\partial j_i}{\partial a}-u_{\pi}f_i\right)da=0. \tag{43}$$

Step 3: Central Bank's HJB In order to find the value of $j(t,\cdot)$, we compute the Gateaux derivative of the Bellman equation (40). If we take the Gateaux derivative at both sides of equation

(41), we obtain

$$\begin{aligned}
\rho\frac{d}{d\alpha}\,\rho J\left[f+\alpha h\right]\Big|_{\alpha=0} &= \frac{d}{d\alpha}\int_{\Phi}\left(u\left(c,\pi\right)\right)^{\mathbf{T}}\left(f+\alpha h\right)da\Bigg|_{\alpha=0}+\frac{d}{d\alpha}\frac{\partial}{\partial t}J\left[f+\alpha h\right]\Bigg|_{\alpha=0}, \\
\sum_{i=1}^{2}\int_{\Phi}\rho j_i\left(t,a\right)h_i\left(a\right)da &= \sum_{i=1}^{2}\int_{\Phi}u\left(c_i,\pi\right)h_i(a)da+\frac{d}{d\alpha}\sum_{i=1}^{2}\int_{\Phi}j_i\left[f+\alpha h\right]\frac{\partial\left(f_i+\alpha h_i\right)}{\partial t}da\Bigg|_{\alpha=0}
\end{aligned} \tag{44}$$

and the last term results in

$$\begin{aligned}
\frac{d}{d\alpha}\sum_{i=1}^{2}\int_{\Phi}j_i\left[f+\alpha h\right]\frac{\partial\left(f_i+\alpha h_i\right)}{\partial t}da\Bigg|_{\alpha=0} &= \sum_{i=1}^{2}\int_{\Phi}\frac{d}{d\alpha}\,j_i\left[f+\alpha h\right]\Big|_{\alpha=0}\frac{\partial f_i}{\partial t}da+\sum_{i=1}^{2}\int_{\Phi}j_i\left(t,a\right)\frac{\partial h_i}{\partial t}da \\
&= \sum_{i=1}^{2}\int_{\Phi}\frac{d}{d\alpha}\,j_i\left[f+\alpha h\right]\Big|_{\alpha=0}\frac{\partial f_i}{\partial t}da+\int_{\Phi}\mathcal{A}j\left(t,a\right)hda.
\end{aligned}$$

If $j\left[f\right]$ is also Fréchet differentiable then $\frac{d}{d\alpha}\,j_i\left[f+\alpha h\right]\big|_{\alpha=0}$ is the second derivative of J:

$$\frac{d}{d\alpha}\,j\left[f+\alpha h\right]\Big|_{\alpha=0}=\sum_{i=1}^{2}\int_{\Phi}\frac{\delta^2 J}{\delta f^2}h_i\left(a\right)da.$$

If we compute

$$\frac{\partial j}{\partial t}=\frac{\partial}{\partial t}\frac{\delta J}{\delta f}=\sum_{i=1}^{2}\int_{\Phi}\frac{\delta^2 J}{\delta f^2}\frac{\partial f}{\partial t}da,$$

and thus

$$\begin{aligned}
\sum_{i=1}^{2}\int_{\Phi}\frac{d}{d\alpha}\,j_i\left[f+\alpha h\right]\Big|_{\alpha=0}\frac{\partial f_i}{\partial t}da &= \sum_{i=1}^{2}\int_{\Phi}\left[\sum_{i=1}^{2}\int_{\Phi}\frac{\delta^2 J}{\delta f^2}h_i\left(a'\right)da'\right]\frac{\partial f_i}{\partial t}da \\
& \sum_{i=1}^{2}\int_{\Phi}\left[\sum_{i=1}^{2}\int_{\Phi}\frac{\delta^2 J}{\delta f^2}\frac{\partial f_i}{\partial t}da\right]h_i\left(a'\right)da' \\
&= \sum_{i=1}^{2}\int_{\Phi}\frac{\partial j_i}{\partial t}h_i\left(a'\right)da'.
\end{aligned}$$

Equation (44) then results in

$$0=\int_{\Phi}\left(u\left(c,\pi\right)+\frac{\partial j}{\partial t}+\mathcal{A}j-\rho j\right)^{\mathbf{T}}h(a)da.$$

For any $h\in L^2\left(\Phi\right)$ we have $\int_{\Phi}\left(u\left(c,\pi\right)+\frac{\partial j}{\partial t}+\mathcal{A}j-\rho j\right)^{\mathbf{T}}hda=0$ and hence $u\left(c,\pi\right)+\frac{\partial j}{\partial t}+$

$\mathcal{A}j - \rho j = 0$, $\forall a \in \Phi$, $y \in \{y_1, y_2\}$:

$$\rho j_i(t,a) = u(c_i, \pi) + \frac{\partial j_i}{\partial t} + s_i(t,a)\frac{\partial j_i}{\partial a} + \lambda_i\left(j_k(t,a) - j_i(t,a)\right), \ i = 1, 2, \ k \neq i. \tag{45}$$

Equation (45) is the same as the individual HJB equation (8). The boundary conditions are also the same (state constraints on the domain Φ) and therefore its solution should be the same: $j(t,a,y) = v(t,a,y)$, that is, the marginal social value to the central bank under discretion $j(\cdot)$ equals the individual value $v(\cdot)$.

Proof of Proposition 2. Solution to the Ramsey problem

Step 1: Statement of the problem The problem of the central bank is given by

$$J\left[f_0(\cdot)\right] = \max_{\{\pi_s, Q_s, v(s,\cdot), c(s,\cdot), f(s,\cdot)\}_{s=0}^{\infty}} \sum_{i=1}^{2} \int_0^{\infty} e^{-\rho s}\left[\int_{\Phi} u\left(c_s, \pi_s\right) f_i(s,a)da\right] ds,$$

subject to the law of motion of the distribution (14), the bond pricing equation (12) and the individual HJB equation (8). This is a problem of constrained optimization in an infinite-dimensional Hilbert space $\hat{\Phi} = [0, \infty) \times \Phi$. We define $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ as the space of functions f such that

$$\int_{\hat{\Phi}} e^{-\rho t}|f|^2 = \int_0^{\infty}\int_{\Phi} e^{-\rho t}|f|^2 dtda = \int_0^{\infty} e^{-\rho t}\|f\|_{\Phi}^2 dt < \infty.$$

Lemma 3 *The space $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ with the inner product*

$$(f,g)_\Phi = \int_{\hat{\Phi}} e^{-\rho t} fg = \int_0^{\infty} e^{-\rho t}\langle f,g\rangle_\Phi dt = \left\langle e^{-\rho t} f, g\right\rangle_{\hat{\Phi}}$$

is a Hilbert space.

Proof. We need to show that $L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$ is complete, that is, that given a Cauchy sequence $\{f_n\}$ with limit $f : \lim_{n\to\infty} f_n = f$ then $f \in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$. If $\{f_n\}$ is a Cauchy sequence then

$$\|f_n - f_m\|_{(\cdot,\cdot)_\Phi} \to 0, \text{ as } n, m \to \infty,$$

or

$$\|f_n - f_m\|_{(\cdot,\cdot)_\Phi}^2 = \int_{\hat{\Phi}} e^{-\rho t}|f_n - f_m|^2 = \left\langle e^{-\frac{\rho}{2}t}\left(f_n - f_m\right), e^{-\frac{\rho}{2}t}\left(f_n - f_m\right)\right\rangle_{\hat{\Phi}} = \left\|e^{-\frac{\rho}{2}t}\left(f_n - f_m\right)\right\|_{\hat{\Phi}}^2 \to 0,$$

as $n,m\to\infty$. This implies that $\left\{e^{-\frac{\rho}{2}t}f_n\right\}$ is a Cauchy sequence in $L^2\left(\hat{\Phi}\right)$. As $L^2\left(\hat{\Phi}\right)$ is a complete space, then there is a function $\hat{f}\in L^2\left(\hat{\Phi}\right)$ such that

$$\lim_{n\to\infty} e^{-\frac{\rho}{2}t}f_n=\hat{f} \tag{46}$$

under the norm $\|\cdot\|^2_{\hat{\Phi}}$. If we define $f=e^{\frac{\rho}{2}t}\hat{f}$ then

$$\lim_{n\to\infty} f_n=f$$

under the norm $\|\cdot\|_{(\cdot,\cdot)_\Phi}$, that is, for any $\varepsilon>0$ there is an integer N such that

$$\|f_n-f\|^2_{(\cdot,\cdot)_\Phi}=\left\|e^{-\frac{\rho}{2}t}\left(f_n-f\right)\right\|^2_{\hat{\Phi}}=\left\|e^{-\frac{\rho}{2}t}f_n-\hat{f}\right\|^2_{\hat{\Phi}}<\varepsilon,$$

where the last inequality is due to (46). It only remains to prove that $f\in L^2\left(\hat{\Phi}\right)_{(\cdot,\cdot)_\Phi}$:

$$\|f\|^2_{(\cdot,\cdot)_\Phi}=\int_{\hat{\Phi}}e^{-\rho t}\left|f\right|^2=\int_{\hat{\Phi}}\left|\hat{f}\right|^2<\infty.$$

■

Step 2: The Lagragian The Lagrangian is

$$\begin{aligned}\mathcal{L}\left[\pi,Q,f,v,c\right] &= \int_0^\infty e^{-\rho t}\left\langle u,f\right\rangle_\Phi dt+\int_0^\infty\left\langle e^{-\rho t}\zeta\left(t,a\right),\mathcal{A}^*f-\frac{\partial f}{\partial t}\right\rangle_\Phi dt\\ &+\int_0^\infty e^{-\rho t}\mu\left(t\right)\left(Q\left(\bar{r}+\pi+\delta\right)-\delta-\dot{Q}\right)dt\\ &+\int_0^\infty\left\langle e^{-\rho t}\theta\left(t,a\right),u+\mathcal{A}v+\frac{\partial v}{\partial t}-\rho v\right\rangle_\Phi dt\\ &+\int_0^\infty\left\langle e^{-\rho t}\eta\left(t,a\right),u_c-\frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_\Phi dt\end{aligned}$$

where $e^{-\rho t}\zeta\left(t,a\right)$, $e^{-\rho t}\eta\left(t,a\right)$, $e^{-\rho t}\theta\left(t,a\right)\in L^2\left(\hat{\Phi}\right)$ and $e^{-\rho t}\mu\left(t\right)\in L^2[0,\infty)$ are the Lagrange multipliers associated to equations (14), (10), (8) and (12), respectively. The Lagragian can be expressed as

$$\begin{aligned}\mathcal{L}\left[\pi,Q,f,v,c\right] &= \int_0^\infty e^{-\rho t}\left\langle u+\frac{\partial\zeta}{\partial t}+\mathcal{A}\zeta-\rho\zeta+\mu\left(Q\left(\bar{r}+\pi+\delta\right)-\delta-\dot{Q}\right),f\right\rangle_\Phi dt\\&+\int_0^\infty e^{-\rho t}\left(\langle\theta,u\rangle_\Phi+\left\langle\mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},v\right\rangle_\Phi+\left\langle\eta,u_c-\frac{1}{Q}\frac{\partial v}{\partial a}\right\rangle_\Phi\right)dt\\&+\langle\zeta\left(0,\cdot\right),f\left(0,\cdot\right)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta\left(T,\cdot\right),f\left(T,\cdot\right)\right\rangle_\Phi\\&+\lim_{T\to\infty}\left\langle e^{-\rho T}\theta\left(T,\cdot\right),v\left(T,\cdot\right)\right\rangle_\Phi-\langle\theta\left(0,\cdot\right),v\left(0,\cdot\right)\rangle+\int_0^\infty e^{-\rho t}\sum_{i=1}^2 v_is_i\theta_i\big|_\phi^\varkappa dt,\end{aligned}$$

where we have applied $\langle\zeta,\mathcal{A}^*f\rangle=\langle\mathcal{A}\zeta,f\rangle$, $\langle\theta,\mathcal{A}v\rangle=\langle\mathcal{A}^*\theta,v\rangle_\Phi+\sum_{i=1}^2 v_is_i\theta_i\big|_\phi^\varkappa$ and integrated by parts

$$\begin{aligned}\int_0^\infty\left\langle e^{-\rho t}\zeta,-\frac{\partial f}{\partial t}\right\rangle_\Phi dt &= -\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}\zeta_i\frac{\partial f_i}{\partial t}dadt\\&= -\sum_{i=1}^2\int_\Phi f_ie^{-\rho t}\zeta_i\big|_0^\infty da+\sum_{i=1}^2\int_0^\infty\int_\Phi f_i\frac{\partial}{\partial t}\left(e^{-\rho t}\zeta_i\right)dadt\\&= \sum_{i=1}^2\int_\Phi f_i\left(0,a\right)\zeta_i\left(0,a\right)da-\lim_{T\to\infty}\sum_{i=1}^2\int_\Phi e^{-\rho T}f_i\left(T,a\right)\zeta_i\left(T,a\right)da\\&+\sum_{i=1}^2\int_0^\infty\int_\Phi e^{-\rho t}f_i\left(\frac{\partial\zeta_i}{\partial t}-\rho\zeta_i\right)dadt\\&= \langle\zeta\left(0,\cdot\right),f\left(0,\cdot\right)\rangle_\Phi-\lim_{T\to\infty}\left\langle e^{-\rho T}\zeta\left(T,\cdot\right),f\left(T,\cdot\right)\right\rangle_\Phi\\&+\int_0^\infty e^{-\rho t}\left\langle\frac{\partial\zeta}{\partial t}-\rho\zeta,f\right\rangle_\Phi dt,\end{aligned}$$

and

$$\begin{aligned}
\int_0^\infty \left\langle e^{-\rho t}\theta, \frac{\partial v}{\partial t} - \rho v \right\rangle dt &= \sum_{i=1}^{2} \int_0^\infty \int_\Phi e^{-\rho t}\theta_i \left(\frac{\partial v_i}{\partial t} - \rho v_i \right) dadt \\
&= \sum_{i=1}^{2} \int_\Phi \theta_i e_i^{-\rho t} v\Big|_0^\infty da - \sum_{i=1}^{2} \int_0^\infty \int_\Phi v_i \left[\frac{\partial}{\partial t}\left(e^{-\rho t}\theta_i\right) + \rho\theta_i \right] dadt \\
&= \lim_{T\to\infty} \sum_{i=1}^{2} \int_\Phi e^{-\rho T} v_i\left(T,a\right)\theta_i\left(T,a\right) da - \sum_{i=1}^{2} \int_\Phi v_i\left(0,a\right)\theta_i\left(0,a\right) da \\
&\quad - \sum_{i=1}^{2} \int_0^\infty \int_\Phi e^{-\rho t} v_i \left(\frac{\partial \theta_i}{\partial t} \right) dadt \\
&= \lim_{T\to\infty} \left\langle e^{-\rho T}\theta\left(T,\cdot\right), v\left(T,\cdot\right) \right\rangle_\Phi - \left\langle \theta\left(0,\cdot\right), v\left(0,\cdot\right) \right\rangle_\Phi \\
&\quad + \int_0^\infty e^{-\rho t} \left\langle -\frac{\partial \theta}{\partial t}, v \right\rangle_\Phi dt,
\end{aligned}$$

Step 3: Necessary conditions In order to find the extrema, we need to take the Gateaux differentials with respect to the controls f, π, Q, v and c.

The Gateaux differential with respect to f is

$$\begin{aligned}
\frac{d}{d\alpha}\mathcal{L}\left[\pi, Q, f + \alpha h\left(t,a\right), v, c\right]\big|_{\alpha=0} &= \left\langle \zeta\left(0,\cdot\right), h\left(0,\cdot\right) \right\rangle_\Phi - \lim_{T\to\infty} \left\langle e^{-\rho T}\zeta\left(T,\cdot\right), h\left(T,\cdot\right) \right\rangle_\Phi \\
&\quad - \int_0^\infty e^{-\rho t} \left\langle u + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta - \rho\zeta, h \right\rangle_\Phi dt,
\end{aligned}$$

which should equal zero for any function $e^{-\rho t}h\left(t,a\right) \in L^2\left(\hat{\Phi}\right)$ such that $h\left(0,\cdot\right) = 0$, as the initial value of $f\left(0,\cdot\right)$ is given. We obtain

$$\rho\zeta = u + \frac{\partial \zeta}{\partial t} + \mathcal{A}\zeta, \tag{47}$$

and taking this into account and considering functions such that $\lim_{T\to\infty} h\left(T,\cdot\right) \neq 0$, we obtain the boundary condition

$$\lim_{T\to\infty} e^{-\rho T}\zeta\left(T,a\right) = 0.$$

We may apply the Feynman–Kac formula to (47) and express $\zeta\left(t,a\right)$ as

$$\zeta\left(t,a\right) = \mathbb{E}_t\left[\int_t^\infty e^{-\rho(s-t)}u(c_s,\pi_s)ds \mid a_t = a\right],$$

subject to the evolution of a_t given by equation (3). This is the expression of the individual value function (7). Therefore, we may conclude that $\zeta\left(\cdot\right) = v\left(\cdot\right)$.

In the case of $c(t,a)$:

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q,f,v,c+\alpha h(t,a)\right]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left\langle \left(u_c-\frac{1}{Q}\frac{\partial\zeta}{\partial a}\right)h,f\right\rangle_\Phi dt\\ +\int_0^\infty e^{-\rho t}\left(\left\langle \theta,\left(u_c-\frac{1}{Q}\frac{\partial v}{\partial a}\right)h\right\rangle_\Phi+\langle\eta,u_{cc}h\rangle_\Phi\right)dt &= 0,\end{aligned}$$

where $\frac{\partial}{\partial a}(\mathcal{A}\zeta)=-\frac{1}{Q}\frac{\partial\zeta}{\partial a}$. Due to the first order conditions (10) and to the fact that $\zeta(\cdot)=v(\cdot)$ this expression reduces to

$$\int_0^\infty e^{-\rho t}\langle\eta(t,a),u_{cc}(t,a)\,h(t,a)\rangle_\Phi\,dt=0,$$

so that $\eta(t,a)=0\;\forall(t,a)\in\hat{\Phi}$, that is, the first order condition (10) is not binding as its associated Lagrange multiplier is zero.

In the case of $v(t,a)$:

$$\begin{aligned}\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q,f,v+\alpha h(t,a),c\right]|_{\alpha=0} &= \int_0^\infty e^{-\rho t}\left(\left\langle \mathcal{A}^*\theta-\frac{\partial\theta}{\partial t},h\right\rangle_\Phi\right)dt\\ +\lim_{T\to\infty}\left\langle e^{-\rho T}\theta(T,\cdot),h(T,\cdot)\right\rangle_\Phi-\langle\theta(0,\cdot),h(0,\cdot)\rangle_\Phi+\sum_{i=1}^{2}h_is_i\theta_i|_\phi^\varkappa &= 0,\end{aligned}$$

where we have already taken into account the fact that $\eta(t,a)=0$. Proceeding as in the case of f, we conclude that this yields a Kolmogorov forward equation

$$\frac{\partial\theta}{\partial t}=\mathcal{A}^*\theta, \tag{48}$$

with boundary conditions

$$\begin{aligned}\sum_{i=1}^{2}s_i(t,\phi)\,\theta_i(t,\phi) &= \sum_{i=1}^{2}s_i(t,\varkappa)\,\theta_i(t,\varkappa)=0,\\ \lim_{T\to\infty}e^{-\rho T}\theta(T,\cdot) &= 0,\\ \theta(0,\cdot) &= 0.\end{aligned}$$

This is exactly the same KF equation than in the case of f, but the initial distribution is $\theta(0,\cdot)=0$. Therefore, the distribution at any point in time should be zero $\theta(\cdot,\cdot)=0$. Both the Lagrange multiplier of the HJB equation θ and that of the first-order condition η are zero, reflecting the fact that the HJB condition is not binding, that is, that the monetary authority would choose the same consumption as the households.

In the case of $\pi(t)$:

$$\frac{d}{d\alpha}\mathcal{L}\left[\pi+\alpha h\left(t\right),Q,f,v,c\right]|_{\alpha=0}=\int_0^\infty e^{-\rho t}\left\langle u_\pi - a\left(\frac{\partial\zeta}{\partial a}\right)+\mu Q,f\right\rangle_\Phi hdt=0,$$

where we have already taken into account that so the fact that $\theta\left(t,a\right)=0$. Taking into account that $\zeta\left(t,a\right)=v\left(t,a\right)$:

$$\mu\left(t\right)Q\left(t\right)=\sum_{i=1}^{2}\int_\Phi\left(a\frac{\partial v_i}{\partial a}-u_\pi\right)f_i\left(t,a\right)da.$$

In the case of $Q\left(t\right)$:

$$\begin{aligned}&\frac{d}{d\alpha}\mathcal{L}\left[\pi,Q+\alpha h\left(t\right),f,v,c\right]|_{\alpha=0}\\ =&\int_0^\infty e^{-\rho t}\left\langle -\frac{\delta h}{Q^2}a\frac{\partial\zeta}{\partial a}-\frac{\left(y-c\right)h}{Q^2}\frac{\partial\zeta}{\partial a}+\mu\left[h\left(\bar{r}+\pi+\delta\right)-\dot{h}\right],f\right\rangle_\Phi dt=0\end{aligned}$$

Integrating by parts

$$\begin{aligned}\int_0^\infty e^{-\rho t}\left\langle -\mu\dot{h},f\right\rangle_\Phi dt &= -\int_0^\infty e^{-\rho t}\mu\dot{h}\left\langle 1,f\right\rangle_\Phi dt=-\int_0^\infty e^{-\rho t}\mu\dot{h}dt\\ &= -\left.e^{-\rho t}\mu h\right|_0^\infty+\int_0^\infty e^{-\rho t}\left(\dot{\mu}-\rho\mu\right)hdt\\ &= \mu\left(0\right)h\left(0\right)+\int_0^\infty e^{-\rho t}\left\langle\left(\dot{\mu}-\rho\mu\right)h,f\right\rangle_\Phi dt.\end{aligned}$$

Therefore,

$$\int_0^\infty e^{-\rho t}\left\langle -\frac{\delta}{Q^2}a\frac{\partial\zeta}{\partial a}-\frac{\left(y-c\right)}{Q^2}\frac{\partial\zeta}{\partial a}+\mu\left(\bar{r}+\pi+\delta-\rho\right)+\dot{\mu},f\right\rangle_\Phi hdt+\mu\left(0\right)h\left(0\right)=0,$$

so that, using similar arguments as in the case of θ above we can show that $\mu\left(0\right)=0$ and

$$\left\langle -\frac{\delta}{Q^2}a\frac{\partial v}{\partial a}-\frac{\left(y-c\right)}{Q^2}\frac{\partial v}{\partial a},f\right\rangle_\Phi+\mu\left(\bar{r}+\pi+\delta-\rho\right)+\dot{\mu}=0,\ t>0.$$

Finally,

$$\frac{d\mu}{dt}=\left(\rho-\bar{r}-\pi-\delta\right)\mu+\frac{1}{Q^2\left(t\right)}\sum_{i=1}^{2}\int_\Phi\frac{\partial v_i}{\partial a}\left[\delta a+\left(y-c\right)\right]f_i\left(t,a\right)da,$$

here we have applied the fact that $\zeta=v$. In steady-state, this results in

$$\mu=\frac{1}{\left(\bar{r}+\pi+\delta-\rho\right)Q^2}\sum_{i=1}^{2}\int_\Phi\frac{\partial v_i}{\partial a}\left[\delta a+\left(y-c\right)\right]f_i\left(a\right)da.$$

Proof of Lemma 2

In order to prove the concavity of the value function we express the model in discrete time for an arbitrarily small Δt. The Bellman equation of a household is

$$\begin{aligned} v_t^{\Delta t}(a,y) &= \max_{a' \in \Gamma(a,y)} \left[u^c \left(\frac{Q(t)}{\Delta t} \left[\left(1 + \left(\frac{\delta}{Q(t)} - \delta - \pi(t) \right) \Delta t \right) a + \frac{y \Delta t}{Q(t)} - a' \right] \right) - u^{\pi}(\pi(t)) \right] \Delta t \\ & \quad + e^{-\rho \Delta t} \sum_{i=1}^{2} v_{t+\Delta t}^{\Delta t}(a', y_i) \mathbb{P}(y' = y_i | y), \end{aligned}$$

where $\Gamma(a,y) = \left[0, \left(1 + \left(\frac{\delta}{Q(t)} - \delta - \pi(t)\right)\Delta t\right) a + \frac{y\Delta t}{Q(t)}\right]$, and $\mathbb{P}(y' = y_i|y)$ are the transition probabilities of a two-state Markov chain. The Markov transition probabilities are given by $\lambda_1 \Delta t$ and $\lambda_2 \Delta t$.

We verify that this problem satisfies the conditions of Theorem 9.8 of Stokey, Lucas and Prescott (1989): (i) Φ is a convex subset of $\mathbb{R}$; (ii) the Markov chain has a finite number of values; (iii) the correspondence $\Gamma(a,y)$ is nonempty, compact-valued and continuous; (iv) the function u^c is bounded, concave and continuous and $e^{-\rho\Delta t} \in (0,1)$; and (v) the set $A^y = \{(a,a') \text{ such that } a' \in \Gamma(a,y)\}$ is convex. We may conclude that $v_t^{\Delta t}(a,y)$ is concave for any $\Delta t > 0$. Finally, for any $a_1,\ a_2 \in \Phi$

$$\begin{aligned} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &\geq \omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y), \\ \lim_{\Delta t \to 0} v_t^{\Delta t}(\omega a_1 + (1-\omega) a_2, y) &\geq \lim_{\Delta t \to 0} \left[\omega v_t^{\Delta t}(a_1, y) + (1-\omega) v_t^{\Delta t}(a_2, y) \right], \\ v(t, \omega a_1 + (1-\omega) a_2, y) &\geq \omega v(t, a_1, y) + (1-\omega) v_t(t, a_2, y), \end{aligned}$$

so that $v(t,a,y)$ is concave.

Proof of Proposition 3: Inflation bias in MPE

As the value function is concave in a, then it satisfies that

$$\frac{\partial v_i(t,\tilde{a})}{\partial a} < \frac{\partial v_i(t,0)}{\partial a} < \frac{\partial v_i(t,\hat{a})}{\partial a},\ \forall \tilde{a} \in (0,\varkappa),\ \hat{a} \in (\phi, 0),\ t \geq 0,\ i = 1,2. \tag{49}$$

In addition, the condition that the country is a net debtor ($\bar{a}_t < 0$) implies

$$\sum_{i=1}^{2} \int_{\phi}^{0} (-a) f_i(t,a) da \geq \sum_{i=1}^{2} \int_{0}^{\varkappa} (a) f_i(t,a) da,\ \forall t \geq 0. \tag{50}$$

Therefore

$$\sum_{i=1}^{2}\int_{0}^{\varkappa} af_i\frac{\partial v_i(t,a)}{\partial a}da \;<\; \frac{\partial v_i(t,0)}{\partial a}\sum_{i=1}^{2}\int_{0}^{\varkappa} af_i da \leq \frac{\partial v_i(t,0)}{\partial a}\sum_{i=1}^{2}\int_{\phi}^{0}(-a)\,f_i(t,a)da \tag{51}$$

$$< \sum_{i=1}^{2}\int_{\phi}^{0}(-a)\,f_i(t,a)\frac{\partial v_i(t,a)}{\partial a}da, \tag{52}$$

where we have applied (49) in the first and last stepts and (50) in the intermediate one. The optimal inflation in the MPE case (22) with separable utility $u = u^c - u^\pi$ is

$$\sum_{i=1}^{2}\int_{\phi}^{\varkappa}\left(af_i\frac{\partial v_i}{\partial a} - u_\pi f_i\right)da = \sum_{i=1}^{2}\int_{\phi}^{\varkappa} af_i\frac{\partial v_i}{\partial a}da + u_\pi^\pi = 0.$$

Combining this expression with (51) we obtain

$$u_\pi^\pi = \sum_{i=1}^{2}\int_{\phi}^{\varkappa}(-a)\,f_i\frac{\partial v_i}{\partial a}da > 0.$$

Finally, taking into account the fact that $u_\pi^\pi > 0$ only for $\pi > 0$ we have that $\pi(t) > 0$.

Proposition 4: optimal long-run inflation under commitment in the limit as $\bar{r} \to \rho$

In the steady state, equations (26) and (28) in the main text become

$$(\rho - \bar{r} - \pi - \delta)\,\mu + \frac{1}{Q^2}\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\,da = 0,$$

$$\mu Q = u_\pi^\pi(\pi) + \sum_{i=1}^{2}\int_{\phi}^{\infty} a\frac{\partial v_i}{\partial a}f_i(a)\,da,$$

respectively. Consider now the limiting case $\rho \to \bar{r}$, and guess that $\pi \to 0$. The above two equations then become

$$\begin{aligned}\mu Q &= \frac{1}{\delta Q}\sum_{i=1}^{2}\int_{\phi}^{\infty}\frac{\partial v_i}{\partial a}\left[\delta a + (y_i - c_i)\right] f_i(a)\,da,\\ \mu Q &= \sum_{i=1}^{2}\int_{\phi}^{\infty} a\frac{\partial v_i}{\partial a}f_i(a)\,da,\end{aligned}$$

as $u_\pi^\pi(0) = 0$ under our assumed preferences in Section 3.4. Combining both equations, and using the fact that in the zero inflation steady state the bond price equals $Q = \frac{\delta}{\delta + \bar{r}}$, we obtain

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \frac{\partial v_i}{\partial a} \left(\bar{r}a + \frac{y_i - c_i}{Q} \right) f_i(a)\, da = 0. \tag{53}$$

In the zero inflation steady state, the value function v satisfies the HJB equation

$$\rho v_i(a) = u^c(c_i(a)) + \left(\bar{r}a + \frac{y_i - c_i(a)}{Q} \right) \frac{\partial v_i}{\partial a} + \lambda_i \left[v_j(a) - v_i(a) \right], \quad i = 1, 2, \ j \neq i, \tag{54}$$

where we have used $u^\pi(0) = 0$ under our assumed preferences. We also have the first-order condition

$$u_c^c(c_i(a)) = Q \frac{\partial v_i}{\partial a} \Rightarrow c_i(a) = u_c^{c,-1}\left(Q \frac{\partial v_i}{\partial a} \right).$$

We guess and verify a solution of the form $v_i(a) = \kappa_i a + \vartheta_i$, so that $u_c^c(c_i) = Q\kappa_i$. Using our guess in (54), and grouping terms that depend on and those that do not, we have that

$$\rho \kappa_i = \bar{r}\kappa_i + \lambda_i (\kappa_j - \kappa_i), \tag{55}$$

$$\rho \vartheta_i = u_c\left(u_c^{c,-1}(Q\kappa_i) \right) + \frac{y_i - u_c^{c,-1}(Q\kappa_i)}{Q} \kappa_i + \lambda_i (\vartheta_j - \vartheta_i), \tag{56}$$

for $i, j = 1, 2$ and $j \neq i$. In the limit as $\bar{r} \to \rho$, equation (55) results in $\kappa_j = \kappa_i \equiv \kappa$, so that consumption is the same in both states. The value of the slope κ can be computed from the boundary conditions.[47] We can solve for $\{\vartheta_i\}_{i=1,2}$ from equations (56),

$$\vartheta_i = \frac{1}{\rho} u_c\left(u_c^{c,-1}(Q\kappa) \right) + \frac{y_i - u_c^{c,-1}(Q\kappa)}{\rho Q} \kappa + \frac{\lambda_i (y_j - y_i)}{\rho (\lambda_i + \lambda_j + \rho) Q} \kappa,$$

for $i, j = 1, 2$ and $j \neq i$. Substituting $\frac{\partial v_i}{\partial a} = \kappa$ in (53), we obtain

$$\sum_{i=1}^{2} \int_{\phi}^{\infty} \left(\bar{r}a + \frac{y_i - c_i}{Q} \right) f_i(a)\, da = 0. \tag{57}$$

[47] The condition that the drift should be positive at the borrowing constraint, $s_i(\phi) \geq 0$, $i = 1, 2$, implies that

$$s_1(\phi) = \bar{r}\phi + \frac{y_1 - u_c^{c,-1}(Q\kappa)}{Q} = 0,$$

and

$$\kappa = \frac{u_c^c(\bar{r}\phi Q + y_1)}{Q}.$$

In the case of state $i = 2$, this guarantees $s_2(\phi) > 0$.

Equation (57) is exactly the zero-inflation steady-state limit of equation (17) in the main text (once we use the definitions of $\bar{a}$, $\bar{y}$ and $\bar{c}$), and is therefore satisfied in equilibrium. We have thus verified our guess that $\pi \to 0$.

B. Computational method: the stationary case

B.1 Exogenous monetary policy

We describe the numerical algorithm used to jointly solve for the equilibrium value function, $v(a, y)$, and bond price, Q, given an exogenous inflation rate π. The algorithm proceeds in 3 steps. We describe each step in turn.

Step 1: Solution to the Hamilton-Jacobi-Bellman equation Given π, the bond pricing equation (12) is trivially solved in this case:

$$Q = \frac{\delta}{\bar{r} + \pi + \delta}. \tag{58}$$

The HJB equation is solved using an *upwind finite difference* scheme similar to Achdou et al. (2015). It approximates the value function $v(a)$ on a finite grid with step $\Delta a : a \in \{a_1, ..., a_J\}$, where $a_j = a_{j-1} + \Delta a = a_1 + (j-1)\Delta a$ for $2 \le j \le J$. The bounds are $a_1 = \phi$ and $a_I = \varkappa$, such that $\Delta a = (\varkappa - \phi) / (J-1)$. We use the notation $v_{i,j} \equiv v_i(a_j)$, $i = 1, 2$, and similarly for the policy function $c_{i,j}$.

Notice first that the HJB equation involves first derivatives of the value function, $v_i'(a)$ and $v_i''(a)$. At each point of the grid, the first derivative can be approximated with a forward (F) or a backward (B) approximation,

$$v_i'(a_j) \approx \partial_F v_{i,j} \equiv \frac{v_{i,j+1} - v_{i,j}}{\Delta a}, \tag{59}$$

$$v_i'(a_j) \approx \partial_B v_{i,j} \equiv \frac{v_{i,j} - v_{i,j-1}}{\Delta a}. \tag{60}$$

In an upwind scheme, the choice of forward or backward derivative depends on the sign of the *drift function* for the state variable, given by

$$s_i(a) \equiv \left(\frac{\delta}{Q} - \delta - \pi\right) a + \frac{(y_i - c_i(a))}{Q}, \tag{61}$$

for $\phi \le a \le 0$, where

$$c_i(a) = \left[\frac{v_i'(a)}{Q}\right]^{-1/\gamma}. \tag{62}$$

Let superscript n denote the iteration counter. The HJB equation is approximated by the following upwind scheme,

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta}+\rho v_{i,j}^{n+1}=\frac{(c_{i,j}^{n})^{1-\gamma}}{1-\gamma}-\frac{\psi}{2}\pi^{2}+\partial_F v_{i,j}^{n+1}s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_B v_{i,j}^{n+1}s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}+\lambda_i\left(v_{-i,j}^{n+1}-v_{i,j}^{n+1}\right),$$

for $i=1,2,\ j=1,...,J$, where $\mathbf{1}(\cdot)$ is the indicator function and

$$s_{i,,jF}^{n} = \left(\frac{\delta}{Q}-\delta-\pi\right)a+\frac{y_i-\left[\frac{Q}{\partial_F v_{i,j}^{n}}\right]^{1/\gamma}}{Q}, \tag{63}$$

$$s_{i,j,B}^{n} = \left(\frac{\delta}{Q}-\delta-\pi\right)a+\frac{y_i-\left[\frac{Q}{\partial_B v_{i,j}^{n}}\right]^{1/\gamma}}{Q}. \tag{64}$$

Therefore, when the drift is positive ($s_{i,,jF}^{n}>0$) we employ a forward approximation of the derivative, $\partial_F v_{i,j}^{n+1}$; when it is negative ($s_{i,j,B}^{n}<0$) we employ a backward approximation, $\partial_B v_{i,j}^{n+1}$. The term $\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta}\to 0$ as $v_{i,j}^{n+1}\to v_{i,j}^{n}$. Moving all terms involving v^{n+1} to the left hand side and the rest to the right hand side, we obtain

$$\frac{v_{i,j}^{n+1}-v_{i,j}^{n}}{\Delta}+\rho v_{i,j}^{n+1}=\frac{(c_{i,j}^{n})^{1-\gamma}}{1-\gamma}-\frac{\psi}{2}\pi^{2}+v_{i,j-1}^{n+1}\alpha_{i,j}^{n}+v_{i,j}^{n+1}\beta_{i,j}^{n}+v_{i,j+1}^{n+1}\xi_{i,j}^{n}+\lambda_i v_{-i,j}^{n+1}, \tag{65}$$

where

$$\begin{aligned}
\alpha_{i,j}^{n} &\equiv -\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a},\\
\beta_{i,j}^{n} &\equiv -\frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a}+\frac{s_{i,j,B}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}}{\Delta a}-\lambda_i,\\
\xi_{i,j}^{n} &\equiv \frac{s_{i,j,F}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}}{\Delta a},
\end{aligned}$$

for $i=1,2,\ j=1,...,J$. Notice that the state constraints $\phi\le a\le 0$ mean that $s_{i,1,B}^{n}=s_{i,J,F}^{n}=0$.

In equation (65), the optimal consumption is set to

$$c_{i,j}^{n}=\left(\frac{\partial v_{i,j}^{n}}{Q}\right)^{-1/\gamma}. \tag{66}$$

where

$$\partial v_{i,j}^{n}=\partial_F v_{i,j}^{n}\mathbf{1}_{s_{i,j,F}^{n}>0}+\partial_B v_{i,j}^{n}\mathbf{1}_{s_{i,j,B}^{n}<0}+\partial\bar{v}_{i,j}^{n}\mathbf{1}_{s_{i,F}^{n}\le 0}\mathbf{1}_{s_{i,B}^{n}\ge 0}.$$

In the above expression, $\partial\bar{v}_{i,j}^{n}=Q(\bar{c}_{i,j}^{n})^{-\gamma}$ where $\bar{c}_{i,j}^{n}$ is the consumption level such that $s(a_i)\equiv$

$s_i^n = 0:$

$$\bar{c}_{i,j}^n = \left(\frac{\delta}{Q} - \delta - \pi\right) a_j Q + y_i.$$

Equation (65) is a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\frac{1}{\Delta}\left(\mathbf{v}^{n+1} - \mathbf{v}^n\right) + \rho \mathbf{v}^{n+1} = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^{n+1}$$

where the matrix $\mathbf{A}^n$ and the vectors v^{n+1} and $\mathbf{u}^n$ are defined by

$$\mathbf{A}^n = -\begin{bmatrix} \beta_{1,1}^n & \xi_{1,1}^n & 0 & 0 & \cdots & 0 & \lambda_1 & 0 & \cdots & 0 \\ \alpha_{1,2}^n & \beta_{1,2}^n & \xi_{1,2}^n & 0 & \cdots & 0 & 0 & \lambda_1 & \ddots & 0 \\ 0 & \alpha_{1,3}^n & \beta_{1,3}^n & \xi_{1,3}^n & \cdots & 0 & 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \alpha_{1,J-1}^n & \beta_{1,J-1}^n & \xi_{1,J-1}^n & 0 & \cdots & \lambda_1 & 0 \\ 0 & 0 & \cdots & 0 & \alpha_{1,J}^n & \beta_{1,J}^n & 0 & 0 & \cdots & \lambda_1 \\ \lambda_2 & 0 & \cdots & 0 & 0 & 0 & \beta_{2,1}^n & \xi_{2,1}^n & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \lambda_2 & 0 & \cdots & \alpha_{2,J}^n & \beta_{2,J}^n \end{bmatrix}, \quad \mathbf{v}^{n+1} = \begin{bmatrix} v_{1,1}^{n+1} \\ v_{1,2}^{n+1} \\ v_{1,3}^{n+1} \\ \vdots \\ v_{1,J-1}^{n+1} \\ v_{1,J}^{n+1} \\ v_{2,1}^{n+1} \\ \vdots \\ v_{2,J}^{n+1} \end{bmatrix} \tag{67}$$

$$\mathbf{u}^n = \begin{bmatrix} \frac{(c_{1,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{1,2}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{1,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \frac{(c_{2,1}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \\ \vdots \\ \frac{(c_{2,J}^n)^{1-\gamma}}{1-\gamma} - \frac{\psi}{2}\pi^2 \end{bmatrix}.$$

The system in turn can be written as

$$\mathbf{B}^n \mathbf{v}^{n+1} = \mathbf{d}^n \tag{68}$$

where , $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{1}{\Delta}\mathbf{v}^n$.

The algorithm to solve the HJB equation runs as follows. Begin with an initial guess $\{v_{i,j}^0\}_{j=1}^J$, $i = 1, 2$. Set $n = 0$. Then:

1. Compute $\{\partial_F v_{i,j}^n, \partial_B v_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (59)-(60).

2. Compute $\{c_{i,j}^n\}_{j=1}^J$, $i = 1, 2$ using (62) as well as $\{s_{i,j,F}^n, s_{i,j,B}^n\}_{j=1}^J$, $i = 1, 2$ using (63) and (64).

3. Find $\{v_{i,j}^{n+1}\}_{j=1}^{J}$, $i = 1, 2$ solving the linear system of equations (68).

4. If $\{v_{i,j}^{n+1}\}$ is close enough to $\{v_{i,j}^{n+1}\}$, stop. If not set $n := n + 1$ and proceed to 1.

Most computer software packages, such as Matlab, include efficient routines to handle sparse matrices such as $\mathbf{A}^n$.

Step 2: Solution to the Kolmogorov Forward equation The stationary distribution of debt-to-GDP ratio, $f(a)$, satisfies the Kolmogorov Forward equation:

$$\begin{aligned} 0 &= -\frac{d}{da}\left[s_i(a) f_i(a)\right] - \lambda_i f_i(a) + \lambda_{-i} f_{-i}(a), \; i = 1, 2. && (69) \\ 1 &= \int_{\phi}^{\infty} f(a) da. && (70) \end{aligned}$$

We also solve this equation using an finite difference scheme. We use the notation $f_{i,j} \equiv f_i(a_j)$. The system can be now expressed as

$$0 = -\frac{f_{i,j} s_{i,j,F} \mathbf{1}_{s_{i,j,F}^n > 0} - f_{i,j-1} s_{i,j-1,F} \mathbf{1}_{s_{i,j-1,F}^n > 0}}{\Delta a} - \frac{f_{i,j+1} s_{i,j+1,B} \mathbf{1}_{s_{i,j+1,B}^n < 0} - f_{i,j} s_{i,,jB} \mathbf{1}_{s_{i,,jB}^n < 0}}{\Delta a} - \lambda_i f_{i,j} + \lambda_{-i} f_{-i,j},$$

or equivalently

$$f_{i,j-1}\xi_{i,j-1} + f_{i,j+1}\alpha_{i,j+1} + f_{i,j}\beta_{i,j} + \lambda_{-i} f_{-i,j} = 0, \tag{71}$$

then (71) is also a system of $2 \times J$ linear equations which can be written in matrix notation as:

$$\mathbf{A}^{\mathbf{T}}\mathbf{f} = \mathbf{0}, \tag{72}$$

where $\mathbf{A}^{\mathbf{T}}$ is the transpose of $\mathbf{A} = \lim_{n\to\infty} \mathbf{A}^n$. Notice that $\mathbf{A}^n$ is the approximation to the operator $\mathcal{A}$ and $\mathbf{A}^{\mathbf{T}}$ is the approximation of the adjoint operator $\mathcal{A}^*$. In order to impose the normalization constraint (70) we replace one of the entries of the zero vector in equation (72) by a positive constant.[48] We solve the system (72) and obtain a solution $\hat{\mathbf{f}}$. Then we renormalize as

$$f_{i,j} = \frac{\hat{f}_{i,j}}{\sum_{j=1}^{J}\left(\hat{f}_{1,j} + \hat{f}_{2,j}\right)\Delta a}.$$

Complete algorithm The algorithm proceeds as follows.

Step 1: Individual economy problem. Given π, compute the bond price Q using (58) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}$ and of the matrix $\mathbf{A}$.

[48] In particular, we have replaced the entry 2 of the zero vector in (72) by 0.1.

Step 2: Aggregate distribution. Given $\mathbf{A}$ find the aggregate distribution $\mathbf{f}$.

B.2 Optimal monetary policy - MPE

In this case we need to find the value of inflation that satisfies condition (22). The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (58) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: Optimal inflation. Given $\mathbf{f}^{(m)}$ and $\mathbf{v}^{(m)}$, iterate steps 1-2 until $\pi^{(m)}$ satisfies[49]

$$\sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2} + \psi\pi^{(m)} = 0.$$

B.3 Optimal monetary policy - Ramsey

Here we need to find the value of the inflation and of the costate that satisfy conditions (26) and (25) in steady-state. The algorith proceeds as follows. We consider an initial guess of inflation, $\pi^{(1)} = 0$. Set $m := 1$. Then:

Step 1: Individual economy problem problem. Given $\pi^{(m)}$, compute the bond price $Q^{(m)}$ using (58) and solve the HJB equation to obtain an estimate of the value function $\mathbf{v}^{(m)}$ and of the matrix $\mathbf{A}^{(m)}$.

Step 2: Aggregate distribution. Given $\mathbf{A}^{(m)}$ find the aggregate distribution $\mathbf{f}^{(m)}$.

Step 3: Costate. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$,compute the costate $\mu^{(m)}$ using condition (25) as

$$\mu^{(m)} = \frac{1}{Q^{(m)}}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1} a_j f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2} + \psi\pi^{(m)}\right].$$

Step 4: Optimal inflation. Given $\mathbf{f}^{(m)}$, $\mathbf{v}^{(m)}$ and $\mu^{(m)}$, iterate steps 1-3 until $\pi^{(m)}$ satisfies

$$\left(\rho - \bar{r} - \pi^{(m)} - \delta\right)\mu^{(m)} + \frac{1}{\left(Q^{(m)}\right)^2}\left[\sum_{i=1}^{2}\sum_{j=2}^{J-1}\left(\delta a_j + y_i - c_{i,j}^{(m)}\right) f_{i,j}^{(m)} \frac{\left(v_{i,j+1}^{(m)} - v_{i,j-1}^{(m)}\right)}{2}\right].$$

[49] This can be done using Matlab's `fzero` function.

C. Computational method: the dynamic case

C.1 Exogenous monetary policy

We describe now the numerical algorithm to analyze the transitional dynamics, similar to the one described in Achdou et al. (2015). With an exogenous monetary policy it just amounts to solve the dynamic HJB equation (8) and then the dynamic KFE equation (14). Define T as the time interval considered, which should be large enough to ensure a converge to the stationary distribution and discretize it in N intervals of lenght

$$\Delta t = \frac{T}{N}.$$

The initial distribution $f(0,a,y) = f_0(a,y)$ and the path of inflation $\{\pi_t\}_{t=0}^{T}$ are given. We proceed in three steps.

Step 0: The asymptotic steady-state The asymptotic steady-state distribution of the model can be computed following the steps described in Section A. Given π_N, the result is a stationary destribution $\mathbf{f}_N$, a matrix $\mathbf{A}_N$ and a bond price Q_N defined at the asymptotic time $T = N\Delta t$.

Step 1: Solution to the Bond Pricing Equation The dynamic bond princing equation (12) can be approximated backwards as

$$(\bar{r} + \pi_n + \delta) Q_n = \delta + \frac{Q_{n+1} - Q_n}{\Delta t}, \Longleftrightarrow Q_n = \frac{Q_{n+1} + \delta \Delta t}{1 + \Delta t \left(\bar{r} + \pi_n + \delta\right)}, \quad n = N-1, .., 0, \tag{73}$$

where Q_N is the asymptotic bond price from Step 0.

Step 2: Solution to the Hamilton-Jacobi-Bellman equation The dynamic HJB equation (8) can approximated using an upwind approximation as

$$\rho \mathbf{v}^n = \mathbf{u}^n + \mathbf{A}^n \mathbf{v}^n + \frac{(\mathbf{v}^{n+1} - \mathbf{v}^n)}{\Delta t},$$

where $\mathbf{A}^n$ is constructing backwards in time using a procedure similar to the one described in Step 1 of Section B. By defining $\mathbf{B}^n = \left(\frac{1}{\Delta} + \rho\right)\mathbf{I} - \mathbf{A}^n$ and $\mathbf{d}^n = \mathbf{u}^n + \frac{\mathbf{v}^{n+1}}{\Delta t}$, we have

$$\mathbf{v}^n = (\mathbf{B}^n)^{-1} \mathbf{d}^n. \tag{74}$$

Step 3: Solution to the Kolmogorov Forward equation Let $\mathbf{A}$ defined in (67) be the approximation to the operator $\mathcal{A}$. Using a finite difference scheme similar to the one employed in the Step 2 of Section A, we obtain:

$$\frac{\mathbf{f}_{n+1}-\mathbf{f}_n}{\Delta t}=\mathbf{A}^{\mathbf{T}}\mathbf{f}_{n+1},\Longleftrightarrow \mathbf{f}_{n+1}=\left(\mathbf{I}-\Delta t\mathbf{A}^{\mathbf{T}}\right)^{-1}\mathbf{f}_n,\quad n=1,..,N \tag{75}$$

where $\mathbf{f}_0$ is the discretized approximation to the initial distribution $f_0(b)$.

Complete algorithm The algorith proceeds as follows:

Step 0: Asymptotic steady-state. Given π_N, compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N.

Step 1: Bond pricing. Given $\{\pi_n\}_{n=0}^{N-1}$, compute the bond price path $\{Q_n\}_{n=0}^{N-1}$ using (73).

Step 2: Individual economy problem. Given $\{\pi_n\}_{n=0}^{N-1}$ and $\{Q_n\}_{n=0}^{N-1}$ solve the HJB equation (74) backwards to obtain an estimate of the value function $\{\mathbf{v}_n\}_{n=0}^{N-1}$, and of the matrix $\{\mathbf{A}_n\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\{\mathbf{A}_n\}_{n=0}^{N-1}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (75).

C.2 Optimal monetary policy - MPE

In this case we need to find the value of inflation that satisfies condition (22)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)}\equiv\{\pi_n^{(0)}\}_{n=0}^{N-1}=\pi_N$ and $k:=1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)}\equiv\{Q_n^{(k)}\}_{n=0}^{N-1}$ using (73).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (74) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)}\equiv\{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)}\equiv\{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (75).

Step 4: Optimal inflation. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, iterate steps 1-3 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)}\equiv\sum_{i=1}^{2}\sum_{j=2}^{J-1}a_j f_{n,i,j}^{(k)}\frac{\left(v_{n,i,j+1}^{(k)}-v_{n,i,j-1}^{(k)}\right)}{2}+\psi\pi_n^{(k)}=0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi \Theta_n^{(k)},$$

with constant $\xi = 0.05$.

C.3 Optimal monetary policy - Ramsey

In this case we need to find the value of the inflation and of the costate that satisfy conditions (26) and (25)

Step 0: Asymptotic steady-state. Compute the stationary destribution $\mathbf{f}_N$, matrix $\mathbf{A}_N$, bond price Q_N and inflation rate π_N. Set $\pi^{(0)} \equiv \{\pi_n^{(0)}\}_{n=0}^{N-1} = \pi_N$ and $k := 1$.

Step 1: Bond pricing. Given $\pi^{(k-1)}$, compute the bond price path $Q^{(k)} \equiv \{Q_n^{(k)}\}_{n=0}^{N-1}$ using (73).

Step 2: Individual economy problem. Given $\pi^{(k-1)}$ and $Q^{(k)}$ solve the HJB equation (74) backwards to obtain an estimate of the value function $\mathbf{v}^{(k)} \equiv \{\mathbf{v}_n^{(k)}\}_{n=0}^{N-1}$ and of the matrix $\mathbf{A}^{(k)} \equiv \{\mathbf{A}_n^{(k)}\}_{n=0}^{N-1}$.

Step 3: Aggregate distribution. Given $\mathbf{A}^{(k)}$ find the aggregate distribution forward $\mathbf{f}^{(k)}$ using (75).

Step 4: Costate. Given $\mathbf{f}^{(k)}$ and $\mathbf{v}^{(k)}$, compute the costate $\mu^{(k)} \equiv \{\mu_n^{(k)}\}_{n=0}^{N-1}$ using (26):

$$\begin{aligned} \mu_{n+1}^{(k)} &= \mu_n^{(k)} \left[1 + \Delta t \left(\rho - \bar{r} - \pi^{(k)} - \delta\right)\right] \\ &+ \frac{\Delta t}{\left(Q_n^{(k)}\right)^2} \left[\sum_{i=1}^{2} \sum_{j=2}^{J-1} \left(\delta a_j + y_i - c_{n,i,j}^{(k)}\right) f_{n,i,j}^{(k+1)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2}\right], \end{aligned}$$

with $\mu_0^{(k)} = 0$.

Step 5: Optimal inflation. Given $\mathbf{f}^{(k)}$, $\mathbf{v}^{(k)}$ and $\mu^{(k)}$ iterate steps 1-4 until $\pi^{(k)}$ satisfies

$$\Theta_n^{(k)} \equiv \sum_{i=1}^{2} \sum_{j=2}^{J-1} a_j f_{n,i,j}^{(k)} \frac{\left(v_{n,i,j+1}^{(k)} - v_{n,i,j-1}^{(k)}\right)}{2} + \psi \pi_n^{(k)} - Q_n^{(k)} \mu_n^{(k)} = 0.$$

This is done by iterating

$$\pi_n^{(k)} = \pi_n^{(k-1)} - \xi \Theta_n^{(k)}.$$

D. An economy with costly price adjustment

In this appendix, we lay out a model economy with the following characteristics: (i) firms are explicitly modelled, (ii) a subset of them are price-setters but incur a convex cost for changing their nominal price, and (iii) the social welfare function and the equilibrium conditions constraining the central bank's problem are the same as in the model economy in the main text.

Final good producer

In the model laid out in the main text, we assumed that output of the consumption good was exogenous. Consider now an alternative setup in which the consumption good is produced by a representative, perfectly competitive final good producer with the following Dixit-Stiglitz technology,

$$y_t = \left(\int_0^1 y_{jt}^{(\varepsilon-1)/\varepsilon} dj \right)^{\varepsilon/(\varepsilon-1)}, \tag{76}$$

where $\{y_{jt}\}$ is a continuum of intermediate goods and $\varepsilon > 1$. Let P_{jt} denote the nominal price of intermediate good $j \in [0,1]$. The firm chooses $\{y_{jt}\}$ to maximize profits, $P_t y_t - \int_0^1 P_{jt} y_{jt} dj$, subject to (76). The first order conditions are

$$y_{jt} = \left(\frac{P_{jt}}{P_t} \right)^{-\varepsilon} y_t, \tag{77}$$

for each $j \in [0,1]$. Assuming free entry, the zero profit condition and equations (77) imply $P_t = (\int_0^1 P_{jt}^{1-\varepsilon} dj)^{1/(1-\varepsilon)}$.

Intermediate goods producers

Each intermediate good j is produced by a monopolistically competitive intermediate-good producer, which we will refer to as 'firm j' henceforth for brevity. Firm j operates a linear production technology,

$$y_{jt} = n_{jt}, \tag{78}$$

where n_{jt} is labor input. At each point in time, firms can change the price of their product but face quadratic price adjustment cost as in Rotemberg (1982). Letting $\dot{P}_{jt} \equiv dP_{jt}/dt$ denote the change in the firm's price, price adjustment costs in units of the final good are given by

$$\Psi_t \left(\frac{\dot{P}_{jt}}{P_{jt}} \right) \equiv \frac{\psi}{2} \left(\frac{\dot{P}_{jt}}{P_{jt}} \right)^2 \tilde{C}_t, \tag{79}$$

where $\tilde{C}_t$ is aggregate consumption. Let $\pi_{jt} \equiv \dot{P}_{jt}/P_{jt}$ denote the rate of increase in the firm's price. The instantaneous profit function in units of the final good is given by

$$\begin{aligned} \Pi_{jt} &= \frac{P_{jt}}{P_t} y_{jt} - w_t n_{jt} - \Psi_t(\pi_{jt}) \\ &= \left(\frac{P_{jt}}{P_t} - w_t\right)\left(\frac{P_{jt}}{P_t}\right)^{-\varepsilon} y_t - \Psi_t(\pi_{jt}), \end{aligned} \tag{80}$$

where w_t is the perfectly competitive real wage and in the second equality we have used (77) and (78). Without loss of generality, firms are assumed to be risk neutral and have the same discount factor as households, ρ. Then firm j's objective function is

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \Pi_{jt} dt,$$

with Π_{jt} given by (80). The state variable specific to firm j, P_{jt}, evolves according to $dP_{jt} = \pi_{jt} P_{jt} dt$. The aggregate state relevant to the firm's decisions is simply time: t. Then firm j's *value function* $V(P_{jt}, t)$ must satisfy the following Hamilton-Jacobi-Bellman (HJB) equation,

$$\rho V(P_j, t) = \max_{\pi_j} \left\{ \left(\frac{P_j}{P_t} - w_t\right)\left(\frac{P_j}{P_t}\right)^{-\varepsilon} y_t - \Psi_t(\pi_j) + \pi_j P_j \frac{\partial V}{\partial P_j}(P_j, t) \right\} + \frac{\partial V}{\partial t}(P_j, t).$$

The first order and envelope conditions of this problem are (we omit the arguments of V to ease the notation),

$$\psi \pi_{jt} \tilde{C}_t = P_j \frac{\partial V}{\partial P_j}, \tag{81}$$

$$\rho \frac{\partial V}{\partial P_j} = \left[\varepsilon w_t - (\varepsilon - 1)\frac{P_j}{P_t}\right]\left(\frac{P_j}{P_t}\right)^{-\varepsilon} \frac{y_t}{P_j} + \pi_j \left(\frac{\partial V}{\partial P_j} + P_j \frac{\partial^2 V}{\partial P_j^2}\right).$$

In what follows, we will consider a symmetric equilibrium in which all firms choose the same price: $P_j = P, \pi_j = \pi$ for all j. After some algebra, it can be shown that the above conditions imply the following pricing Euler equation,[50]

$$\left[\rho - \tilde{C}'(t)/\tilde{C}(t)\right]\pi(t) = \frac{\varepsilon - 1}{\psi}\left(\frac{\varepsilon}{\varepsilon - 1} w(t) - 1\right)\frac{1}{\tilde{C}_t} + \pi'(t). \tag{82}$$

Equation (82) determines the market clearing wage $w(t)$.

[50] The proof is available upon request.

Households

The preferences of household $k \in [0,1]$ are given by

$$\mathbb{E}_0 \int_0^\infty e^{-\rho t} \log\left(\tilde{c}_{kt}\right) dt,$$

where $\tilde{c}_{kt}$ is household consumption of the final good. We now define the following object,

$$c_{kt} \equiv \tilde{c}_{kt} + \frac{\tilde{c}_{kt}}{\tilde{C}_t} \int_0^1 \Psi_t\left(\pi_{jt}\right) dj,$$

i.e. household k's consumption plus a fraction of total price adjustment costs ($\int \Psi_t(\cdot)\, dj$) equal to that household's share of total consumption ($\tilde{c}_{kt}/\tilde{C}_t$). Using the definition of Ψ_t (eq. 79) and the symmetry across firms in equilibrium ($\dot{P}_{jt}/P_{jt} = \pi_t, \forall j$), we can write

$$c_{kt} = \tilde{c}_{kt} + \tilde{c}_{kt}\frac{\psi}{2}\pi_t^2 = \tilde{c}_{kt}\left(1 + \frac{\psi}{2}\pi_t^2\right). \tag{83}$$

Therefore, household k's instantaneous utility can be expressed as

$$\begin{aligned} \log(\tilde{c}_{kt}) &= \log\left(c_{kt}\right) - \log\left(1 + \frac{\psi}{2}\pi_t^2\right) \\ &= \log\left(c_{kt}\right) - \frac{\psi}{2}\pi_t^2 + O\left(\left\|\frac{\psi}{2}\pi_t^2\right\|^2\right), \end{aligned} \tag{84}$$

where $O(\|x\|^2)$ denotes terms of order second and higher in x. Expression (84) is the same as the utility function in the main text (eq. 30), up to a first order approximation of $\log(1+x)$ around $x = 0$, where $x \equiv \frac{\psi}{2}\pi^2$ represents the percentage of aggregate spending that is lost to price adjustment. For our baseline calibration ($\psi = 5.5$), the latter object is relatively small even for relatively high inflation rates, and therefore so is the approximation error in computing the utility losses from price adjustment. Therefore, the utility function used in the main text provides a fairly accurate approximation of the welfare losses caused by inflation in the economy with costly price adjustment described here.

Households can be in one of two idiosyncratic states. Those in state $i = 1$ do not work. Those in state $i = 2$ work and provide z units of labor inelastically. As in the main text, the instantaneous transition rates between both states are given by λ_1 and λ_2, and the share of households in each state is assumed to have reached its ergodic distribution; therefore, the fraction of working and non-working households is $\lambda_1/\left(\lambda_1+\lambda_2\right)$ and $\lambda_2/\left(\lambda_1+\lambda_2\right)$, respectively. Hours per worker z are such that total labor supply $\frac{\lambda_1}{\lambda_1+\lambda_2}z$ is normalized to 1.

An exogenous government insurance scheme imposes a (total) lump-sum transfer τ_t from working to non-working households. All households receive, in a lump-sum manner, an equal share of aggregate firm profits *gross* of price adjustment costs, which we denote by $\hat{\Pi}_t \equiv P_t^{-1} \int_0^1 P_{jt} y_{jt} dj - w_t \int_0^1 n_{jt} dj$. Therefore, disposable income (gross of price adjustment costs) for non-working and working households are given respectively by

$$I_{1t} \equiv \frac{\tau_t}{\lambda_2 / (\lambda_1 + \lambda_2)} + \hat{\Pi}_t,$$

$$I_{2t} \equiv w_t z - \frac{\tau_t}{\lambda_1 / (\lambda_1 + \lambda_2)} + \hat{\Pi}_t.$$

We assume that the transfer τ_t is such that gross disposable income for households in state i equals a constant level y_i, $i = 1, 2$, with $y_1 < y_2$. As in our baseline model, both income levels satisfy the normalization

$$\frac{\lambda_2}{\lambda_1 + \lambda_2} y_1 + \frac{\lambda_1}{\lambda_1 + \lambda_2} y_2 = 1.$$

Also, later we show that in equilibrium gross income equals one: $\hat{\Pi}_t + w_t \frac{\lambda_1}{\lambda_1 + \lambda_2} z = 1$. It is then easy to verify that implementing the gross disposable income allocation $I_{it} = y_i$, $i = 1, 2$, requires a transfer equal to $\tau_t = \frac{\lambda_2}{\lambda_1 + \lambda_2} y_1 - \frac{\lambda_2}{\lambda_1 + \lambda_2} \hat{\Pi}_t$. Finally, total price adjustment costs are assumed to be distributed in proportion to each household's share of total consumption, i.e. household k incurs adjustment costs in the amount $(\tilde{c}_{kt} / \tilde{C}_t)(\frac{\psi}{2} \pi_t^2 \tilde{C}_t) = \tilde{c}_{kt} \frac{\psi}{2} \pi_t^2$. Letting $I_{kt} \equiv y_{kt} \in \{y_1, y_2\}$ denote household k's gross disposable income, the law of motion of that household's real net wealth is thus given by

$$\begin{aligned} da_{kt} &= \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t \right) a_{kt} + \frac{I_{kt} - \tilde{c}_{kt} - \tilde{c}_{kt} \psi \pi_t / 2}{Q_t} \right] dt \\ &= \left[\left(\frac{\delta}{Q_t} - \delta - \pi_t \right) a_{kt} + \frac{y_{kt} - c_{kt}}{Q_t} \right] dt, \end{aligned} \tag{85}$$

where in the second equality we have used (83). Equation (85) is exactly the same as its counterpart in the main text, equation (3). Since household's welfare criterion is also the same, it follows that so is the corresponding maximization problem.

Aggregation and market clearing

In the symmetric equilibrium, each firm's labor demand is $n_{jt} = y_{jt} = \bar{y}_t$. Since labor supply $\frac{\lambda_1}{\lambda_1 + \lambda_2} z = 1$ equals one, labor market clearing requires

$$\int_0^1 n_{jt} dj = \bar{y}_t = 1.$$

Therefore, in equilibrium aggregate output is equal to one. Firms' profits gross of price adjustment costs equal

$$\hat{\Pi}_t = \int_0^1 \frac{P_{jt}}{P_t} y_{jt} dj - w_t \int_0^1 n_{jt} dj = \bar{y}_t - w_t,$$

such that gross income equals $\hat{\Pi}_t + w_t = \bar{y}_t = 1$.

Central bank and monetary policy

We have shown that households' welfare criterion and maximization problem are as in our baseline model. Thus the dynamics of the net wealth distribution continue to be given by equation (14). Foreign investors can be modelled exactly as in Section 2.2. Therefore, the central bank's optimal policy problems, both under commitment and discretion, are exactly as in our baseline model.