Rationalizable Implementation of Correspondences*

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Abstract

A new condition, which we call uniform monotonicity, is shown to be necessary and almost sufficient for rationalizable implementation of correspondences. Uniform monotonicity is much weaker than Maskin monotonicity and reduces to it in the case of functions. Maskin monotonicity, the key condition for Nash implementation, had also been shown to be necessary for rationalizable implementation of social choice functions. Our conclusion is that the conditions for rationalizable implementation are not only starkly different from, but also much weaker than those for Nash implementation, when we consider social choice correspondences. Thus, dropping rational expectations significantly expands the class of rules that can be decentralized by communication-based economic institutions.

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1 Introduction

The design of institutions to be used by rational agents has been an important research agenda in economic theory. As captured by the notion of Nash equilibrium, rationality is encapsulated in two aspects: these are (i) the best responses of agents to their beliefs, and (ii) that those beliefs are correct, the so-called rational expectations assumption. One can drop the latter and retain the former, moving then into the realm of rationalizability. One would conjecture that the design of institutions under rationalizable behavior, i.e., without insisting on rational expectations, should leave room for significantly different results than the theory based on equilibrium.[1] Settling this important question is our task in this paper. We show that dropping rational expectations significantly expands the class of rules that can be decentralized by communication-based institutions designed by the Central Authority for participating agents in the system.

The theory of Nash implementation has uncovered the conditions under which one can design a mechanism (or game form) such that the set of its Nash equilibrium outcomes coincides with a given *social choice correspondence* (henceforth, SCC). Indeed, Maskin (1999) proposes a well-known monotonicity condition, which we refer to as *Maskin monotonicity*. Maskin's (1999) main result shows that Maskin monotonicity is necessary and almost sufficient for Nash implementation.[2]

Nash implementation is concerned with complete information environments, in which all agents know the underlying state and this fact is commonly certain among them. As a foundation of Nash equilibrium, Aumann and Brandenburger (1995) delineate a set of epistemic conditions under which the agents' strategic interaction always leads to a Nash equilibrium. Furthermore, Polak (1999) shows that when the agents' payoffs are commonly certain, as complete information environments prescribe, the Aumann-Brandenburger epistemic conditions imply common certainty of rationality.

Bernheim (1984) and Pearce (1984) independently propose rationalizability, a weaker solution concept than Nash equilibrium, by asking what are the strategic implications that come solely from common certainty of rationality. Brandenburger and Dekel (1987) allow for the agents' beliefs to be correlated and propose an even weaker version of rationalizability. Lipman (1994) extends the concept of rationalizability to games with infinite action sets. In this case, the set of all

[1] On the one hand, from the existence point of view, since rationalizability is a weaker solution concept, one would conjecture a more permissive theory. On the other hand, uniqueness would be harder to establish. Hence, the answer, a priori, is far from clear.

[2] Maskin (1999) uses deterministic mechanisms, but allows mixed-strategy equilibria in an *ex post* sense (each outcome in the support of the equilibrium must be in the SCC). For us, given the importance of disagreements in beliefs, random outcomes are central in our mechanisms, but see footnote 15 as a point of comparison.

rationalizable strategies is fully characterized in terms of the strategies that survive the (possibly transfinite) iterative deletion of never best responses, taking limits as needed. Throughout the current paper, our discussion is entirely based upon Lipman's (1994) extension of correlated rationalizability of Brandenburger and Dekel (1987).

In a paper that was our starting point and motivation, Bergemann, Morris, and Tercieux (2011) –BMT in the sequel– recently consider the implementation of *social choice functions* (henceforth, SCFs) under complete information in rationalizable strategies. By an SCF we mean a single-valued SCC. They show that Maskin monotonicity is necessary and almost sufficient for rationalizable implementation. This essentially would imply that rationalizable implementation is similar to Nash implementation. However, their result has one important caveat: BMT focus only on SCFs in their analysis (we note that rationalizability and single-valuedness amount to uniqueness of Nash equilibrium). In any attempt to extend their result, one should ponder the following observations: (1) Maskin's characterization on Nash implementation holds true regardless of whether we consider SCFs or SCCs; (2) Maskin monotonicity can be quite restrictive in the case of SCFs (see, e.g., Mueller and Satterthwaite (1977) and Saijo (1987)); and (3) Many interesting SCCs are Maskin monotonic, including the Pareto, Core, envy-free, constrained Walrasian or Lindhal correspondences, while any SCF selected from a Maskin monotonic SCC no longer inherits the property.[3]

Therefore, what we set out to resolve here is the question of how close rationalizable implementation really is to Nash implementation, without imposing the straightjacket of single-valuedness. We interpret characterizations of implementable correspondences as descriptions of all that is feasible for the mechanism designer, and in this sense, multivaluedness strikes us as being quite plausible. In dealing with correspondences, we identify a new condition, which we call *uniform monotonicity*, basically closing the gap between necessity and sufficiency.[4] We show that uniform monotonicity is necessary (Theorem 1) and almost sufficient (Theorem 2) for rationalizable implementation of SCCs.[5]

A comparison between Maskin monotonicity and uniform monotonicity is in-

[3]The results in BMT (2011) contrasts with the much more permissive findings in Abreu and Matsushima (1994) for implementation in iterative elimination of weakly dominated strategies, or those in Abreu and Matsushima (1992), even though the latter are obtained for virtual or approximate implementation.

[4]A weaker version of this condition, based on the strict lower contour sets, first surfaced in Cabrales and Serrano (2011) under the name *weak quasimonotonicity;* see also its corrigendum, posted at http://www.econ.brown.edu/faculty/serrano/pdfs/2011GEB73-corrigendum.pdf.

[5]Theorem 2 assumes at least three agents and three additional conditions, "strong no worst alternative," "minimal conflict-of-interests" and "responsiveness" (See definitions in Section 6 below).

structive. Our uniform monotonicity requires the lower contour sets to be nested across states "uniformly" over all outcomes in the range of the SCC. This set-wise definition of monotonicity exhibits a clear contrast with Maskin monotonicity, which is a "pointwise" condition, in the sense that it requires the nestedness of the lower contour sets across states at any fixed outcome in the range of the SCC (see Subsection 4.1). Uniform monotonicity is logically weaker than Maskin monotonicity, and it is likely to be much weaker if the SCC contains many values in its range. However, both become equivalent in the case of SCFs. We also construct an example in which an SCC is rationalizably implementable by a finite mechanism, while it violates Maskin monotonicity at almost any outcome in the range of the SCC. In this sense, the SCC in the example is "very far from" being Nash implementable. Of course, as expected from our necessity result, we confirm that uniform monotonicity is satisfied for this SCC (Lemma 1).

Thus, rationalizable implementation is generally quite different from Nash implementation, and their alleged resemblance in BMT arose as an artifact of the assumption that only SCFs were being considered. This allows us to conclude that the design of economic institutions that rely on agents as best-responders, but which drop the rational expectations assumption, is possible for a significantly wider class of socially desirable rules.

In drawing that landscape of possibilities, we have relied on a canonical mechanism that heavily exhibits the violation of rational expectations. The mechanism features a novel use of a modulo game. In it, the election of a king is conducted, and the task of the king is to dictate the outcome. However, agents have different beliefs about who the elected king will be; for instance, agent i hopes for agent $(i+1)$ to be a generous king that will award agent i her most preferred outcome in the SCC, which implies in particular that agent i can announce that she would implement agent $(i-1)$'s most preferred outcome were agent i elected. It turns out that the messages involved, with the corresponding beliefs, can be made consistent with rationalizability. We remark that we do not require the existence of Nash equilibrium in the mechanism, unlike BMT; see footnote 15 again.

The rest of the paper is organized as follows. In Section 2, we introduce the general notation for the paper. Section 3 introduces rationalizability as our solution concept and defines the concept of rationalizable implementation. In Section 4, we propose and discuss uniform monotonicity, and show it to be necessary for rationalizable implementation. Section 5 illustrates by an example the conditions for rationalizable implementation and Nash implementation. In Section 6, we propose sufficient conditions for full implementation in rationalizable strategies, and provide a sketch of the proof to highlight the intuition behind our mechanism. Section 7 concludes. In the Appendix, we provide the proof of a claim (omitted from the main body of the paper), discuss the ordinal approach to rationalizable

implementation as well as the role of finite mechanisms, extend our results to the case of weak implementation, and evaluate the roles of the additional sufficient conditions used.

2 Preliminaries

Let $N = \{1, \ldots, n\}$ denote the finite set of agents and Θ be the finite set of states. It is assumed that the underlying state $\theta \in \Theta$ is common knowledge among the agents. Let A denote the set of social alternatives, which are assumed to be independent of the information state. We shall assume that A is countable, and denote by $\Delta(A)$ the set of probability distributions over A.[6] Note that $\Delta(A)$ has a countable support because A is countable. For any arbitrary set X (countable or not), we denote by $\Delta(X)$ the set of all Borel-measurable probability distributions over X endowed with the weak-* topology, and by $\Delta^*(X)$ its subset of distributions with countable support. Agent i's state dependent von Neumann-Morgenstern utility function is denoted $u_i : \Delta(A) \times \Theta \to \mathbb{R}$. We can now define an *environment* as $\mathcal{E} = (A, \Theta, (u_i)_{i \in N})$, which is implicitly understood to be common knowledge among the agents.

A (stochastic) *social choice correspondence* $F : \Theta \rightrightarrows \Delta(A)$ is a mapping from Θ to a nonempty compact subset of $\Delta(A)$.[7] The mapping F is called a *social choice function* if it is a single-valued social choice correspondence. In this case, we denote it by $f : \Theta \to \Delta(A)$. We henceforth use the acronyms SCC and SCF for both objects, respectively.

A *mechanism* (or *game form*) $\Gamma = ((M_i)_{i \in N}, g)$ describes a nonempty message space M_i for each agent $i \in N$ and an outcome function $g : M \to \Delta(A)$ where $M = M_1 \times \cdots \times M_n$.

3 Implementation in Rationalizable Strategies

We adopt *correlated rationalizability*, allowing the agents' beliefs to be correlated, as a solution concept and investigate the implications of implementation in *rationalizable* strategies. We fix a mechanism $\Gamma = (M, g)$ and define a message correspondence profile $S = (S_1, \ldots, S_n)$, where each $S_i \in 2^{M_i}$, and we write $\mathcal{S}$ for

[6] It is easy to see that one can extend our arguments to a separable metric space of alternatives, focusing on its countable dense subset.

[7] The compact-valuedness of the SCC is used in our sufficiency results. We note, for instance, that it is consistent with the environment in Mezzetti and Renou (2012), who consider Nash implementation in terms of the support of the equilibrium, with finite A and deterministic SCCs. In their footnote 4 (p. 2360), they argue that their results extend to the case in which A is a separable metric space and the SCC maps Θ into a countable dense subset of A.

the collection of message correspondence profiles. The collection $\mathcal{S}$ is a lattice with the natural ordering of set inclusion: $S \leq S^{'}$ if $S_i \subseteq S_i^{'}$ for all $i \in N$. The largest element is $\bar{S} = (M_1, \ldots, M_n)$. The smallest element is $\underline{S} = (\emptyset, \ldots, \emptyset)$.

We define an operator $b^\theta : \mathcal{S} \to \mathcal{S}$ to iteratively eliminate never best responses with $b^\theta = (b_1^\theta, \ldots, b_n^\theta)$ and b_i^θ is now defined as:

$$b_i^\theta(S) \equiv \left\{ m_i \in M_i \;\middle|\; \begin{array}{l} \exists \lambda_i \in \Delta^*(M_{-i}) \text{ such that} \\ (1)\lambda_i(m_{-i}) > 0 \Rightarrow m_j \in S_j \;\forall j \neq i; \\ (2) m_i \in \arg\max_{m_i'} \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i', m_{-i}); \theta) \end{array} \right\}$$

Recall that $\Delta^*(M_{-i})$ denotes the set of all Borel-measurable probability distributions over M_{-i} with countable support, endowed with the weak-* topology. Here we argue how we obtain $\Delta^*(M_{-i})$. We define the weak-* topology on M as follows: $m^k \to m$ as $k \to \infty$ if $g(m^k)$ converges to $g(m)$ pointwise. Since the set of lotteries $\Delta(A)$ has a countable dense subset, we can define an equivalence class on M such that for any $m, m^{'} \in M$, we say that $m \sim m^{'}$ if the closest lottery to $g(m)$ is equivalent to the closest lottery to $g(m^{'})$ within the countable dense subset of $\Delta(A)$. Since expected utility is continuous on $\Delta(A)$, this equivalence class on M can be taken and not affect the agents' behavior. Therefore, we can assume without loss of generality that the agents' beliefs have a countable support.

Observe that b^θ is increasing by definition: i.e., $S \leq S^{'} \Rightarrow b^\theta(S) \leq b^\theta(S^{'})$. By Tarski's fixed point theorem, there is a largest fixed point of b^θ, which we label $S^{\Gamma(\theta)}$. Thus, (i) $b^\theta(S^{\Gamma(\theta)}) = S^{\Gamma(\theta)}$ and (ii) $b^\theta(S) = S \Rightarrow S \leq S^{\Gamma(\theta)}$. We can also construct the fixed point $S^{\Gamma(\theta)}$ by starting with $\bar{S}$ – the largest element of the lattice – and iteratively applying the operator b^θ. If the message sets are finite, we have

$$S_i^{\Gamma(\theta)} \equiv \bigcap_{k \geq 1} b_i^\theta \left(\left[b^\theta \right]^k (\bar{S}) \right)$$

In this case, the solution coincides with iterated deletion of strictly dominated strategies. But because the mechanism Γ may be infinite, transfinite induction may be necessary to reach the fixed point. It is useful to define

$$S_{i,k}^{\Gamma(\theta)} \equiv b_i^\theta \left(\left[b^\theta \right]^{k-1} (\bar{S}) \right),$$

using transfinite induction if necessary. Thus, $S_i^{\Gamma(\theta)}$ is the set of messages surviving (transfinite) iterated deletion of never best responses of agent i. We refer the reader to Lipman (1994) for the formal treatment.

This is the central definition of implementability that we use in this paper:

Definition 1 (Full Rationalizable Implementation) *An SCC F is **fully implementable in rationalizable strategies** if there exists a mechanism $\Gamma =$*

(M, g) *such that for each* $\theta \in \Theta$,

$$\bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} = F(\theta).$$

Remark: This is the definition of implementability that Maskin (1999) adopts for Nash implementation. We believe that this is the right paradigm if we want to compare the permissiveness of Nash implementation theory versus a theory based on rationalizability. However, we also consider a weaker notion of implementation: an SCC F is *weakly* implementable in rationalizable strategies if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$, we have (i) $S^{\Gamma(\theta)} \neq \emptyset$ and (ii) $g(m) \in F(\theta)$ for each $m \in S^{\Gamma(\theta)}$. The reader is referred to Section A.4 for the details of the analysis in this case.

4 Uniform Monotonicity

In this section, we introduce a central condition to our results, which we term *uniform monotonicity*. We motivate it by comparing it to Maskin monotonicity, and we later show that uniform monotonicity is necessary for rationalizable implementation.

For the domain of complete information environments, Maskin (1999) proposes a monotonicity condition for *Nash implementation* where the set of Nash equilibrium outcomes is required to coincide with the SCC. This condition is often called *Maskin monotonicity*.

Definition 2 *An SCC* F *satisfies* ***Maskin monotonicity*** *if, for any states* $\theta, \theta' \in \Theta$ *and any* $a \in F(\theta)$*, if*

$$u_i(a, \theta) \geq u_i(z, \theta) \Rightarrow u_i(a, \theta') \geq u_i(z, \theta') \ \ \forall i \in N, \ \forall z \in \Delta(A),$$

then $a \in F(\theta')$.

Let D denote a countable subset of $\Delta(A)$ with a generic element d being a lottery over A. We denote the convex hull of D by

$$\text{co}(D) = \left\{ \{\alpha_d\}_{d \in D} \middle| \ \alpha_d \geq 0 \ \forall d \in D \ \text{ and } \ \sum_{d \in D} \alpha_d = 1 \right\}.$$

Definition 3 *An SCC* F *satisfies* ***weak uniform monotonicity*** *if, for every pair of states* $\theta, \theta' \in \Theta$*, if*

$$u_i(a; \theta) \geq u_i(z; \theta) \Rightarrow u_i(a; \theta') \geq u_i(z; \theta') \quad \forall a \in co(F(\theta)), \forall i \in N, \ \forall z \in \Delta(A),$$

then, $F(\theta) \subseteq F(\theta')$.

Remark: When we consider SCFs, $\text{co}(F(\theta))$ becomes a singleton set. Therefore, in this case, the condition just defined reduces to Maskin monotonicity.

We slightly strengthen weak uniform monotonicity into the following:

Definition 4 *An SCC F satisfies* ***uniform monotonicity*** *if, for every pair of states $\theta, \theta' \in \Theta$, if*

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \quad \forall a \in F(\theta),\ \forall i \in N,\ \forall z \in \Delta(A),$$

then $F(\theta) \subseteq F(\theta')$.

Remark: Note how, under expected utility, both conditions amount to the same thing, as requiring the nestedness of the lower contour sets over all $a \in F(\theta)$ or their convex hull is equivalent. However, it will be convenient to use the weak version for the proof of the necessity result, and the strong version for the proof of sufficiency.

4.1 Intuition and Examples

The comparison between Maskin monotonicity and uniform monotonicity is instructive. Maskin monotonicity always implies uniform monotonicity. The former checks for the "pointwise" inclusion, at an alternative $a \in F(\theta)$, of the lower contour sets of agents' preferences in state θ into those in θ', in order to determine whether that same alternative a should still remain in $F(\theta')$. The latter takes the entire set of alternatives $F(\theta)$ and checks "uniformly" whether, for each agent and $a \in F(\theta)$, his lower contour set at a in θ is contained in the lower contour set of a at θ', in order to determine that all outcomes in $F(\theta)$ should still be in $F(\theta')$. In other words, for an outcome $a \in F(\theta)$ to fall out of the SCC at θ' a preference reversal involving outcome a and another outcome $b \in \Delta(A)$ is required if the SCC is Maskin monotonic. If the SCC is uniformly monotonic, for $a \in F(\theta)$ and $a \notin F(\theta')$ to happen, all that is required is a preference reversal involving some pair $x \in F(\theta)$ and $y \in \Delta(A)$ and, importantly, x need not be the same as a. In this sense, uniform monotonicity is likely to be extremely weak in many settings because such "uniform inclusions" of lower contour sets will just be impossible, and the condition will be vacuously satisfied: for example, in a standard convex exchange economy (before extending it to expected utility preferences), if an SCC contains outcomes in which each agent is assigned bundles on different indifference curves (say a_i and b_i), it will generally be very difficult that the indifference curve through a_i at θ be nested into the one through the same bundle at θ', and at the same time, that the same nestedness happens for the indifference curves through bundle b_i. The reader is referred to Figure 1 for an illustration of this difficulty.

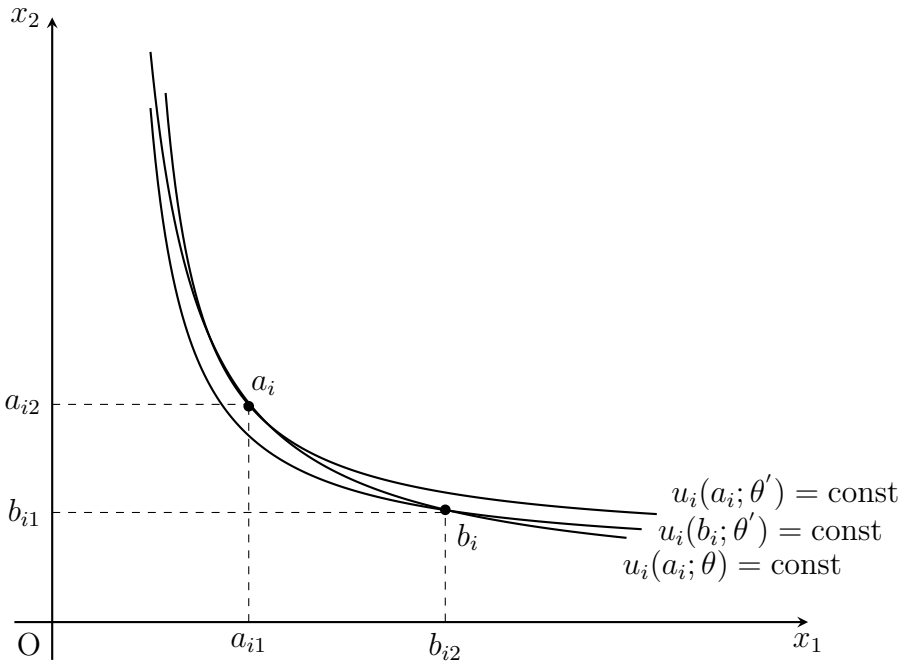

Figure 1: The Permissiveness of Uniform Monotonicity

In the figure, one can see that the nestedness of the lower contour sets at a_i from θ to θ' is satisfied, whereas the nestedness of the lower contour sets at b_i from θ to θ' is violated.

The same logic applies if one uses the probability simplex of lotteries over alternatives. With expected utility, the indifference map under any state consists of parallel straight lines. Maskin monotonicity is a trivial condition at points in the interior of the simplex, as the lower contour sets at any point are never nested (this was the key insight behind the very permissive results of virtual implementation (Abreu and Sen (1991), Matsushima (1988)), for instance). Thus, to make the argument of the relative permissiveness of uniform monotonicity, one should consider SCC's whose outcomes are at the boundaries of the simplex. Again, it will not be generally easy to have that *all* the lower contour sets at multiple boundary points in the simplex at state θ be nested into the corresponding lower contour sets at θ'.

This is not to say that uniform monotonicity is universally satisfied by all SCCs. Indeed, some SCCs may violate it. For instance, consider the *egalitarian-equivalent* allocation correspondence (henceforth, the EEA rule) in an exchange economy with continuous, convex, and strictly monotone preferences (define feasible allocations

with equality between total consumption and aggregate endowment).[8] Pazner and Schmeidler (1978) originally propose such an allocation rule and characterize it as the subset of feasible allocations for each of which there is a "reference" bundle on the ray that goes from the origin to the aggregate endowment vector such that each agent is indifferent between his assigned bundle and the reference bundle. Given the assumptions we imposed on the economy, the EEA rule is always nonempty, as the equal-division rule is egalitarian-equivalent. First we confirm that the EEA rule violates Maskin monotonicity. Let a_θ be an allocation specified by the EEA rule in state θ. Even if the nestedness of lower contour sets at a_θ across states is satisfied, as long as an agent's indifference curves at a_θ are not identical between two states, the original allocation a_θ no longer remains egalitarian-equivalent in the new state. Second, we argue that the EEA rule even violates uniform monotonicity (recall that uniform monotonicity is logically weaker than Maskin monotonicity). For the sake of expositional simplicity, consider the case where there are two agents and two commodities, each with the same aggregate amount. Assume further that agents have different Cobb-Douglas utility functions so that the contract curve (i.e., the set of Pareto efficient allocations) always lies either above or below the diagonal of the Edgeworth box. Then, we know that the equal-division rule is "not" Pareto efficient but as Pazner and Schmeidler (1978) show, there is a unique egalitarian equivalent allocation that is Pareto efficient. This implies that the EEA rule is genuinely a multi-valued correspondence consisting of these two allocations. Suppose the nestedness of the lower contour sets across states "over both outcomes in the EEA rule" needed for uniform monotonicity is satisfied. Note that the equal-division allocation continues to be egalitarian equivalent in the new state trivially. Let $\bar{z}_\theta$ be the reference bundle that corresponds to the unique Pareto efficient and egalitarian-equivalent allocation in state θ. In order for uniform monotonicity to hold, one must have that, given the reference bundle $\bar{z}_{\theta'}$, all agents' indifference curves through the assigned bundle in the new state θ' must continue to intersect at $\bar{z}_{\theta'}$. However, this cannot be guaranteed by the monotonic transformation of preferences we have for the hypothesis of uniform monotonicity. Therefore, the EEA rule violates uniform monotonicity.[9]

[8] An allocation $(x_i)_{i\in N}$ is said to be *egalitarian-equivalent* if there is a bundle z such that z is indifferent to x_i for every $i \in N$.

[9] Dutta and Vohra (1993) show in their Theorem 2 that the EEA rule satisfies a condition of weak positive association, denoted by WPA_h, which is weaker than Maskin monotonicity. Clarifying the connection between WPA_h and uniform monotonicity might be an interesting open question, left for future research.

4.2 Necessity for Rationalizable Implementation

We proceed to state and prove our first result, which identifies a necessary condition for rationalizable implementation[10]:

Theorem 1 *If an SCC F is fully implementable in rationalizable strategies, it satisfies weak uniform monotonicity.*

Proof: Suppose F is fully implementable in rationalizable strategies by a mechanism $\Gamma = (M, g)$. Fix two states $\theta, \theta^{'} \in \Theta$ satisfying the following property:

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta^{'}) \geq u_i(z;\theta^{'}) \ \forall a \in \text{co}(F(\theta)), \ \forall i \in N, \ \forall z \in \Delta(A) \ (*)$$

Then, due to the hypothesis that F is implementable by Γ, we fix $m^* \in S^{\Gamma(\theta)}$, and we have that $g(m^*) \in F(\theta)$.

Fix $i \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}$, there exists $\lambda_i^{m_i^*,\theta} \in \Delta^*(M_{-i})$ satisfying the following two properties: (i) $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{\Gamma(\theta)}$ and $g(m_i^*, m_{-i}) \in F(\theta)$; and (ii) $\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^{'}, m_{-i});\theta)$ for each $m_i^{'} \in M_i$.

We focus on the best response property of m_i^* summarized by inequality (ii). Fix $m_i^{'} \in M_i$. Due to the construction of $\lambda_i^{m_i^*,\theta}$, we have that

$$\begin{aligned} \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) &\geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^{'}, m_{-i});\theta) \\ u_i(a;\theta) &\geq u_i(z^a;\theta), \end{aligned}$$

where the two lotteries a and z^a are defined as

$$a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i^*, m_{-i}) \text{ and } z^a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i^{'}, m_{-i}).$$

Since $g(m_i^*, m_{-i}) \in F(\theta)$ for each m_{-i} with $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0$, we have $a \in \text{co}(F(\theta))$. Using Property $(*)$, we also obtain

$$u_i(a;\theta^{'}) \geq u_i(z^a;\theta^{'}).$$

[10]Our first working paper version dates back to May 2016. On the substance, the contents of that version and the current version are not significantly different. In a related model, using an implementability notion based on Mezzetti and Renou (2012), Jain – draft dated June 2016 – independently proves a necessity result that is close to our result in this section. That draft, however, contained a sufficiency result that was far from closing the gap between necessary and sufficient conditions. In a recent draft dated May 2017, Jain (2017) produces a sufficiency result that comes closer to closing that gap in his framework.

Due to the choice of a and z^a and the hypothesis that $u_i(\cdot)$ is a von-Neumann-Morgenstern expected utility, we obtain the following:

$$\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta') \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta').$$

Since this argument does not depend upon the choice of m_i', this shows that m_i^* is a best response to $\lambda_i^{m_i^*,\theta}$ in state θ' as well. Therefore, $m_i^* \in S_i^{\Gamma(\theta')}$. Since the choice of agent i is arbitrary, we can conclude that $m^* \in S^{\Gamma(\theta')}$. Furthermore, since the choice of $m^* \in S^{\Gamma(\theta)}$ is also arbitrary, we have $S^{\Gamma(\theta)} \subseteq S^{\Gamma(\theta')}$. Finally, by full implementability, this implies that

$$F(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} \subseteq \bigcup_{m \in S^{\Gamma(\theta')}} \{g(m)\} = F(\theta').$$

The proof is thus complete. ■

5 An Example

In this section, we show by example that rationalizable implementation can be very different from Nash implementation. We consider the following example. There are two agents $N = \{1, 2\}$; two states $\Theta = \{\alpha, \beta\}$; and a finite number K of pure outcomes $A = \{a_1, a_2 \ldots, a_K\}$ where $K \geq 4$.[11] Assume that it is commonly certain that both agents know the state, i.e., it is a complete information environment. Agent 1's utility function is given as follows: for each $k = 1, \ldots, K$,

$$u_1(a_k, \alpha) = u_1(a_k, \beta) = \begin{cases} 1 + K\varepsilon & \text{if } k = K, \\ 1 + (K - k)\varepsilon & \text{if } k \neq K, \end{cases}$$

where $\varepsilon \in (0, 1)$. Hence, agent 1 has state-uniform preferences over A and a_K is the best outcome in both states; a_1 is the second best outcome in both states; ...; and a_{K-1} is the worst outcome in both states for agent 1.

Agent 2's utility function in state α is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \alpha) = \begin{cases} 1 + (K + 1)\varepsilon & \text{if } k = K, \\ 1 + K\varepsilon & \text{if } k = 2, \\ 1 + k\varepsilon & \text{otherwise.} \end{cases}$$

[11] This example builds upon the one discussed in the Concluding Remarks section of BMT (2011).

In state β, agent 2's utility function is defined as follows: for each $k = 1, \ldots, K$,

$$u_2(a_k, \beta) = \begin{cases} 1 + (K+1)\varepsilon & \text{if } k = K, \\ 1 & \text{if } k = 2, \\ 1 + k\varepsilon & \text{otherwise.} \end{cases}$$

Note that a_K is the best outcome for agent 2 in both states; a_2 is his second best outcome in state α but it is his worst outcome in state β; and a_{K-1} is his third best outcome in state α and it is his second best outcome in state β.

We consider the following SCC F: $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$ and $F(\beta) = \{a_K\}$.

Claim 1 *For every outcome $a_k \in A$ with $a_k \neq a_2$,*

$$u_i(a_k, \alpha) \geq u_i(y, \alpha) \Rightarrow u_i(a_k, \beta) \geq u_i(y, \beta) \ \ \forall i = \{1, 2\}, \ \forall y \in \Delta(A).$$

Proof: Since agent 1 has state-uniform preferences, this claim is trivially true for agent 1. Thus, in what follows, we focus on agent 2. Take any lottery in the lower contour set of $a_k \in A \setminus \{a_2\}$ in state α. If that lottery did not contain a_2 in its support, it is still in the lower contour set of a_k in state β as no utilities have changed, and if it did contain a_2 in its support, since the utility of a_2 has decreased, it will also be in the lower contour set at β. This completes the proof. ■

Fix $a_k \in A \backslash \{a_2, a_K\}$ arbitrarily. If F were to satisfy Maskin monotonicity, we would have $a_k \in F(\beta)$, which is not the case. Therefore, we confirm the violation of Maskin monotonicity by the SCC F at every $a_k \in A \backslash \{a_2, a_K\}$. As is clear from the construction, we can choose K arbitrarily large. Therefore, the violation of Maskin monotonicity is severe, measured by the number of alternatives that should remain in the social choice in state β given the relevant nestedness of agents' preferences across the two states. In this sense, this correspondence is "very far" from being Maskin monotonic.

Nevertheless, we claim that the SCC F is implementable in rationalizable strategies using a finite mechanism. Consider the following mechanism $\Gamma = (M, g)$ where $M_i = \{m_i^1, m_i^2, \ldots, m_i^K\}$ for each $i = 1, 2$ and the deterministic outcome function $g(\cdot)$ is given in the table below:

$g(m)$		Agent 2						
		m_2^1	m_2^2	m_2^3	m_2^4	$\cdots$	m_2^{K-1}	m_2^K
Agent 1	m_1^1	a_1	a_1	a_{K-2}	a_{K-3}	$\cdots$	a_2	a_{K-1}
	m_1^2	a_2	a_1	a_1	a_{K-2}	$\cdots$	a_3	a_{K-1}
	m_1^3	a_3	a_2	a_1	a_1	$\cdots$	a_4	a_{K-1}
	m_1^4	a_4	a_3	a_2	a_1	$\cdots$	a_5	a_{K-1}
	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
	m_1^{K-1}	a_1	a_{K-2}	a_{K-3}	a_{K-4}	$\cdots$	a_1	a_{K-1}
	m_1^K	a_{K-1}	a_{K-1}	a_{K-1}	a_{K-1}	$\cdots$	a_{K-1}	a_K

Claim 2 *The SCC F is fully implementable in rationalizable strategies by the mechanism Γ.*

Proof: In state α, all messages can be best responses. Therefore, no message can be discarded via the iterative elimination of never best responses. That is, the set of rationalizable message profiles $S^{\Gamma}(\alpha) = M$. This implies that the set of rationalizable outcomes in state α is $F(\alpha) = \{a_1, a_2, \ldots, a_K\}$.

In state β, message m_2^K strictly dominates all other messages, $m_2^1, \ldots, m_2^{K-1}$ for agent 2. On the other hand, all messages for agent 1 can be a best response. In the second round of elimination of never best responses, m_1^K strictly dominates all other messages $m_1^1, \ldots, m_1^{K-1}$ for agent 1. Thus, we have $S^{\Gamma(\beta)} = \{(m_1^K, m_2^K)\}$. This implies that we have $F(\beta) = \{a_K\}$ as the unique rationalizable outcome in state β. This completes the proof. ■

BMT (2011) show in their Proposition 1 that strict Maskin monotonicity is necessary for implementation in rationalizable strategies under complete information. It follows from the previous example that this crucially relies on the assumption that only SCFs were considered in BMT's main result. More specifically, we show that, while the failure of Maskin monotonicity is severe, implementation in rationalizable strategies is still possible by a finite mechanism. For completeness, we provide the following lemma.

Lemma 1 *The SCC F satisfies uniform monotonicity.*

Proof: Since agent 1 has state-uniform preferences, we only focus on agent 2 in the following argument. First, we set $\theta = \alpha$ and $\theta' = \beta$ in the definition of uniform monotonicity. We know that $F(\alpha) = \{a_1, \ldots, a_K\}$ and by Claim 1, for any $a \in F(\alpha) \backslash \{a_2, a_K\}$ and $i \in \{1, 2\}$, we have the corresponding monotonic transformation from α to β. For $a_2 \in F(\alpha)$ and $a_3 \in A$, however, we have

$$u_2(a_2; \alpha) > u_2(a_3; \alpha) \text{ and } u_2(a_2; \beta) < u_2(a_3; \beta).$$

Therefore, the condition needed for the monotonic transformation from α to β under uniform monotonicity is not satisfied. Hence, in this case, uniform monotonicity imposes no conditions on SCCs.

Second, we set $\theta = \beta$ and $\theta' = \alpha$ in the definition of uniform monotonicity. Since $F(\beta) = \{a_K\}$ and a_K is the best outcome for agent 2 in both states, we have that for any $y \in \Delta(A)$,

$$u_2(a_K; \beta) \geq u_2(y; \beta) \Rightarrow u_2(a_K; \alpha) \geq u_2(y; \alpha).$$

In this case, uniform monotonicity implies that $a_K \in F(\alpha)$, which is indeed the case. Thus, F satisfies uniform monotonicity. ■

6 Sufficient Conditions for Full Implementation in Rationalizable Strategies

We turn in this section to our general sufficiency result. Before that, we introduce three additional conditions, the first of which follows immediately.

Definition 5 *An SCC F satisfies the* ***strong no-worst-alternative*** *condition (henceforth, SNWA) if, for each $\theta \in \Theta$ and $i \in N$, there exists $z_i^\theta \in \Delta(A)$ such that, for each $a \in F(\theta)$,*

$$u_i(a; \theta) > u_i(z_i^\theta; \theta).$$

Remark: This condition is introduced by Cabrales and Serrano (2011). In words, SNWA says that the SCC never assign the worst outcome to any agent at any state. BMT (2011) use its SCF-version and call it the no-worst-alternative condition (NWA).

Lemma 2 *If an SCC F satisfies SNWA, then for each $i \in N$, there exists a function $z_i : \Theta \times \Theta \to \Delta(A)$ such that for all $\theta, \theta' \in \Theta$:*

$$u_i(a; \theta') > u_i(z_i(\theta, \theta'); \theta') \ \ \forall a \in F(\theta')$$

and whenever $\theta \neq \theta'$,

$$u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta).$$

Proof: The proof is an appropriate extension of Lemma 2 of BMT (2011). For any agent $i \in N$, by SNWA, we are given the set of lotteries $\{z_i^\theta\}_{\theta \in \Theta}$. Then, define the average lottery as

$$\underline{z}_i \equiv \frac{1}{|\Theta|} \sum_{\theta \in \Theta} z_i^\theta.$$

Fix $i \in N$. For all $\theta' \in \Theta$, we define

$$z_i(\theta', \theta') \equiv (1-\varepsilon) z_i^{\theta'} + \varepsilon \underline{z}_i,$$

where $\varepsilon \in (0,1)$. For all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$:

$$z_i(\theta, \theta') \equiv (1-\varepsilon) z_i^{\theta'} + \frac{\varepsilon}{|\Theta|} \left(\sum_{\hat{\theta} \neq \theta} z_i^{\hat{\theta}} + a \right),$$

where $a \in F(\theta)$. By SNWA and the finiteness of Θ, we can choose $\varepsilon \in (0,1)$ sufficiently small so that $u_i(a'; \theta') > u_i(z_i(\theta, \theta'); \theta')$ for all $\theta, \theta' \in \Theta$ and $a' \in F(\theta')$. This establishes the first inequality.

Observe that the only difference between $z_i(\theta', \theta')$ and $z_i(\theta, \theta')$ lies in the fact that the lottery z_i^{θ} is replaced by some lottery $a \in F(\theta)$. But by SNWA, this is clearly increasing the expected utility of agent i in state θ, and hence we have that for all $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$:

$$u_i(z_i(\theta, \theta'); \theta) > u_i(z_i(\theta', \theta'); \theta).$$

This establishes the second inequality. This completes the proof. ■

We introduce next a second additional condition for the sufficiency result:

Definition 6 *An SCC F satisfies the* ***minimal conflict-of-interests condition*** *(henceforth, MCI) if there do not exist $\theta, \theta' \in \Theta$ with $\theta \neq \theta'$ and $a \in \Delta(A)$ such that $|F(\theta')| \geq 2$ and $a \in \arg\max_{b \in F(\theta')} u_i(b; \theta)$ for all $i \in N$.*

Remark: Note that MCI becomes a vacuous constraint when we consider SCFs. As the example in Section 5 shows, the outcome a_K is the best outcome for both agents in state α and $F(\alpha) = \{a_1, \ldots, a_K\}$. Therefore, the SCC F does not satisfy MCI, but it is implementable in rationalizable strategies by a finite mechanism. This implies that MCI is not necessary for rationalizable implementation. The same example shows that SNWA is not necessary either, as the SCC violates SNWA.

Next we introduce a natural extension of the responsiveness condition used by BMT (2011) to the case of SCCs.

Definition 7 *An SCC F is* ***responsive*** *if there do not exist two distinct states $\theta, \theta' \in \Theta$ such that $F(\theta) = F(\theta')$.*

Remark: Responsiveness is not necessary for rationalizable implementation either (think, for example, of the case of a constant correspondence). We discuss the role of this condition further in the last section of the Appendix.

For the sufficiency result we establish below, we propose the following mechanism $\Gamma = (M, g)$: each agent i sends a message $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6)$, where

- $m_i^1 \in \Theta$, i.e., a state;
- $m_i^2 = \{m_i^2[\theta]\}_{\theta \in \Theta}$ where $m_i^2[\theta] \in F(\theta)$, i.e., a state-dependent menu of socially desirable alternatives, understood as a recommendation to the designer;
- $m_i^3 = \{(m_i^3[\theta, 1], m_i^3[\theta, 2])\}_{\theta \in \Theta}$ where $m_i^3[\theta, 1] \in \Delta(A)$ and $m_i^3[\theta, 2] \in F(\theta)$, i.e., a state-dependent pair of alternatives, one of them in the SCC, understood as potential arguments for a challenge to the designer;
- $m_i^4 \in \Delta(A)$, i.e., a state-independent alternative, also understood as a challenge to the designer;
- $m_i^5 \in N$, i.e., a number chosen from $\{1, \ldots, n\}$, understood as a vote for some person to be the king;
- and $m_i^6 \in \mathbb{N}$, i.e., a positive integer.

The outcome function $g : M \to \Delta(A)$ is defined as follows: for each $m \in M$:

Rule 1. Consensus implements the recommendation made by the elected king: If there exists $\theta' \in \Theta$ such that $m_i^1 = \theta'$ and $m_i^6 = 1$ for all $i \in N$, then $g(m) = m_t^2[\theta']$ where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$.

Rule 2. An odd man out: If there exist $\theta' \in \Theta$ and $i \in N$ such that [a] $m_j^1 = \theta'$ and $m_j^6 = 1$ for all $j \neq i$, and [b] either $m_i^6 > 1$ or $m_i^1 \neq \theta'$, then the following subrules apply:

Rule 2-1. A nongreedy odd man out is heard in his challenge, although some bad outcomes are also implemented in the appeal process: If $u_i(m_t^2[\theta']; \theta') \geq u_i(m_i^3[\theta', 1]; \theta')$ and $m_t^2[\theta'] = m_i^3[\theta', 2]$ where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$, then

$$g(m) = \begin{cases} m_i^3[\theta', 1] & \text{with probability } m_i^6/(m_i^6 + 1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6 + 1) \end{cases}$$

Rule 2-2. A greedy odd man out is not heard in his challenge, although some bad outcomes are also implemented in the appeal process: Otherwise,

$$g(m) = \begin{cases} m_t^2[\theta'] & \text{with probability } m_i^6/(m_i^6+1) \\ z_i(\theta', \theta') & \text{with probability } 1/(m_i^6+1) \end{cases}$$

where $t = (\sum_{j \in N} m_j^5) \pmod{n+1}$.

Rule 3. Stronger disagreements lead to the integer game, implementing potential disarray in the appeal/challenge process: In all other cases,

$$g(m) = \begin{cases} m_1^4 & \text{with probability } \frac{m_1^6}{n(m_1^6+1)} \\ m_2^4 & \text{with probability } \frac{m_2^6}{n(m_2^6+1)} \\ \vdots & \vdots \\ m_n^4 & \text{with probability } \frac{m_n^6}{n(m_n^6+1)} \\ \underline{z} & \text{with the remaining probability,} \end{cases}$$

where

$$\underline{z} = \frac{1}{n} \sum_{i \in N} \underline{z}_i \text{ and } \underline{z}_i = \frac{1}{|\Theta|} \sum_{\theta \in \Theta} z_i^\theta.$$

We are finally ready to state the general sufficiency result for full implementation in rationalizable strategies.

Theorem 2 *Suppose that there are at least three agents ($n \geq 3$). If an SCC F satisfies uniform monotonicity, SNWA, MCI, and responsiveness, it is **fully** implementable in rationalizable strategies.*

Proof: We use the mechanism $\Gamma = (M, g)$ constructed above. The proof consists of Steps 1 through 4. Before going into the details of the proof, we briefly sketch its basic logic. We discuss the properties of our mechanism after providing the formal proof.

In Step 1, we show that any rationalizable message m_i involves $m_i^6 = 1$, that is, there must be at least consensus in the integer chosen. If this were not the case, either Rule 2 (odd man out) or Rule 3 (stronger disagreements) is triggered with probability one. By choosing the third and fourth components of the message appropriately, it is strictly better for agent i to announce an integer even higher than m_i^6, which contradicts the hypothesis that m_i is rationalizable.

In Step 2, we prove that any outcome in the range of the SCC can be supported by a rationalizable message profile. Let θ be the true state and fix $a \in F(\theta)$ arbitrarily. We construct the following message profile m: $m_i^1 = \theta$; $m_i^5 = 1$; and $m_i^6 = 1$ for every $i \in N$ and $m_1^2[\theta] = a$. Note first that no agent has an incentive

to become the odd man out and induce Rule 2 by unilaterally deviating from m. Thus, the specification of the third and fourth components of the messages do not matter. Then, m induces Rule 1 with probability one and $g(m) = a$ where agent 1 is the king, the winner of the modulo game. The novelty of the argument is that we can make m_1 rationalizable because agent 1 believes that agent 2 is a "generous king" so as to choose agent 1's best outcome from $F(\theta)$. Similarly, we can also make m_2 rationalizable because agent 2 believes that agent 3 is a "generous king" so as to choose agent 2's best outcome from $F(\theta)$. We extend this argument to all agents so that we can make m rationalizable.

In Step 3, we show that every agent believes that all rationalizable message profiles induce Rule 1 (consensus also in the announced state) with probability one. Suppose, by way of contradiction, that agent i believes with positive probability that Rule 2 (odd man out) or Rule 3 (stronger disagreements) is triggered. By choosing the third and fourth components of the message appropriately and an integer in its sixth component sufficiently high, agent i is able to find an even better response against his belief. This is a contradiction. Step 3 implies that one can partition the set of rationalizable message profiles into separate components, $\theta, \theta', \theta'', \ldots$. For instance, in the θ' component, this is the announced state by each agent in the first item of their messages, which also determines the event to which each of them assigns probability 1. That is, in that component, each agent i believes that all the others are using strategies of the form $(\theta', \cdot, \cdot, \cdot, \cdot, 1)$ with probability 1.

In Step 4, we prove that if m is a message profile such that $m_i = (\theta', \cdot, \cdot, \cdot, \cdot, 1) \in S_i^{\Gamma(\theta)}$ for each $i \in N$, then $g(m) \in F(\theta)$. If $\theta' = \theta$, this is trivially true. So, we assume $\theta' \neq \theta$. First, using the features of the canonical mechanism, a technical claim –Claim 3– shows that if one has a rationalizable message profile, one can modify it slightly in order to support any outcome in the range of the social choice correspondence. After that claim, the proof is by contradiction. If in state θ there were a rationalizable message profile whose outcome is not in $F(\theta)$ where all agents are coordinating in a deception in which they are reporting state θ', given previous steps in the proof, the outcome must be actually in $F(\theta')$. We then claim $F(\theta') \subseteq F(\theta)$. If this were not the case, uniform monotonicity would allow us to use the preference reversal for at least an agent and at least an alternative $a^* \in F(\theta')$. By the technical claim, this should also be supported by rationalizable messages, but, using MCI, we show it cannot.

Finally, by responsiveness, if $m_i = (\theta', \cdot, \cdot, \cdot, \cdot, 1) \in S_i^{\Gamma(\theta)}$ for each $i \in N$, we must have $\theta' = \theta$. This means that it is commonly certain that all agents announce the true state θ under rationalizability. This completes the proof.

Now, we proceed to the formal proof. Throughout, we denote the true state by θ

Step 1: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow m_i^6 = 1$.

Proof of Step 1: Let $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6) \in S_i^{\Gamma(\theta)}$. Suppose by way of contradiction that $m_i^6 > 1$. Then, for any profile of messages m_{-i} that agent i's opponents may play, (m_i, m_{-i}) will trigger either Rule 2 or Rule 3. We can partition the message profiles of all agents but i as follows:

$$M_{-i}^2 \equiv \left\{ m_{-i} \in M_{-i} \middle| \ \exists \theta^{'} \in \Theta \text{ s.t. } m_j^1 = \theta^{'}, m_j^2[\theta^{'}] \in F(\theta^{'}), \text{ and } m_j^6 = 1 \ \forall j \neq i \right\}$$

denotes the set of messages of all agents but i in which Rule 2 is triggered, and

$$M_{-i}^3 \equiv M_{-i} \backslash M_{-i}^2$$

denotes the set of messages of all agents but i in which Rule 3 is triggered.

Suppose first that agent i has a belief $\lambda_i \in \Delta^*(M_{-i})$ under which Rule 3 is triggered with positive probability, so that $\sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) > 0$. If $u_i(m_i^4; \theta) > u_i(z_i^\theta; \theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}_i^6$ is chosen to be larger than m_i^6. In doing so, agent i decreases the probability that $\underline{z}$ is chosen in Rule 3.

Note that, under Rule 3, by choosing an appropriate lottery, each agent has a strict incentive to reduce the probability that $\underline{z}$ occurs. To see this, fix $\theta \in \Theta$ and $a \in F(\theta)$. Then, define

$$\hat{z}_i(\theta) \equiv \frac{1}{|\Theta|} \sum_{\hat{\theta} \neq \theta} z_i^{\hat{\theta}} + \frac{1}{|\Theta|} a.$$

By SNWA, we obtain

$$u_i(\hat{z}_i(\theta); \theta) > u_i(\underline{z}_i; \theta).$$

Define

$$z_i^*(\theta) \equiv \frac{1}{n} \sum_{j \neq i} \underline{z}_j + \frac{1}{n} \hat{z}_i(\theta).$$

Since $u_i(\hat{z}_i(\theta); \theta) > u_i(\underline{z}_i; \theta)$, we have

$$u_i(z_i^*(\theta); \theta) > u_i(\underline{z}; \theta).$$

So, conditional on Rule 3, we have

$$\sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i}); \theta) > \sum_{m_{-i} \in M_{-i}^3} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta).$$

If $u_i(m_i^4;\theta) \leq u_i(z_i^\theta;\theta)$, we define $\hat{m}_i$ as the same as m_i except that $\hat{m}_i^4 \in F(\theta)$ and $\hat{m}_i^6$ is chosen to be larger than m_i^6. Similarly, conditional on Rule 3, we obtain the same inequality.

Now suppose that agent i believes that Rule 2 will be triggered with positive probability, so that $\sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) > 0$. We again consider a deviation from m_i to $\hat{m}_i$ and observe that the choice of $\hat{m}_i^4$ does not affect the outcome of the mechanism conditional on Rule 2.

First, assume that $m_j^1 = \theta' \neq \theta$ for each $j \neq i$. Suppose $u_i(m_i^3[\theta',1];\theta) \geq u_i(z_i(\theta,\theta');\theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^6$ larger than m_i^6 and keeping m_i unchanged otherwise. Since $u_i(m_i^3[\theta',1];\theta) \geq u_i(z_i(\theta,\theta');\theta) > u_i(z_i(\theta',\theta');\theta)$, we have that conditional on Rule 2,

$$\sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i});\theta) > \sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i});\theta).$$

Otherwise, suppose that $u_i(m_i^3[\theta',1];\theta) < u_i(z_i(\theta,\theta');\theta)$. In this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^3[\theta',1] = z_i(\theta,\theta')$, $\hat{m}_i^6 > m_i^6 > 1$, and keeping m_i unchanged otherwise. Since $u_i(z_i(\theta,\theta');\theta) > u_i(z_i(\theta',\theta');\theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i});\theta) > \sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i});\theta).$$

Second, assume that $m_j^1 = \theta$ for each $j \neq i$. We choose $t^* \neq i$ and $m_{-i}^* \in \text{supp}(\lambda_i(\cdot))$ such that for each $j \neq i$ and $m_{-i} \in \text{supp}(\lambda_i(\cdot))$,

$$u_i(m_{t^*}^{*2}[\theta];\theta) \geq u_i(m_j^2[\theta];\theta).$$

Then, in this case, agent i could change m_i to $\hat{m}_i$ by having $\hat{m}_i^3[\theta,1] = m_{t^*}^{*2}[\theta]$ and $\hat{m}_i^6 > m_i^6 > 1$, keeping m_i unchanged otherwise. Since $u_i(m_{t^*}^{*2}[\theta];\theta) > u_i(z_i(\theta,\theta);\theta)$, we have that, conditional on Rule 2,

$$\sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(\hat{m}_i, m_{-i});\theta) > \sum_{m_{-i}\in M_{-i}^2} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i});\theta).$$

It follows that, in all cases, these choices of $\hat{m}_i$ strictly improve the expected payoff of agent i if either Rule 2 or Rule 3 is triggered. This implies that m_i is never a best response to any belief λ_i, which contradicts our hypothesis that $m_i \in S_i^{\Gamma(\theta)}$. ■

Step 2: For any $\theta \in \Theta$ and $a \in F(\theta)$, there exists $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a$.

Proof of Step 2: Fix $\theta \in \Theta$ as the true state, and fix $a \in F(\theta)$. Define $m_1^* = (\theta, m_1^{*2}, m_1^{*3}, m_1^{*4}, 1, 1)$, where $m_1^{*2}[\theta] = a$. For each $j \in \{2, \ldots, n\}$, define $m_j^* = (\theta, m_j^{*2}, m_j^{*3}, m_j^{*4}, 1, 1)$, where $m_j^{*2}[\theta] = a_{j-1}(\theta)$, which denotes one of the maximizers of $u_{j-1}(\cdot; \theta)$ within all the outcomes in $F(\theta)$ – recall that F is compact-valued and $u_{j-1}(\cdot; \theta)$ is continuous in probability. Then, the constructed message profile m^* induces Rule 1 and agent 1 becomes the winner of the modulo game. We thus have $g(m^*) = a$ by construction. What remains to show is that $m^* \in S^{\Gamma(\theta)}$.

By construction of the mechanism, Rule 3 cannot be triggered by any unilateral deviation from Rule 1. So, the specification of m_i^{*4} does not affect our argument. Moreover, also by construction of the mechanism, no agent has an incentive to induce Rule 2 with a unilateral deviation from a truthful profile under Rule 1. So, effectively, the specification of m_i^{*3} does not affect our argument either.

We first show that m_1^* can be made a best response to some belief. Define $\lambda_1^* \in \Delta^*(M_{-1})$ as follows: for any $m_{-1} \in M_{-1}$, if $\lambda_1^*(m_{-1}) > 0$,

$$\begin{aligned} m_j^1 &= \theta; \\ m_j^2[\theta] &= a_{j-1}(\theta); \\ m_j^5 &= \begin{cases} 2 & \text{if } j = 2, \\ 1 & \text{otherwise;} \end{cases} \\ m_j^6 &= 1. \end{aligned}$$

for all $j \in \{2, \ldots, n\}$. Given this belief λ_1^* and m_1^*, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is generated. Therefore, m_1^* is a best response to λ_1^* so that it survives the first round of deletion of never best responses.

We next show that the support of λ_1^* is rationalizable. Assume $j \neq 1$. Define

$$\bar{m}_j = \begin{cases} (\theta, \bar{m}_2^2, \bar{m}_2^3, \bar{m}_2^4, 2, 1) & \text{if } j = 2, \\ (\theta, \bar{m}_j^2, \bar{m}_j^3, \bar{m}_j^4, 1, 1) & \text{otherwise.} \end{cases}$$

where $\bar{m}_j^2[\theta] = a_{j-1}(\theta)$. Define $\bar{\lambda}_2 \in \Delta^*(M_{-2})$ as follows: for any $m_{-2} \in M_{-2}$, if $\bar{\lambda}_2(m_{-2}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= a_{k-1}(\theta); \\ m_k^5 &= \begin{cases} 2 & \text{if } k = 1, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1. \end{aligned}$$

for all $k \neq 2$. Then, given this belief $\bar{\lambda}_2$ and $\bar{m}_2$, agent 3 becomes the winner of the modulo game so that the outcome $a_2(\theta)$, which is the best one for agent 2, is

realized. Therefore, $\bar{m}_2$ is a best response to $\bar{\lambda}_2$ so that it survives the first round of deletion of never best responses. Assume $j \in N\backslash\{1,2\}$. Define $\bar{\lambda}_j \in \Delta^*(M_{-j})$ as follows: for any $m_{-j} \in M_{-j}$, if $\bar{\lambda}_j(m_{-j}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1, \\ a_{k-1}(\theta) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = 1 \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Assume $j < n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent $j+1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Assume, on the other hand, that $j = n$. Then, given the belief $\bar{\lambda}_j$ and $\bar{m}_j$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best one for agent n, is realized. Therefore, $\bar{m}_j$ is a best response to $\bar{\lambda}_j$ so that it survives the first round of deletion of never best responses. We can repeat this argument iteratively so that m_1^* survives the iterative deletion of never best responses. Hence, $m_1^* \in S_i^{\Gamma(\theta)}$.

Third, we shall show that, for each $j \neq 1$, m_j^* can be made a best response to some belief. For each $j \in \{2, \ldots, n\}$, define $\lambda_j^* \in \Delta^*(M_{-j})$ with support as follows:

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= \begin{cases} a_n(\theta) & \text{if } k = 1, \\ a_{k-1}(\theta) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = 1, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Given this belief λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the outcome $a_j(\theta)$, which is the best one for agent j, is realized. Therefore, for each $j \neq 1$, m_j^* is a best response to λ_j^* so that it survives the first round of deletion of never best responses.

Fourth, we will show that the support of λ_j^* is rationalizable. Consider $\bar{m}_1 = (\theta, \bar{m}_1^2, \bar{m}_1^3, \bar{m}_1^4, j+1, 1)$, where $\bar{m}_1^2[\theta] = a_n(\theta)$. Define $\bar{\lambda}_1 \in \Delta^*(M_{-1})$ as follows: for

any $m_{-1} \in M_{-1}$, if $\bar{\lambda}_1(m_{-1}) > 0$,

$$\begin{aligned} m_k^1 &= \theta; \\ m_k^2[\theta] &= a_{k-1}(\theta); \\ m_k^5 &= \begin{cases} n+2-j & \text{if } k=2, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq 1$. Given this belief $\bar{\lambda}_1$ and $\bar{m}_1$, agent 2 becomes the winner of the modulo game so that the outcome $a_1(\theta)$, which is the best one for agent 1, is realized. Therefore, $\bar{m}_1$ is a best response to $\bar{\lambda}_1$ so that it survives the first round of deletion of never best responses.

Consider agent $k \in N \backslash \{1, j\}$. We first assume $k < n$. Define $\bar{m}_k = (\theta, \bar{m}_k^2, \bar{m}_k^3, \bar{m}_k^4, 1, 1)$, where $\bar{m}_k^2[\theta] = a_{k-1}(\theta)$. Define $\bar{\lambda}_k \in \Delta^*(M_{-k})$ as follows: for any $m_{-k} \in M_{-k}$, if $\bar{\lambda}_k(m_{-k}) > 0$,

$$\begin{aligned} m_i^1 &= \theta; \\ m_i^2[\theta] &= a_{i-1}(\theta); \\ m_i^6 &= 1, \end{aligned}$$

for all $i \neq k$ and $\sum_{i \neq k} m_i^5 = n + k - 1$. Given this belief $\bar{\lambda}_k$ and $\bar{m}_k$, agent $k+1$ becomes the winner of the modulo game so that the outcome $a_k(\theta)$, which is the best one for agent k, is realized. Therefore, $\bar{m}_k$ is a best response to $\bar{\lambda}_k$ so that it survives the first round of deletion of never best responses.

Assume $n \neq j$. We define $\bar{m}_n = (\theta, \bar{m}_n^2, \bar{m}_n^3, \bar{m}_n^4, 1, 1)$ and $\bar{\lambda}_n \in \Delta^*(M_{-n})$ as follows: for any $m_{-n} \in M_{-n}$, if $\bar{\lambda}_n(m_{-n}) > 0$,

$$\begin{aligned} m_i^1 &= \theta; \\ m_i^2 &= \begin{cases} a_n(\theta) & \text{if } i=1 \\ a_{i-1}(\theta) & \text{otherwise;} \end{cases} \\ m_i^5 &= 1; \\ m_i^6 &= 1, \end{aligned}$$

for all $i \neq n$. Given this belief $\bar{\lambda}_n$ and $\bar{m}_n$, agent 1 becomes the winner of the modulo game so that the outcome $a_n(\theta)$, which is the best for agent n, is realized. Therefore, $\bar{m}_n$ is a best response to $\bar{\lambda}_n$ so that it survives the first round of deletion of never best responses.

We conclude that the support of λ_j^* is rationalizable. So, we can repeat this argument iteratively so that for each $j \neq 1$, m_j^* survives the iterative deletion of never best responses. Therefore, $m_j^* \in S_j^{\Gamma(\theta)}$ for each $j \neq 1$. Since $m_1^* \in S_1^{\Gamma(\theta)}$, we obtain $m^* \in S^{\Gamma(\theta)}$. This completes the proof of Step 2. ■

Step 3: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow \lambda_i(m_{-i}) = 0$ for any profile (m_i, m_{-i}) under Rules 2 or 3, where $\lambda_i \in \Delta^*(M_{-i})$ represents the belief held by i to which m_i is a best response.

Proof of Step 3: Suppose $m_i \in S_i^{\Gamma(\theta)}$. By Step 1, m_i has the form of $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1)$ for some $\theta' \in \Theta$, where the state θ' announced by different agents might be different. Given the message m_i, we define the set of messages of the remaining agents which trigger Rule 1, 2, or 3. Let M_{-i}^1 be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 1 and $M_{-i}^{2,i}$ be the set of $m_{-i} \in M_{-i}$ such that (m_i, m_{-i}) triggers Rule 2 with agent i as the deviating player (odd man out).

We consider a given belief λ_i of agent i. If $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 0$, then Rule 2 or 3 will be triggered with probability one. Although Rule 2 can now be triggered with a "deviating agent (odd man out)" being different from i, it is easily checked that a similar argument to that in Step 1 applies so that the message m_i cannot be a best reply to λ_i. So, suppose that

$$0 < \sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) < 1.$$

For each $\tilde{\theta} \in \Theta$, define

$$\hat{m}_i^3(\tilde{\theta}) = \begin{cases} (m_{j^*}^2[\theta'], m_i^3[\theta', 2]) & \text{if } \tilde{\theta} = \theta' \\ m_i^3[\tilde{\theta}] & \text{otherwise,} \end{cases}$$

where $j^* = \arg\max_{j \in N} u_i(m_j^2[\theta']; \theta)$. Define $\hat{m}_i^4 = \arg\max_{y \in \Delta(A)} u_i(y; \theta)$. We set $\hat{m}_i^6$ to be an integer sufficiently large. Define $\hat{m}_i = (\theta', m_i^2, \hat{m}_i^3, \hat{m}_i^4, m_i^5, \hat{m}_i^6)$ as i's alternative message in which we keep $m_i^1 = \theta', m_i^2$ and m_i^5 unchanged. Then, as $\hat{m}_i^6$ tends to infinity, agent i's expected utility from choosing $\hat{m}_i$ is approximately at least as high as

$$\sum_{m_{-i} \in M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(g(m_i, m_{-i}); \theta) + \sum_{m_{-i} \notin M_{-i}^1 \cup M_{-i}^{2,i}} \lambda_i(m_{-i}) u_i(\hat{m}_i^4; \theta),$$

which is strictly larger than i's expected payoff from choosing m_i. Hence, by choosing $\hat{m}_i^6$ large enough, $\hat{m}_i$ is a better response to λ_i (in words, the loss in Rule 2 can always be offset by a bigger gain in Rule 3). This is a contradiction.

So, if $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1) \in S_i^{\Gamma(\theta)}$, it follows that agent i must be convinced that each $j \neq i$ is choosing a message of the form $(\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1)$ and hence $\sum_{m_{-i} \in M_{-i}^1} \lambda_i(m_{-i}) = 1$. ■

We introduce an additional piece of notation. For any $\theta, \theta' \in \Theta$ and $i \in N$, define

$$S_i^{\Gamma(\theta)}[\theta'] = \left\{ m_i \in S_i^{\Gamma(\theta)} \middle| \; m_i^1 = \theta' \text{ and } m_i^6 = 1 \right\}.$$

For each $i \in N$ and $\theta' \in \Theta$, consider the sets $S_i^{\Gamma(\theta)}[\theta']$. Now define:

$$S^{\Gamma(\theta)}[\theta'] = \prod_{i \in N} S_i^{\Gamma(\theta)}[\theta'].$$

We also define

$$S_i^{\Gamma(\theta)} = \bigcup_{\theta' \in \Theta} S_i^{\Gamma(\theta)}[\theta'].$$

And, of course,

$$S^{\Gamma(\theta)} = \prod_{i \in N} S_i^{\Gamma(\theta)}.$$

Step 4: $S^{\Gamma(\theta)}[\theta'] \subseteq F(\theta)$.

Proof of Step 4: If $\theta = \theta'$, we are done because $m \in S^{\Gamma(\theta)}[\theta]$ implies $g(m) \in F(\theta)$. So, let $\theta \neq \theta'$. Suppose we have $m_i = (\theta', m_i^2, m_i^3, m_i^4, m_i^5, 1) \in S_i^{\Gamma(\theta)}[\theta']$. By Step 3, agent i believes with probability one that Rule 1 is triggered, implying that for every agent $k \neq i$, the sets$S_k^{\Gamma(\theta)}[\theta']$ is nonempty. Moreover, for any $k \neq i$, $m_k \in S_k^{\Gamma(\theta)}[\theta']$ implies that it is a best response to $\lambda_k \in \Delta(M_{-k})$, where the support of this belief consists of strategies that yield outcomes under Rule 1. That is, every agent $k \neq i$ also believes that Rule 1 will be triggered with probability one.

Take now the profile $\bar{m} = (\bar{m}_i, \bar{m}_{-i})$ such that $\bar{m}_j \in S_j^{\Gamma(\theta)}[\theta']$ for each $j \in N$. Clearly, by construction, $g(\bar{m}) \in F(\theta')$. If $F(\theta') \subseteq F(\theta)$, we immediately conclude that $g(\bar{m}) \in F(\theta)$. Therefore, we must assume that $F(\theta')$ is not a subset of $F(\theta)$.

So, suppose, contradicting the claim in Step 4, that $g(\bar{m}) \notin F(\theta)$. Since $F(\theta')$ is not a subset of $F(\theta)$, by uniform monotonicity, there exist an agent $j \in N$, an outcome $a^* \in F(\theta')$, and $z^* \in \Delta(A)$ such that $u_j(a^*; \theta') \geq u_j(z^*; \theta')$; and $u_j(a^*; \theta) < u_j(z^*; \theta)$. We begin with the following auxiliary claim, whose proof is relegated to the Appendix:

Claim 3 *If there exists $\bar{m} \in S^{\Gamma(\theta)}[\theta']$, for any $a^* \in F(\theta')$, there also exists $m^* \in S^{\Gamma(\theta)}[\theta']$ such that $a^* = g(m^*)$.*

We thus proceed with the proof. By Claim 3, there exists $m^* \in S^{\Gamma(\theta)}[\theta']$ such that $g(m^*) = a^*$ and $m_j^* = (\theta', m_j^{*2}, m_j^{*3}, m_j^{*4}, m_j^{*5}, 1) \in S_j^{\Gamma(\theta)}[\theta']$ for any $j \in N$.[12] Since $m_i^* \in S_i^{\Gamma(\theta)}[\theta']$, there exists $\lambda_i \in \Delta^*(M_{-i})$ such that (i) $\lambda_i(m_{-i}) > 0 \Rightarrow m_j = (\theta', m_j^2, m_j^3, m_j^4, m_j^5, 1) \in S_j^{\Gamma(\theta)}[\theta']$ for any $j \neq i$ and (ii) $\sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta) \geq \sum_{m_{-i}} \lambda_i(m_{-i}) u_i(g(\tilde{m}_i, m_{-i}); \theta)$ for all $\tilde{m}_i \in M_i$. Over profiles m_{-i} to which m_i^* is one of i's best responses in state θ, define

$$\hat{m}_{-i}(m_i^*) \in \arg \max_{(m_i^*, m_{-i}) \in S^{\Gamma(\theta)}[\theta']} u_i(g(m_i^*, m_{-i}); \theta).$$

[12] If $|F(\theta')| = 1$, we have $g(\bar{m}) = g(m^*) = a^*$.

Without loss of generality, we assume that the winner of the modulo game that yields the outcome $g(m_i^*, \hat{m}_{-i}(m_i^*))$ is actually not agent i himself.[13]

The rest of the proof is intended to establish that $m_i^* \notin S_i^{\Gamma(\theta)}[\theta']$. By Claim 3, this will imply $\bar{m}_i \notin S_i^{\Gamma(\theta)}[\theta']$, which will contradict our initial hypothesis. Therefore, we will have that either $\bar{m} \notin S^{\Gamma(\theta)}[\theta']$ or $\bar{m} \in S^{\Gamma(\theta)}[\theta']$ and $g(\bar{m}) \in F(\theta)$. This will complete the proof of Step 4.

We proceed to detail. Assume that $g(m_i^*, \hat{m}_{-i}(m_i^*)) \neq a^*$.[14] First, we observe that this assumption implies that $|F(\theta')| \geq 2$. Since the SCC F satisfies MCI and $|F(\theta')| \geq 2$, the profile $(m_i^*, \hat{m}_{-i}(m_i^*))$ is *not* a Nash equilibrium in state θ, since there must exist at least one agent $j \in N\backslash\{i\}$ who has a different strategy that is a better response to the profile $(m_i^*, \hat{m}_{-i}(m_i^*))$. Since every agent believes that Rule 1 is triggered with probability one, as we have established in Steps 1 to 3, this further implies that there are no message profiles in $S^{\Gamma(\theta)}[\theta']$ that are Nash equilibria in state θ. (In particular, m is not a Nash equilibrium either in state θ.)[15]

The preceding argument confirms that $S_{-i}^{\Gamma(\theta)}[\theta']$ contains multiple message profiles, which together with m_i^*, lead to distinct outcomes. Recall that $S_{-i}^{\Gamma(\theta)}[\theta']$ denotes the set of all rationalizable message profiles of all agents other than i in state θ where all other agents coordinate on θ' for the first component of their message. This is consistent with our assumption that $g(m_i^*, \hat{m}_{-i}(m_i^*))$ and a^* are two distinct outcomes, each of which is induced by some rationalizable message profile (m_i^*, m_{-i}) with $m_{-i} \in S_{-i}^{\Gamma(\theta)}[\theta']$. In particular, for each $j \in N\backslash\{i\}$, we have $m_j^* \in S_j^{\Gamma(\theta)}[\theta']$. Recall that $g(m^*) = a^*$. We define $\tilde{\lambda}_i \in \Delta^*(M_{-i})$ as follows: $\tilde{\lambda}_i(m_{-i}) = 0$ if and only if $m_{-i} \neq \hat{m}_{-i}(m_i^*)$. We now define $\lambda_i^\varepsilon \in \Delta^*(M_{-i})$ as the belief that assigns probability $1-\varepsilon$ to $\tilde{\lambda}_i$ and assigns probability ε to the event that all agents other than i use m_{-i}^*. By construction, the support of λ_i^ε is concentrated

[13] This is indeed confirmed in the proof of Claim 3, where we explicitly construct m_i^* from $\bar{m}_i$.

[14] Later we consider the case where $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$ and argue that it can also be handled by the same argument we are about to construct.

[15] Although there are no pure-strategy Nash equilibria in state θ, there might exist a mixed-strategy equilibrium whose support belongs to the set of rationalizable message profiles triggering Rule 1. For example, suppose there are three agents and $F(\theta) = \{a_1, a_2, a_3\}$, with $u_1(a_1;\theta) = 2$, $u_1(a_2;\theta) = 1$, $u_1(a_3;\theta) = 0$, $u_2(a_2;\theta) = 2$, $u_2(a_3;\theta) = 1$, $u_2(a_1;\theta) = 0$, $u_3(a_3;\theta) = 2$, $u_3(a_1;\theta) = 1$, $u_3(a_2;\theta) = 0$. Then, a profile of mixed-strategies, all of them inducing Rule 1, in which each agent announces the true state θ, asks for her most preferred outcome in $F(\theta)$, announces integer 1 in her sixth component, and randomizes with equal probability over all three names in her fifth component, is a Nash equilibrium. The probability that each agent is elected to be the king is 1/3, and cannot be affected by unilateral deviations. The outcome is the uniform probability distribution over the three alternatives in $F(\theta)$. Note how for each of the pure strategies in the support of the equilibrium, the resulting outcome is one of the alternatives in $F(\theta)$. This is the only canonical form that Nash equilibria can take in this mechanism.

on $S_{-i}^{\Gamma(\theta)}[\theta']$.

By construction of $\tilde{\lambda}_i$, m_i^* must be a best response to the redefined belief $\tilde{\lambda}_i$. We assume without loss of generality that the lottery z^* is the best lottery for agent i in state θ such that $u_i(a^*;\theta') \geq u_i(z^*;\theta')$ and $u_i(z^*;\theta) > u_i(a^*;\theta)$.[16] Fix $\varepsilon > 0$ small enough. Since $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) \geq u_i(g(m_i^{'}, \hat{m}_{-i}(m_i^*));\theta)$ for any $m_i^{'}$ and $\varepsilon > 0$ is chosen to be small, the best possible deviation by agent i is to choose $m_i^3[\theta', 1] = z^*$; $m_i^3[\theta', 2] = a^*$; and $m_i^6 \to \infty$, keeping the rest of her announcement the same as m_i^* so that the outcome is changed only when a^* used to occur under Rule 1. Therefore, we must have that for any $m_i^{'} \in M_i$,

$$\begin{aligned}
& \sum_{m_{-i}} \lambda_i^{\varepsilon}(m_{-i}) u_i(g(m_i^{'}, m_{-i});\theta) \\
\leq\ & (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) + \varepsilon u_i(z^*;\theta) \\
=\ & (1-\varepsilon) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) + \varepsilon u_i(a^*;\theta) + \varepsilon[u_i(z^*;\theta) - u_i(a^*;\theta)] \\
=\ & \sum_{m_{-i}} \lambda_i^{\varepsilon}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) + \varepsilon[u_i(z^*;\theta) - u_i(a^*;\theta)].
\end{aligned}$$

Thus, we obtain that for any $m_i^{'} \in M_i$,

$$\sum_{m_{-i}} \lambda_i^{\varepsilon}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) \geq \sum_{m_{-i}} \lambda_i^{\varepsilon}(m_{-i}) u_i(g(m_i^{'}, m_{-i});\theta) - \varepsilon[u_i(z^*;\theta) - u_i(a^*;\theta)].$$

Set $\varepsilon^{'} = \varepsilon[u_i(z^*;\theta) - u_i(a^*;\theta)]$. Hence, if we choose $\varepsilon > 0$ small enough (and hence, $\varepsilon^{'}$ also small enough), we have argued that m_i^* is an $\varepsilon^{'}$-best response to λ_i^{ε}, even including deviations to messages that trigger Rule 2.

However, we show this is not the case. Consider the already described deviation, m_i, by agent i, who chooses m_i^6 arbitrarily large, $m_i^3[\theta', 1] = z^*$, and $m_i^3[\theta', 2] = a^*$ but keeps the rest of her announcement the same as m_i^* so that the outcome is changed only when a^* used to occur under Rule 1. The construction of λ_i^{ε} guarantees that given m_i^*, the outcome a^* is realized with probability $\varepsilon > 0$. We define $\{\varepsilon(m_i^6)\}$ as a sequence on $\mathbb{R}$ such that (i) $\varepsilon(m_i^6) > 0$ for each m_i^6; (ii) $\varepsilon(m_i^6) \to 0$ as $m_i^6 \to \infty$; and (iii)

$$\frac{\frac{1}{m_i^6+1}}{\varepsilon(m_i^6)} \to 0 \text{ as } m_i^6 \to \infty.$$

For example, we can set $\varepsilon(m_i^6) = 1/\sqrt{m_i^6 + 1}$, which satisfies the three properties.

[16] This is indeed without loss of generality: one could take z^* to be the supremum lottery over the set of alternatives satisfying these inequalities, and then construct the argument below using a sequence of alternatives converging to z^*.

Recall that m_i^* is a best response to $\hat{m}_{-i}(m_i^*)$. This implies in particular that $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(z^*; \theta)$.

We next show that there exists $\varepsilon > 0$ small enough such that m_i^* is *not* an $\varepsilon^{'}$-best response to λ_i^{ε}, where $\varepsilon^{'} = \varepsilon[u_i(z^*; \theta) - u_i(a^*; \theta)]$. Indeed, we confirm this as follows:

$$
\begin{aligned}
& \sum_{m_{-i}} \lambda_i^{\varepsilon(m_i^6)}(m_{-i}) u_i(g(m_i, m_{-i}); \theta) - \varepsilon^{'}(m_i^6) \\
= \; & (1 - \varepsilon(m_i^6)) \left[\frac{m_i^6}{m_i^6 + 1} u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \frac{1}{m_i^6 + 1} u_i(z_i(\theta^{'}, \theta^{'}) \right] \\
& + \varepsilon(m_i^6) \left[\frac{m_i^6}{m_i^6 + 1} u_i(z^*; \theta) + \frac{1}{m_i^6 + 1} u_i(z_i(\theta^{'}, \theta^{'}); \theta) \right] - \varepsilon^{'}(m_i^6) \\
& (\because \text{ agent } i \text{ is not the winner of the modulo game under } (m_i, \hat{m}_{-i}(m_i^*)).) \\
\geq \; & \frac{m_i^6}{m_i^6 + 1} \left[(1 - \varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon(m_i^6) u_i(z^*; \theta) \right] \\
& + \frac{1}{m_i^6 + 1} u_i(z_i(\theta^{'}, \theta^{'}); \theta) - \varepsilon(m_i^6) \left[u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) - u_i(a^*; \theta) \right] \\
& (\because \; \varepsilon^{'}(m_i^6) = \varepsilon(m_i^6) \left[u_i(z^*; \theta) - u_i(a^*; \theta) \right] \text{ and } u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) \geq u_i(z^*; \theta)) \\
\approx \; & (1 - \varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon(m_i^6) u_i(z^*; \theta) \\
& - \varepsilon(m_i^6) \left[u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) - u_i(a^*; \theta) \right] \\
& (\text{if we choose } m_i^6 \text{ large enough so that } 1/(m_i^6 + 1) \to 0 \text{ but } \varepsilon(m_i^6) > 0.) \\
= \; & (1 - 2\varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon(m_i^6)(u_i(z^*; \theta) + u_i(a^*; \theta)) \\
= \; & (1 - 2\varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + 2\varepsilon(m_i^6) u_i(z^*; \theta) - \varepsilon(m_i^6)[u_i(z^*; \theta) - u_i(a^*; \theta)] \\
\approx \; & (1 - 2\varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + 2\varepsilon(m_i^6) u_i(z^*; \theta) \\
& (\text{if we choose } \varepsilon(m_i^6) \text{ small enough, noting } 2u_i(z^*; \theta) > u_i(z^*; \theta) - u_i(a^*; \theta)) \\
\approx \; & (1 - \varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon(m_i^6) u_i(z^*; \theta) \\
& (\text{if we choose } m_i^6 \text{ large enough so that } \varepsilon(m_i^6) \to 0) \\
> \; & (1 - \varepsilon(m_i^6)) u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) + \varepsilon(m_i^6) u_i(a^*; \theta) \\
& (\because \; u_i(z^*; \theta) > u_i(a^*; \theta), \text{ and } \varepsilon(m_i^6) > 0.) \\
= \; & \sum_{m_{-i}} \lambda_i^{\varepsilon(m_i^6)}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta).
\end{aligned}
$$

Hence, we have established the desired opposite inequality, showing that m_i^* is not an $\varepsilon^{'}$-best response to λ_i^{ε}.[17]

[17]To make our argument more transparent, we could divide it into the following two cases. We first assume $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*)); \theta) > u_i(z^*; \theta)$. In this case, we immedi-

For the case where $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$, we first observe that m_i^* is a best response to λ_i^ε independently of the size of ε because $g(m_i^*, \hat{m}_{-i}(m_i^*)) = g(m^*) = a^*$. We next claim that if we choose m_i^6 large enough, the same deviation strategy m_i constructed above is a better response to λ_i^ε than m_i^*. Specifically, given the belief λ_i^ε, m_i induces the outcome z^* with probability $m_i^6/(m_i^6+1)$ and the outcome $z_i(\theta', \theta')$ with the rest of probability. Since $u_i(z^*;\theta) > u_i(a^*;\theta)$, by choosing m_i^6 large enough, we obtain

$$\sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(m_i, m_{-i});\theta) > \sum_{m_{-i}} \lambda_i^\varepsilon(m_{-i}) u_i(g(m_i^*, m_{-i});\theta).$$

Thus, even if $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$, we obtain the desired contradiction, as in the previous case. Hence, regardless of whether $g(m_i^*, \hat{m}_{-i}(m_i^*)) \neq a^*$ or $g(m_i^*, \hat{m}_{-i}(m_i^*)) = a^*$, we conclude that $m_i^* \notin S_i^{\Gamma(\theta)}[\theta']$. This concludes the proof of Step 4. ■

Now we shall conclude the proof of Theorem 2. By Step 4 and the auxiliary claim (Claim 3) in its proof, for all $\theta \neq \theta'$, we know that $F(\theta') \subseteq F(\theta)$. Simply by reversing the roles of θ and θ' in the proof, we would reach the opposite inclusion, which would imply that $F(\theta) = F(\theta')$, contradicting responsiveness.

Therefore, it must be the case that, for any $\theta, \theta' \in \Theta$, whenever $\theta \neq \theta'$, the set $S_i^{\Gamma(\theta)}[\theta']$ is empty for all $i \in N$. It then follows that for any $\theta \in \Theta$,

$$S^{\Gamma(\theta)} = \prod_{i \in N} S_i^{\Gamma(\theta)} = \prod_{i \in N} S_i^{\Gamma(\theta)}[\theta].$$

This together with Step 2 further implies

$$\bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} = F(\theta).$$

This concludes the proof of Theorem 2. ■

We make some comments on the properties of the mechanism we constructed. One novelty of the mechanism lies in the way we use the modulo game under the consensus Rule 1. Modulo games have been often used in the literature, but they are viewed as a substitute for the integer games where the person who announces the highest integer becomes a dictator. In fact, our Rule 3 on strong disagreements is a stochastic version of an integer game, used to knock out bad message profiles

ately obtain a strict inequality even before we take $1/(m_i^6+1) \to 0$. Otherwise, i.e., if $u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) = u_i(z^*;\theta)$, when we obtain $(1-2\varepsilon(m_i^6))u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) + 2\varepsilon(m_i^6)u_i(z^*;\theta)$, this is equivalent to $(1-\varepsilon(m_i^6))u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) + \varepsilon(m_i^6)u_i(z^*;\theta)$. This is larger than $(1-\varepsilon(m_i^6))u_i(g(m_i^*, \hat{m}_{-i}(m_i^*));\theta) + \varepsilon(m_i^6)u_i(a^*;\theta)$, regardless of the size of $\varepsilon(m_i^6)$.

so that the agents end up making a unanimous announcement in equilibrium, as only Rule 1 (consensus) prevails. However, in our mechanism, the modulo game in Rule 1 is used to select an outcome rather than knock out bad message profiles. Since the rational expectations hypothesis is not needed here, each agent believes that the modulo game under Rule 1 always works in his favor but the resulting outcome is not necessarily what he expects to happen.

Our paper follows the classic implementation literature in allowing for arbitrary mechanisms. This is often justified in order to obtain a tight characterization, i.e., to have a small gap between necessary and sufficient conditions. Jackson (1992), however, rightly argues that some of the power of implementation results derive from the fact that we have not imposed any restrictions on the mechanisms. In order to restrict attention to reasonable mechanisms, Jackson, Palfrey, and Srivastava (henceforth, JPS, 1994) propose the best response property, which requires that there be a best response for every possible belief that an agent might hold about other agents' strategies. Indeed, our mechanism does not satisfy this property as there exist no best responses when Rule 3 (strong disagreements) is triggered. As we discuss in Section 5, finite mechanisms sometimes suffice for rationalizable implementation. Finite mechanisms clearly satisfy the JPS best response property. We will have more to say about rationalizable implementation by finite mechanisms for the case of SCFs and postpone this discussion until we state our result for that case.

When focusing only on SCFs, we obtain the following result as a corollary of Theorem 2.

Corollary 1 *Suppose that there are at least three agents ($n \geq 3$). If an SCF f satisfies Maskin monotonicity, NWA, and responsiveness, it is fully implementable in rationalizable strategies.*

Remark: This result is the same as Proposition 2 of BMT (2011). Using the example proposed by Jain (2017, Appendix A), we can show that responsiveness cannot be dispensed with for Corollary 1 if we were to achieve rationalizable implementation by our canonical mechanism. We discuss this in the Appendix. However, it remains an open question whether one can dispense with responsiveness for rationalizable implementation. Of course, if this is possible, we need to devise a different mechanism than the mechanism proposed in Theorem 2.

Proof: This follows because MCI becomes a vacuous constraint and uniform monotonicity and SNWA reduce to Maskin monotonicity and NWA (the SCF-version used by BMT (2011)), respectively, as long as the social choice rule is single-valued. ■

We recall that BMT (2011) introduce their *best-response property*, and restrict attention to the mechanisms satisfying it when considering nonresponsive SCFs (See Definition 6 of BMT (p.1267)). It is easy to see that our canonical mechanism used in Theorem 2 satisfies BMT's best-response property.[18]

7 Concluding Remarks

By relying on a setwise condition requiring the nestedness of lower contour sets, a condition that we term uniform monotonicity, we have shown that rationalizable implementation of correspondences leads to a significantly more permissive theory than its counterpart using Nash equilibrium. This has been established for environments with at least three agents. The two-agent general sufficiency argument is likely handled by adding the usual requirement of nonempty intersections of lower contour sets; we chose instead to focus on a simple finite mechanism for a useful example. The extension to incomplete information environments should be our natural next step. Our conclusion is that, in the comparison with Nash equilibrium, dropping the rational expectations assumption while still retaining best-replies to beliefs, expands significantly the range of socially desirable rules that can be potentially decentralized. For a specific rule in a concrete environment, one should aim to construct a less abstract mechanism than our proposed canonical one, but we view our contribution as a way to draw the landscape of rules that could be in principle implemented.

Appendix

In this Appendix, we first provide the proof of Claim 3, which is part of Step 4 in the proof of Theorem 2. Second, we discuss ordinality issues. Third, we comment

[18] In their proposition 3, BMT show that strict Maskin monotonicity* (BMT (p.1265)) is a necessary condition for rationalizable implementation of SCFs by a mechanism satisfying their best response property. Maskin monotonicity*, together with the modified version of NWA, implies strict Maskin monotonicity*, which is itself logically stronger than strict Maskin monotonicity. See p.1269 of BMT for the details. When considering environments with monetary transfers, Chen *et al.* (2018) show that an SCF is rationalizably implementable by a finite mechanism if and only if it satisfies Maskin monotonicity*. That paper also obtains the following result: an SCF is Nash implementable by a finite mechanism if and only if it satisfies Maskin monotonicity. Hence, in some classes of economic environments, when we only deal with SCFs and finite mechanisms, Nash implementation yields results that are more permissive than rationalizable implementation. For the case of SCCs, the result for Nash implementation is extended. In particular, if an SCC is deterministic and the set of pure outcomes is finite, the SCC is Nash implementable by a finite mechanism if and only if it satisfies Maskin monotonicity. What is unknown is whether their rationalizable implementation result can be extended to SCCs.

on the role of finite mechanisms. Fourth, we outline how one can extend our results (Theorems 1 and 2) to the case of weak implementation. And fifth, we discuss the roles of responsiveness and MCI, used in Theorem 2 and Corollary 1.

A.1. Proof of Claim 3

We set $n+1 \equiv 1$ and $0 \equiv n$. We construct a message profile m^*, which induces Rule 1 with probability one, and in which all agents unanimously announce θ' in the first component of their message, and agent $i+1$ becomes the winner of the modulo game, as follows. First, for agent $i+1$, define $m^*_{i+1} = (\theta', m^{*2}_{i+1}, \bar{m}^3_{i+1}, \bar{m}^4_{i+1}, i+1, 1)$, where $m^{*2}_{i+1}[\theta'] = a^*$ and $m^{*2}_{i+1}[\tilde{\theta}] = \bar{m}^2_{i+1}[\tilde{\theta}]$ for every $\tilde{\theta} \neq \theta'$. Second, for each $j \in N\backslash\{i+1\}$, define $m^*_j = (\theta', m^{*2}_j, \bar{m}^3_j, \bar{m}^4_j, 1, 1)$ such that $m^{*2}_j[\tilde{\theta}] = \bar{m}^2_j[\tilde{\theta}]$ for every $\tilde{\theta} \neq \theta'$ and

$$m^{*2}_j[\theta'] = \begin{cases} a_{j-1}(\theta, \theta') & \text{if } j \neq 1 \\ a_n(\theta, \theta') & \text{if } j = 1, \end{cases}$$

where $a_{j-1}(\theta, \theta') \in \arg\max_{a \in F(\theta')} u_{j-1}(a; \theta)$, which denotes one of the maximizers of $u_{j-1}(\cdot; \theta)$ within all the outcomes in $F(\theta')$. What remains to show is that $m^* \in S^{\Gamma(\theta)}$. We proceed to do so.

First, we show that m^*_{i+1} is a best response to some belief. Define $\lambda^*_{i+1} \in \Delta^*(M_{-(i+1)})$ with support as follows:

$$\begin{aligned} m^1_j &= \theta'; \\ m^2_j[\theta'] &= a_{j-1}(\theta, \theta'); \\ m^5_j &= \begin{cases} 2 & \text{if } j = i+2, \\ 1 & \text{otherwise;} \end{cases} \\ m^6_j &= 1, \end{aligned}$$

for all $j \in N\backslash\{i+1\}$. Given λ^*_{i+1} and m^*_{i+1}, agent $(i+2)$ becomes the winner of the modulo game. Thus, it generates the best possible outcome for agent $(i+1)$ conditional on Rule 1. Since there exists $\bar{m}_{i+1} \in S^{\Gamma(\theta)}_{i+1}[\theta']$ and m^*_{i+1} differs from $\bar{m}_{i+1}$ only in the second and fifth components of the message, m^*_{i+1} is a best response to λ^*_{i+1}.

Second, we show that $\lambda^*_{i+1}(m_{-(i+1)}) > 0 \Rightarrow m_{-(i+1)} \in S^{\Gamma(\theta)}_{-(i+1)}[\theta']$, which means that each m_j in the support of λ^*_{i+1} is rationalizable. Define $\lambda^*_{i+2} \in \Delta^*(M_{-(i+2)})$

with support as follows:

$$\begin{aligned} m_k^1 &= \theta^{'}; \\ m_k^2[\theta^{'}] &= a_{k-1}(\theta, \theta^{'}); \\ m_k^5 &= \begin{cases} i+3 & \text{if } k = i+1, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq i+2$. Then, given λ_{i+2}^* and m_{i+2}^*, agent $(i+3)$ becomes the winner of the modulo game so that the best outcome for agent $(i+2)$ conditional on Rule 1 is realized. Assume $j \in N \backslash \{i+1, i+2\}$. Define $\lambda_j^* \in \Delta^*(M_{-j})$ with support as follows:

$$\begin{aligned} m_k^1 &= \theta^{'}; \\ m_k^2[\theta^{'}] &= \begin{cases} a_n(\theta, \theta^{'}) & \text{if } k = 1, \\ a_{k-1}(\theta, \theta^{'}) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = i+1, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Assume $j < n$. Then, given λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the best outcome for agent j is realized conditional on Rule 1. Assume, on the other hand, that $j = n$. Then, given λ_n^* and m_n^*, agent 1 becomes the winner of the modulo game so that the best outcome for agent n is realized conditional on Rule 1. We know that (i) by our hypothesis and Step 3, there exist $\bar{m}_j \in S_j^{\Gamma(\theta)}[\theta^{'}]$ together with the belief $\bar{\lambda}_j$ to which $\bar{m}_j$ is a best response and which induces Rule 1 with probability one; (ii) m_j^* differs from $\bar{m}_j$ only in the second and fifth components of the message; and (iii) m_j^* generates the best outcome for himself conditional on Rule 1 given the belief λ_j^*. Therefore, we have that the support of λ_{i+1}^* is rationalizable. That is, for each $j \neq i+1$, $\lambda_{i+1}^*(m_{-(i+1)}) > 0 \Rightarrow m_{-(i+1)} \in S_{-(i+1)}^{\Gamma(\theta)}[\theta^{'}]$. Thus, $m_{i+1}^* \in S_{i+1}^{\Gamma(\theta)}$.

Third, we show that, for each $j \neq i+1$, m_j^* can be made a best response to some belief. Fix $j \neq i+1$. Define $\lambda_j^* \in \Delta^*(M_{-j})$ with support as follows:

$$\begin{aligned} m_k^1 &= \theta^{'}; \\ m_k^2[\theta^{'}] &= \begin{cases} a_n(\theta, \theta^{'}) & \text{if } k = 1, \\ a_{k-1}(\theta, \theta^{'}) & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} j+1 & \text{if } k = i+1, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq j$. Given λ_j^* and m_j^*, agent $j+1$ becomes the winner of the modulo game so that the best outcome for agent j conditional on Rule 1 is realized. Since we know that there exists $\bar{m}_j \in S_j^{\Gamma(\theta)}[\theta']$ together with $\bar{\lambda}_j$ to which $\bar{m}_j$ is a best response and which induces Rule 1 with probability one and m_j^* differs from $\bar{m}_j$ only in the second and fifth components of the message, m_j^* is a best response to λ_j^*.

Fourth, we claim that for each $j \neq i+1$, the support of λ_j^* is rationalizable, i.e., $\lambda_j^*(m_{-j}) > 0 \Rightarrow m_{-j} \in S_{-j}^{\Gamma(\theta)}[\theta']$. Consider $\hat{m}_{i+1} = (\theta', \hat{m}_{i+1}^2, \bar{m}_{i+1}^3, \bar{m}_{i+1}^4, j+1, 1)$, as a generic element in the support of λ_j^*, where $\hat{m}_{i+1}^2[\theta'] = a_n(\theta, \theta')$ if $i+1 = 1$ (i.e., $i = n$) or $a_i(\theta, \theta')$ otherwise. Define $\hat{\lambda}_{i+1} \in \Delta^*(M_{-(i+1)})$ with support as follows:

$$\begin{aligned} m_k^1 &= \theta'; \\ m_k^2[\theta'] &= \begin{cases} a_n(\theta, \theta') & \text{if } k = 1, \\ a_{k-1}(\theta, \theta') & \text{otherwise;} \end{cases} \\ m_k^5 &= \begin{cases} n+2-j & \text{if } k = i+2, \\ 1 & \text{otherwise;} \end{cases} \\ m_k^6 &= 1, \end{aligned}$$

for all $k \neq i+1$. Given $\hat{m}_{i+1}$ and $\hat{\lambda}_{i+1}$, agent $(i+2)$ becomes the winner of the modulo game so that the best outcome for agent $(i+1)$ conditional on Rule 1 is realized. Fix $k \in N \backslash \{i+1, j\}$. Consider $\hat{m}_k = (\theta', \hat{m}_k^2, \bar{m}_k^3, \bar{m}_k^4, 1, 1)$, as a generic element in the support of λ_j^* where $\hat{m}_k^2[\theta'] = a_n(\theta, \theta')$ if $k = 1$ or $a_{k-1}(\theta, \theta')$ otherwise. Define $\hat{\lambda}_k \in \Delta^*(M_{-k})$ with support as follows:

$$\begin{aligned} m_h^1 &= \theta'; \\ m_h^2[\theta'] &= \begin{cases} a_n(\theta, \theta') & \text{if } h = 1, \\ a_{k-1}(\theta, \theta') & \text{otherwise;} \end{cases} \\ m_h^5 &= \begin{cases} k+1 & \text{if } h = i+1, \\ 1 & \text{otherwise;} \end{cases} \\ m_h^6 &= 1, \end{aligned}$$

for all $h \neq k$. Given $\hat{m}_k$ and $\hat{\lambda}_k$, agent $(k+1)$ becomes the winner of the modulo game so that the best outcome for agent k conditional on Rule 1 is realized. We know that: (i) by our hypothesis and Step 3, there exists $\bar{m}_k \in S_k^{\Gamma(\theta)}[\theta']$ together with the belief $\bar{\lambda}_k$ to which $\bar{m}_k$ is a best response and which induces Rule 1 with probability one; (ii) $\hat{m}_k$ differs from $\bar{m}_k$ only in the second and fifth components of the message; and (iii) $\hat{m}_k$ generates the best outcome for himself conditional on Rule 1 given the belief $\hat{\lambda}_k$. Therefore, for each $j \neq i+1$, we have that the support of λ_j^* is rationalizable. That is, for each $j \neq i+1$, $\lambda_j^*(m_{-j}) > 0 \Rightarrow m_{-j} \in S_{-j}^{\Gamma(\theta)}[\theta']$. This implies that $m_j^* \in S_j^{\Gamma(\theta)}[\theta']$ for each $j \neq i+1$.

In sum, we conclude that we have $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a^*$, as desired. ■

A.2. Ordinality

The theory of implementation often associates each state with a profile of *ordinal* preferences and does not introduce any cardinal representation. In this subsection, we identify a state θ with a profile of ordinal preferences $(\succeq_i^\theta)_{i\in N}$ over A. This is the ordinal approach considered in Mezzetti and Renou (2012) in the context of Nash implementation. BMT (2011) also discuss the ordinal approach to rationalizable implementation. Specifically, we refer the reader to Section 6 of Mezzetti and Renou (2012) and Section 5 of BMT (2011). We say that $u = (u_1, \ldots, u_n)$ is a cardinal representation of $(\succeq_i^\theta)_{i\in N,\theta\in\Theta}$ if, for each $a, a^{'} \in A, i \in N$, and $\theta \in \Theta$, we have $u_i(a;\theta) \geq u_i(a^{'};\theta) \Leftrightarrow a \succeq_i^\theta a^{'}$. A deterministic SCC F is *ordinally* fully implementable in rationalizable strategies if it is fully implementable in rationalizable strategies "independently of the cardinal representation."

We come to the next couple of definitions:

Definition 8 *A deterministic SCC F satisfies* ***ordinal*** *(weak) uniform monotonicity if it satisfies (weak) uniform monotonicity for any cardinal representation.*

With its weak version, we can show the following necessity result, whose proof is omitted:

Proposition 1 *If a deterministic SCC F is ordinaly fully implementable in rationalizable strategies, it satisfies ordinal weak uniform monotonicity.*

Here is our next definition:

Definition 9 *A deterministic SCC F satisfies* ***ordinal*** *SNWA if, for each $\theta \in \Theta$ and $i \in N$, there exists a pure alternative $z_i^\theta \in A$ such that $a \succ_i^\theta z_i^\theta$ for each $a \in F(\theta)$.*

We also propose an ordinal version of MCI.

Definition 10 *A deterministic SCC F satisfies the* ***ordinally minimal conflict-of-interests condition*** *(henceforth, ordinal MCI) if there do not exist $\theta, \theta^{'} \in \Theta$ and $a \in A$ such that $|F(\theta^{'})| \geq 2$ and $a \succeq_i^\theta b$ for all $b \in F(\theta^{'})$ for all $i \in N$.*

The proof of this sufficiency result is also omitted:

Proposition 2 *Suppose that there are at least three agents ($n \geq 3$). If a deterministic SCC F satisfies ordinal uniform monotonicity, ordinal MCI, ordinal SNWA, and responsiveness, it is ordinally fully implementable in rationalizable strategies.*

A.3. Finite Mechanisms

Recall that our sufficiency result for rationalizable implementation (Theorem 2) employs an *infinite* implementing mechanism. In this section, we elaborate on the role of finite implementing mechanisms. We show by example that there is a finite mechanism that achieves Nash implementation but fails rationalizable implementation. Of course, this does not necessarily imply that one cannot achieve rationalizable implementation by a finite mechanism. Rather, the point is that a Nash-implementing finite mechanism need not achieve rationalizable implementation.

There are two agents $N = \{1, 2\}$; two states $\Theta = \{\alpha, \beta\}$; and a finite number of pure outcomes $A = \{a_1, a_2, a_3\}$. Agent 1's strict preference relation over A is given as follows:

$$a_1 \succ_1^{\alpha} a_2 \succ_1^{\alpha} a_3 \text{ and } a_2 \succ_1^{\beta} a_1 \succ_1^{\beta} a_3.$$

Agent 2's strict preference relation over A is given as follows:

$$a_1 \succ_2^{\alpha} a_3 \succ_2^{\alpha} a_2 \text{ and } a_1 \succ_2^{\beta} a_3 \succ_2^{\beta} a_2.$$

Note that Agent 2 has state-independent preferences. We consider the following SCC: $F(\alpha) = \{a_1, a_3\}$ and $F(\beta) = \{a_3\}$. Note that F is responsive. We first claim that the SCC F satisfies Maskin monotonicity. When we move from β to α, we know that $F(\beta) \subseteq F(\alpha)$. So, we consider the case of moving from α to β. Fix $a_1 \in F(\alpha)$. Since there is a preference reversal around a_1 such that $a_1 \succ_1^{\alpha} a_2$ and $a_2 \succ_1^{\beta} a_1$, going from α to β is not a monotonic transformation of preferences around a_1. Thus, no conditions are imposed on F so that it satisfies Maskin monotonicity. This further implies that F satisfies uniform monotonicity as well.

We construct a finite mechanism that implements the SCC F in Nash equilibrium. Consider the following mechanism $\Gamma = (M, g)$ where $M_1 = \{m_1^1, m_1^2, m_1^3\}$; $M_2 = \{m_2^1, m_2^2\}$; and the deterministic outcome function $g(\cdot)$ is given in the table below:

$g(m)$		Agent 2	
		m_2^1	m_2^2
Agent 1	m_1^1	a_1	a_3
	m_1^2	a_2	a_3
	m_1^3	a_3	a_3

It can be easily shown that $(m_1^1, m_2^1), (m_1^2, m_2^2)$, and (m_1^3, m_2^2) are pure strategy Nash equilibria in the game $\Gamma(\alpha)$ and (m_1^2, m_2^2) and (m_1^3, m_2^2) are pure strategy Nash equilibria in the game $\Gamma(\beta)$. As long as we are concerned with pure strategy Nash equilibria, we can see that the mechanism Γ Nash implements the SCC F.

Suppose that there is a nontrivial mixed strategy Nash equilibrium in the game $\Gamma(\alpha)$. Assume that agent 2 chooses a pure strategy in the supposed mixed equilibrium. Then, agent 1 has an incentive to randomize over the messages only if agent 2 chooses m_2^2 where a_3 is the resulting outcome no matter what agent 1 does. Therefore, any such mixed strategy equilibrium outcome, if it exists, is consistent with the SCC. Now, let us consider a mixed strategy equilibrium where agent 2 uses a genuinely mixed strategy: choose m_2^1 with probability q and m_2^2 with probability $1-q$ where $q \in (0,1)$. In the game $\Gamma(\alpha)$, for any $q \in (0,1)$, m_1^2 cannot be a best response to agent 2's strategy $(q, 1-q)$. Hence, in such a mixed strategy equilibrium, m_1^2 is never played with positive probability. Then, any such mixed strategy equilibrium outcome, if it exists, only generates a_1 or a_3, which are consistent with the SCC F.

Suppose that there is a nontrivial mixed strategy equilibrium in the game $\Gamma(\beta)$. Assume that agent 2 chooses a pure strategy in the supposed mixed equilibrium. Then, agent 1 has an incentive to randomize over the messages only if agent 2 chooses m_2^2 where a_3 is the resulting outcome no matter what agent 1 does. Therefore, any such mixed strategy equilibrium outcome, if it exists, is consistent with the SCC. Now, let us consider a mixed strategy equilibrium where agent 2 uses a genuinely mixed strategy: choose m_2^1 with probability q and m_2^2 with probability $1-q$ where $q \in (0,1)$. In the game $\Gamma(\beta)$, for any $q \in (0,1)$, m_1^2 is a strict best response to agent 2's strategy $(q, 1-q)$. In other words, agent 2 should choose m_1^2 with probability one. However, if agent 1 chooses m_1^2 for sure, agent 2 is better off by switching to choosing a pure strategy m_2^2. Therefore, there is no such mixed strategy equilibrium. This implies that the SCC F is Nash implementable by the finite mechanism Γ even in the sense of Mezzetti and Renou (2012), who require every outcome in the support of Nash equilibria to be consistent with the SCC.

Next, we show that the mechanism Γ does not implement the SCC F in rationalizable strategies. Consider the game $\Gamma(\beta)$. First, m_1^2 is a best response to the belief that agent 2 chooses m_2^1. Second, m_2^1 is a best response to the belief that agent 1 chooses m_1^1. Third, m_1^1 is a best response to the belief that agent 2 chooses m_2^2. Therefore, both m_1^2 and m_2^1 survive the first round of deletion of never best responses. We can repeat this argument so that $m_1^2 \in S_1^{\Gamma(\beta)}$ and $m_2^1 \in S_2^{\Gamma(\beta)}$. Since both m_1^2 and m_2^1 are rationalizable messages, we obtain that $g(m_1^2, m_2^1) = a_2$, which is inconsistent with $F(\beta) = \{a_3\}$. Therefore, the SCC F is *not* rationalizably implementable by the mechanism Γ.

Of course, we do not know whether one could achieve rationalizable implementation using a different finite mechanism. Since this example does not satisfy MCI, we do not even know whether one could achieve rationalizable implementation by an *infinite* mechanism, although we conjecture that one can modify this example so that our Theorem 2 applies. Hence, what we have shown is that a

Nash-implementing finite mechanism need not achieve rationalizable implementation. On the other hand, the example in Section 5 is also based on a finite mechanism, and it re-enforces the paper's main message of rationalizable implementation being significantly more permissive than Nash implementation, even for a two-agent environment.

In closing, we recall that BMT (2011) introduce the *best-response property*, and restrict attention to the mechanisms satisfying it when considering nonresponsive SCFs. It is easy to see that our canonical mechanism used in Theorem 2 satisfies their best-response property. Of course, as long as we have a finite mechanism that achieves rationalizable implementation, that finite mechanism will satisfy the best response property as well. This is confirmed in the example we discussed in Section 5.

A.4. Weak Implementation in Rationalizable Strategies

Next, we provide the definition of *weak rationalizable implementation.*

Definition 11 (Weak Rationalizable Implementation) *An SCC F is **weakly implementable in rationalizable strategies** if there exists a mechanism $\Gamma = (M, g)$ such that for each $\theta \in \Theta$, the following two conditions hold: (1) $S^{\Gamma(\theta)} \neq \emptyset$; and (2) for each $m \in S^{\Gamma(\theta)}$, then $g(m) \in F(\theta)$.*

We begin by proposing a condition for weak implementation in rationalizable strategies:

Definition 12 *An SCC F satisfies **weak K-uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, there exists a nonempty set $K(\theta) \subseteq F(\theta)$ such that if*

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \quad \forall a \in co(K(\theta)),\ \forall i \in N,\ \forall z \in \Delta(A),$$

then, $K(\theta) \subseteq F(\theta')$.

Remark: When we consider SCFs, co$(K(\theta))$ becomes a singleton set. Therefore, in this case, the condition just defined also reduces to Maskin monotonicity.

We slightly strengthen weak K-uniform monotonicity into the following:

Definition 13 *An SCC F satisfies **K-uniform monotonicity** if, for every pair of states $\theta, \theta' \in \Theta$, there exists a nonempty set $K(\theta) \subseteq F(\theta)$ such that, if*

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \quad \forall a \in K(\theta),\ \forall i \in N,\ \forall z \in \Delta(A),$$

then, $K(\theta) \subseteq F(\theta')$.

The same comment we made after the definition of uniform monotonicity applies here. Therefore, under expected utility, weak K-uniform monotonicity is equivalent to K-uniform monotonicity. Note also that K-uniform monotonicity is logically weaker than uniform monotonicity, itself weaker than Maskin monotonicity. All three reduce to the same condition when one considers single-valued rules.

The necessity result for weak implementation follows:

Theorem 3 *If an SCC F is weakly implementable in rationalizable strategies, it satisfies weak K-uniform monotonicity.*

Proof: Suppose F is weakly implementable in rationalizable strategies by a mechanism $\Gamma = (M, g)$. Fix two states $\theta, \theta' \in \Theta$. Define

$$K(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\}.$$

Assume the following property:

$$u_i(a;\theta) \geq u_i(z;\theta) \Rightarrow u_i(a;\theta') \geq u_i(z;\theta') \ \forall a \in \text{co}(K(\theta)),\ \forall i \in N,\ \forall z \in \Delta(A) \ (*)$$

Then, due to the hypothesis that F is weakly implementable by Γ, we fix $m^* \in S^{\Gamma(\theta)}$, and we have that $g(m^*) \in K(\theta)$.

Fix $i \in N$. Since $m_i^* \in S_i^{\Gamma(\theta)}$, there exists $\lambda_i^{m_i^*,\theta} \in \Delta(M_{-i})$ satisfying the following two properties: (i) $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0 \Rightarrow m_{-i} \in S_{-i}^{\Gamma(\theta)}$ and $g(m_i^*, m_{-i}) \in F(\theta)$; and (ii) $\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i});\theta)$ for each $m_i' \in M_i$.

We focus on the best response property of m_i^* summarized by inequality (ii). Fix $m_i' \in M_i$. Due to the construction of $\lambda_i^{m_i^*,\theta}$, we have that

$$\begin{aligned} \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i});\theta) &\geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i});\theta) \\ u_i(a;\theta) &\geq u_i(z^a;\theta), \end{aligned}$$

where the two lotteries a and z^a are defined as

$$a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i^*, m_{-i}) \text{ and } z^a = \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) g(m_i', m_{-i}).$$

Since $g(m_i^*, m_{-i}) \in K(\theta)$ for each m_{-i} with $\lambda_i^{m_i^*,\theta}(m_{-i}) > 0$, we have $a \in \text{co}(K(\theta))$. Using Property $(*)$, we also obtain

$$u_i(a;\theta') \geq u_i(z^a;\theta').$$

Due to the choice of a and z^a and the hypothesis that $u_i(\cdot)$ is a von-Neumann-Morgenstern expected utility, we obtain the following:

$$\sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i^*, m_{-i}); \theta') \geq \sum_{m_{-i}} \lambda_i^{m_i^*,\theta}(m_{-i}) u_i(g(m_i', m_{-i}); \theta').$$

Since this argument does not depend upon the choice of m_i', this shows that m_i^* is a best response to $\lambda_i^{m_i^*,\theta}$ in state θ' as well. Therefore, $m_i^* \in S_i^{\Gamma(\theta')}$. Since the choice of agent i is arbitrary, we can conclude that $m^* \in S^{\Gamma(\theta')}$. Furthermore, since the choice of $m^* \in S^{\Gamma(\theta)}$ is also arbitrary, we have $S^{\Gamma(\theta)} \subseteq S^{\Gamma(\theta')}$. Finally, by weak implementability, this implies that

$$K(\theta) = \bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} \subseteq \bigcup_{m \in S^{\Gamma(\theta')}} \{g(m)\} \subseteq F(\theta').$$

The proof is thus complete. ■

We also propose a version of MCI, which we call K-minimal conflict-of-interests condition.

Definition 14 *An SCC F satisfies the K-**minimal conflict-of-interests condition** (henceforth, K-MCI) if there do not exist $\theta, \theta' \in \Theta$, a nonempty set $K(\theta') \subseteq F(\theta')$, and $a \in \Delta(A)$ such that $|K(\theta')| \geq 2$ and $a \in \arg\max_{b \in K(\theta')} u_i(b; \theta)$ for all $i \in N$.*

We also adapt responsiveness to weak implementation.

Definition 15 *An SCC F is K-**responsive** if, for each $\theta \in \Theta$, there exists a nonempty subset $K(\theta) \subseteq F(\theta)$ such that $\theta' \neq \theta''$ implies $K(\theta') \neq K(\theta'')$.*

Next, we state the general sufficiency result for weak implementation in rationalizable strategies.

Theorem 4 *Suppose that there are at least three agents ($n \geq 3$) and the set of pure outcomes A is finite. If a deterministic SCC F satisfies K-uniform monotonicity, K-MCI, SNWA, and K-responsiveness, it is **weakly** implementable in rationalizable strategies.*

Remark: In the sufficiency result below, we need the compactness of $K(\theta)$. To guarantee this, we assume A is finite and only consider deterministic SCCs. We view this as a technical requirement for the result.

Proof: By K-uniform monotonicity, for each $\theta \in \Theta$, we have a nonempty set $K(\theta) \subseteq F(\theta)$. Since the SCC F is deterministic and the set of pure outcomes A is finite, we have that $K(\theta)$ is a finite set for each $\theta \in \Theta$. We construct a mechanism $\Gamma = (M, g)$ such that each agent i sends a message $m_i = (m_i^1, m_i^2, m_i^3, m_i^4, m_i^5, m_i^6)$ where $m_i^1 \in \Theta, m_i^2 = \{m_i^2[\theta]\}_{\theta \in \Theta}$ where $m_i^2[\theta] \in K(\theta)$, $m_i^3 = \{(m_i^3[\theta, 1], m_i^3[\theta, 2])\}_{\theta \in \Theta}$ where $m_i^3[\theta, 1] \in \Delta(A)$ and $m_i^3[\theta, 2] \in K(\theta)$, $m_i^4 \in \Delta(A), m_i^5 \in N$, and $m_i^6 \in \mathbb{N}$. This mechanism is essentially the same as the one proposed in the proof of Theorem 2. The only change we have made from the message space proposed in the proof of Theorem 2 is that we use $K(\theta)$ in the spaces of $m_i^2[\theta]$ and $m_i^3[\theta, 2]$. The outcome function $g : M \to \Delta(A)$ is defined exactly as for the case of full implementation in the proof of Theorem 2. The proof consists of Steps I through IV, parallel to the steps for the full implementation proof.

Step I: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow m_i^6 = 1$.

Step II: For any $\theta \in \Theta$ and $a \in K(\theta)$, there exists $m^* \in S^{\Gamma(\theta)}$ such that $g(m^*) = a$.

Step III: $m_i \in S_i^{\Gamma(\theta)} \Rightarrow \lambda_i(m_{-i}) = 0$ for any profile (m_i, m_{-i}) under Rules 2 or 3, where $\lambda_i \in \Delta^*(M_{-i})$ represents the belief held by agent i to which m_i is a best response.

Step IV: $S^{\Gamma(\theta)}[\theta'] \subseteq K(\theta)$.

Claim III: If there exists $\bar{m} \in S^{\Gamma(\theta)}[\theta']$, for any $a^* \in K(\theta')$, there also exists $m^* \in S^{\Gamma(\theta)}[\theta']$ such that $a^* = g(m^*)$.

Now we shall conclude the proof of Theorem 2. By Step IV and the auxiliary claim (Claim III) in its proof, for all $\theta \neq \theta'$, we know that $K(\theta') \subseteq K(\theta)$. Simply by reversing the roles of θ and θ' in the proof, we would reach the opposite inclusion, which would imply that $K(\theta) = K(\theta')$, contradicting K-responsiveness.

Therefore, it must be the case that, for any $\theta, \theta' \in \Theta$, whenever $\theta \neq \theta'$, the set $S_i^{\Gamma(\theta)}[\theta']$ is empty for all $i \in N$. It then follows that for any $\theta \in \Theta$,

$$S^{\Gamma(\theta)} = \prod_{i \in N} S_i^{\Gamma(\theta)} = \prod_{i \in N} S_i^{\Gamma(\theta)}[\theta].$$

This together with Step II further implies

$$\bigcup_{m \in S^{\Gamma(\theta)}} \{g(m)\} = K(\theta) \subseteq F(\theta).$$

This concludes the proof of Theorem 4. ■

A.5. The Roles of Responsiveness and MCI

On responsiveness. We borrow this example from Appendix A of Jain (2017). The reader is referred to that paper for more details. Let $A = \{a, b, c, d\}$, $N = \{1, 2, 3\}$, and $\Theta = \{\theta, \theta', \theta'', \theta'''\}$. The preference profile for each state is given in the table below. Numbers in the parentheses specify the von-Neumann and Morgenstern utilities. Consider the following SCF $f(\theta) = f(\theta') = f(\theta'') = a$ and $f(\theta''') = b$. This SCF f clearly violates responsiveness.

θ			θ'			θ''			θ'''		
1	2	3	1	2	3	1	2	3	1	2	3
$a(3)$	$a(3)$	$a(3)$	$b(3)$	$c(3)$	$d(3)$	$c(3)$	$d(3)$	$b(3)$	$b(3)$	$c(3)$	$b(3)$
$b(2)$	$c(2)$	$b(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$	$a(2)$
$c(1)$	$b(1)$	$c(1)$	$c(1)$	$d(1)$	$c(1)$	$d(1)$	$c(1)$	$c(1)$	$c(1)$	$d(1)$	$c(1)$
$d(0)$	$d(0)$	$d(0)$	$d(0)$	$b(0)$	$b(0)$	$b(0)$	$b(0)$	$d(0)$	$d(0)$	$b(0)$	$d(0)$
$f(\theta) = a$			$f(\theta') = a$			$f(\theta'') = a$			$f(\theta) = b$		

Claim 4 *The SCF f satisfies strict Maskin monotonicity.*

Proof. We simply check all the possible pair of states with different social outcome.

- $(\theta, \theta''') : u_1(a, \theta) > u_1(b, \theta)$ but $u_1(a, \theta''') < u_1(b, \theta''')$.
- $(\theta', \theta''') : u_2(a, \theta') > u_2(c, \theta')$ but $u_2(a, \theta''') < u_2(c, \theta''')$.
- $(\theta'', \theta''') : u_1(a, \theta'') > u_1(b, \theta'')$ but $u_1(a, \theta''') < u_1(b, \theta''')$.
- $(\theta''', \theta) : u_1(b, \theta''') > u_1(a, \theta''')$ but $u_1(b, \theta) < u_1(a, \theta)$.
- $(\theta''', \theta') : u_2(b, \theta''') > u_2(d, \theta''')$ but $u_2(b, \theta') < u_2(d, \theta')$.
- $(\theta''', \theta'') : u_1(b, \theta''') > u_1(a, \theta''')$ but $u_1(b, \theta'') < u_1(a, \theta'')$. ■

Claim 5 *The SCF f satisfies NWA and MCI.*

Proof: This is trivial. ■

Thus far, in this example, we have three agents and confirm that the SCF f satisfies strict Maskin monotonicity, NWA, and MCI. But the SCF f is not responsive. Suppose by way of contradiction that even if the SCF f is not responsive, it is implementable in rationalizable strategies by the canonical mechanism used in

Theorem 2. Then, by Steps 1 through 3 of our Theorem 2, we have shown that one can partition the set of rationalizable message profiles into separate components. In each component, each agent i believes that all the others are using strategies of the form $(\theta^{'}, \cdot, \cdot, \cdot, \cdot, 1)$ with probability one. Since we focus on the case of SCFs, it does not matter how we specify $m_i^2, m_i^3[\theta, 2], m_i^4$, and m_i^5 in our mechanism.

In the following two claims, we characterize $S^{\Gamma(\theta)}$ in our mechanism.

Claim 6 $S^{\Gamma(\theta)}[\theta^{'}] \neq \emptyset$

Proof: By Step 2 of our Theorem 2, we have shown that there exists $m \in S^{\Gamma(\theta)}$ such that $m_i = (\theta, m_i^2, m_i^3, m_i^4, m_i^5, 1)$ for each $i \in N$. If we simply replace θ with $\theta^{'}$ and keep the rest the same as before, we propose a minimally modified message profile $m_i^{'} = (\theta^{'}, m_i^2, m_i^3, m_i^4, m_i^5, 1)$. Fix any $i \in N$ and any such $m_i^{'} \in S_i^{\Gamma(\theta)}$. Then, for any belief λ_i whose support is concentrated on $S_{-i}^{\Gamma(\theta)}[\theta^{'}]$, agent i obtains the best outcome a in state θ by playing $m_i^{'}$, which together with λ_i induces Rule 1 with probability one. The only deviation player i has is to trigger Rule 2 but by doing so, agent i's expected payoff becomes strictly lower than what he gets under $m_i^{'}$, no matter how large an integer in the 6th component of the message is chosen. This implies that $m_i^{'} \in S^{\Gamma(\theta)}[\theta^{'}]$. ■

Claim 7 $S^{\Gamma(\theta)}[\theta^{''}] \neq \emptyset$

Proof: By Step 2 of our Theorem 2, we have shown that there exists $m \in S^{\Gamma(\theta)}$ such that $m_i = (\theta, m_i^2, m_i^3, m_i^4, m_i^5, 1)$ for each $i \in N$. If we simply replace θ with $\theta^{''}$ and keep the rest the same as before, we propose a minimally modified message profile $m_i^{''} = (\theta^{''}, m_i^2, m_i^3, m_i^4, m_i^5, 1)$. Fix any $i \in N$ and any such $m_i^{''} \in S_i^{\Gamma(\theta)}$. Then, for any belief λ_i whose support is concentrated on $S_{-i}^{\Gamma(\theta)}[\theta^{''}]$, agent i obtains the best outcome a in state θ by playing $m_i^{''}$, which together with λ_i induces Rule 1 with probability one. The only deviation player i has is to trigger Rule 2 but by doing so, agent i's expected payoff becomes strictly lower than what he gets under $m_i^{''}$, no matter how large an integer in the 6th component of the message is chosen. This implies that $m_i^{''} \in S^{\Gamma(\theta)}[\theta^{''}]$. ■

In sum, we obtain the following structure of $S^{\Gamma(\theta)}$:

$$S^{\Gamma(\theta)} = S^{\Gamma(\theta)}[\theta] \cup S^{\Gamma(\theta)}[\theta^{'}] \cup S^{\Gamma(\theta)}[\theta^{''}]$$

For example, we take $m_1 = (\theta, m_1^2, m_1^3, m_1^4, m_1^5, 1)$; $m_2 = (\theta^{'}, m_1^2, m_1^3, m_1^4, m_1^5, 1)$; and $m_3 = (\theta^{''}, m_3^2, m_3^3, m_3^4, m_3^5, 1)$. Due to the structure of $S^{\Gamma(\theta)}$ above, we know that $m_1 \in S_1^{\Gamma(\theta)}$, $m_2 \in S_2^{\Gamma(\theta)}$, and $m_3 \in S_3^{\Gamma(\theta)}$. Therefore, we have $m = (m_1, m_2, m_3) \in S^{\Gamma(\theta)}$. Since m triggers Rule 3 in our mechanism, we claim $g(m) \neq f(\theta) = a$. This shows that our mechanism fails to achieve rationalizable implementation. Under

Rule 3, the lottery $\underline{z}$ always occurs with positive probability. Since the SCF f satisfies NWA and a is the common best outcome in state θ, we have that $\underline{z} \neq a$.

On MCI. Consider the following variant of the example in Section 5. We assume $K = 4$ and define $F(\alpha) = \{a_2, a_4\}$ and $F(\beta) = \{a_1\}$. We also add Agent 3, who has state-uniform preferences identical to Agent 1's. We note that: (i) there are three agents; (ii) the given SCC F satisfies uniform monotonicity, SNWA, and responsiveness; but (iii) no mechanism implements F in rationalizable strategies because the message profile whose outcome is a_4 is a Nash equilibrium in state β. It follows that we need an extra condition to establish the sufficiency result for rationalizable implementation, and MCI is such a condition. We could use Example 8.1 in Jain (2017) to make the same point.

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