Aggregate Dynamics in Lumpy Economies*

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Abstract

We develop a new framework to analyze the aggregate implications of lumpiness in microeconomic adjustment, which is pervasive in many economic environments. We derive structural relationships between the steady state moments and the business cycle dynamics of lumpy economies, and we show how to discipline these relationships using panel micro data. As an application, we study capital misallocation and investment dynamics by implementing our tools on establishment–level data from Chile and Colombia. Our framework is very flexible and can accommodate a large set of inaction models, stochastic processes, and higher order dynamics.

JEL: D30, D80, E20, E30

Keywords: inaction, transitional dynamics, adjustment costs, aggregate shocks, deviations from steady state, sufficient statistics, (S,s) models.

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1 Introduction

Lumpiness in microeconomic adjustment is pervasive in many economic environments. For instance, capital investment, inventory management, consumption of durable goods, price setting, portfolio adjustment, and many other economic decisions faced by firms and households are characterized by periods of inaction followed by bursts of activity. Recurrent questions that arise in these environments relate to the macroeconomics consequences of this micro lumpiness. How does lumpiness in microeconomic adjustment affect aggregate dynamics? After a policy change or an aggregate shock, how long do transitions last until the lumpy economy reaches its new long-run equilibrium? Understanding these issues is key for the design and implementation of policies aiming at stabilizing the business cycle or generating long-run growth.

Our work contributes by developing new theoretical tools to study transitional dynamics in lumpy economies. We consider environments with heterogeneous agents that make decisions subject to adjustment frictions. These frictions can take the form of fixed adjustment costs, random opportunities of adjustment, fixed dates of adjustments, among many other. Following Álvarez, Le Bihan and Lippi (2014), we measure transitional dynamics through the cumulative impulse-response function (CIR), which equals the cumulative deviations of a variable with respect to its steady state value during the transition. We contribute by formally characterizing three properties of the CIR that hold in any inaction model, whether it is state-dependent, time-dependent, or a hybrid of both.

The first property, aggregation, expresses transitional dynamics as the solution to a representative agent's recursive problem. The second property, representation, expresses the representative agent's problem as a function of steady state moments. The third property, observation, recovers the steady state moments and parameters using information about observable actions, such as the frequency and size of adjustments. Taking the three results in conjunction, our theory provides a tight link between observable actions (in micro panel data), steady state moments, and business cycle dynamics.

With our theoretical results at hand, we illustrate the potential of our framework with an application to capital misallocation and investment dynamics. For this purpose, we set-up a canonical model of lumpy investment à la Khan and Thomas (2008), Bachmann, Caballero and Engel (2013) and Winberry (2016), that features idiosyncratic productivity shocks, depreciation and productivity growth, asymmetric policies, and random capital adjustments costs. Defining the capital gap as the log difference between the current capital and its static optimum, the CIR of average capital depends on two cross-sectional steady state moments: the variance of capital gaps and the covariance between capital gaps and the time since their last adjustment. We show analytically how to measure these *unobservable* statistics using *observable* micro data on manufacturing plants in Chile and Colombia. A key contribution lies in demonstrating that certain micro statistics that were never computed in the data and ignored when calibrating models, such as the covariance above, are crucial determinants of aggregate dynamics. We show that the canonical investment model with adjustment costs misses these statistics. In this spirit, our tools can aid researchers in improving their models for the aggregate implications of inaction.

Let us now discuss in more detail the three properties of the CIR.

1. Aggregation. Transitional dynamics towards steady state are characterized as the solution to the recursive problem of a representative agent. By requiring certain degree of history independence in the processes and policies, we are able to characterize all ex-post heterogeneity due to idiosyncratic shocks through the problem of one agent. Intuitively, consider an environment in which upon adjustment, all agents adjust to bring their state to the same value. Then, agents become ex-ante identical at the moment of adjustment, and they will only differ ex-post due to the idiosyncratic shocks. Thus any ex-post heterogeneity due to different initial conditions or conditional dynamics can be summarized by one of these ex-ante agents. Importantly, our aggregation result *does not imply* that heterogeneity is irrelevant for aggregate dynamics; it says that all heterogeneity can be summarized in a compact way.

2. Representation. The solution to the representative agent's problem can be represented through a combination of steady state cross-sectional moments. The idea behind this result lies in that, in an approximation around the steady state, the ergodic moments encode information about agents' responsiveness to idiosyncratic shocks, and thus these moments inform us about the representative agent's policy. We show that in certain cases, the ergodic moments are not sufficient to fully characterize the CIR and additional information about micro-level elasticities is needed. In the case of the canonical lumpy investment model, this structural relation becomes a function of the two cross-sectional moments of the distribution discussed above.

3. Observation. Our previous results express transitional dynamics as a function of steady state moments. However, such moments, in many cases, may be hard to observe in the data. For instance, firms' markups are unobservable, since we only observe average cost and not marginal costs, and it is even harder to think about capital gaps, the distance between firms' capital and their optimal level. Then, how we can discipline steady state moments of these variables? Our third and most applicable result shows how to recover the steady state moments with observable actions, namely, adjustments and stopping times, both of which are very likely to be observable in micro datasets. The logic behind this mapping is that, by assuming a structure for the state's evolution during inaction and merging it with information revealed through agents' adjustments, we can back out ergodic moments and the process of shocks affecting them.

In our baseline analysis, we focus on the CIR of the first moment following a horizontal shift of the distribution, and assume that the idiosyncratic shocks are described by a random walk with drift. Then, we extend our results to study the transition of any moment of the distribution (first, second, etc.), for any initial condition away from the steady state (mean preserving spreads, etc.), and general stochastic processes (e.g. with mean-reversion). While the specific mappings between the micro data, the steady state moments and the aggregate dynamics change in each of these environments, the existence of a structural relationship among them continues to hold.

General equilibrium effects. Our analysis takes as a premise that the steady state policies hold along the transition path. This assumption is valid as long as the general equilibrium feedback from the aggregate distribution to the individual policies though prices is quantitatively insignificant. There are several general equilibrium frameworks in which this is the case.[1] When general equilibrium effects are quantitative relevant, the tools developed in this paper do not fully characterize aggregate dynamics. Nevertheless, there exist lumpy environments in which the partial equilibrium mechanisms continue to hold in general equilibrium.

A concrete example of this logic is found in the context of pricing literature with Calvo-type adjustments. In a model with negligible first order general equilibrium effects, Álvarez, Le Bihan and Lippi (2014) show analytically that the effectiveness of monetary policy is a function of the average duration of pricing spells, independent of any type of heterogeneity. Following this result, Blanco and Cravino (2018) reach a similar conclusion in a model with large general equilibrium effects (arising from real rigidities) in the context of real-exchange dynamics. Therefore, the role of heterogeneity and inaction in shaping aggregate dynamics is not altered by general equilibrium forces.

Related literature. Aggregate dynamics in inaction models has been widely studied. The groundbreaking work of Caplin and Spulber (1987), Caballero and Engel (1991) and Caplin and Leahy (1991) provided theoretical guidelines in stylized models to understand the role of micro lumpiness in shaping aggregate dynamics. With the surge of micro data, more realistic models that incorporated idiosyncratic shocks were developed, such as Cooper and Haltiwanger (2006), Golosov and Lucas (2007), Midrigan (2011), Berger and Vavra (2015), Carvalho and Schwartzman (2015) and Alvarez, Lippi and Paciello (2016), with the objective of understanding how the interaction of heterogeneity and lumpiness mattered for aggregate dynamics. We contribute by providing novel theoretical insights and an empirical strategy that exploits the micro data to its maximum while imposing a minimum structure to the inaction model.

Our paper is closely related to the work by Álvarez, Le Bihan and Lippi (2014), who consider a multi-product menu cost model with random opportunities to freely adjust and Brownian innovations to markup gaps. In that setup, they study the real effects of monetary shocks. In our view, one of the key results in their paper is that the cumulative impulse-response (CIR) for average markup gaps—a measure of the real effects of a money shock—equals the kurtosis of price changes times the average duration of prices divided by 6. They show this result analytically for the case of one product ($n = 1$) and infinite products ($n = \infty$), and more generally, they construct power series of each of the terms in the equality and confirm numerically that the relationship holds. Our contribution lies in proving a formal proof to their result in a more general framework using the structural relation between the CIR and the steady state moments. Moreover, our strategy allows to extend the results to richer environments.

Structure of the paper. Section 2 presents a standard model of lumpy investment that allows us to introduce the objects of interest. Section 3 develops the theory and explains the logic behind the aggregation, representation and observation properties of the CIR. Section 4 applies the theory using micro-level data. Section 5 generalizes and extends the results.

[1] For the effect of monetary shocks, see Woodford (2009), Golosov and Lucas (2007), and the vast literature that builds on them. For real exchange dynamics, see Carvalho and Nechio (2011) and Kehoe and Midrigan (2008). Regarding investment models, Bachmann, Caballero and Engel (2013) and Winberry (2016), building on Khan and Thomas (2008), show that partial equilibrium dynamics are not undone by general equilibrium effects whenever the model is calibrated to match the cyclical properties of aggregate investment or interest rates. Web Appendix B describes some of these frameworks.

2 Baseline Model: Lumpy Investment

This section describes the economic environment in which we apply the theory we develop. We build a partial equilibrium lumpy investment model in the spirit of Khan and Thomas (2008), Bachmann, Caballero and Engel (2013) and Winberry (2016), with a few simplifications that are discussed below.

2.1 Environment

Time is continuous and infinite. There is a representative household and a continuum of ex-ante identical firms. There is no aggregate uncertainty and firms face two types of idiosyncratic shocks: productivity shocks and a random adjustment cost to be paid with each capital adjustment. We denote with $\omega \in \Omega$ the full history of these two shocks and consider $(\Omega, P, \mathcal{F})$ to be a probability space equipped with the filtration $\mathcal{F} = \{\mathcal{F}_t : t \geq 0\}$. We use the notation $g_{\omega,t} : \Omega \times \mathbb{R} \to \mathbb{R}$ to denote an adapted process (a function $\mathcal{F}_t$-measurable for any $t \geq 0$) and $\mathbb{E}[g_{\omega,t}]$ to denote its expectation under P.

Firms. Firms operate in competitive markets. They produce output Y using capital K as the only input through a decreasing returns technology:

$$Y_{\omega,t} = E_{\omega,t}^{1-\alpha} K_{\omega,t}^{\alpha}, \tag{1}$$

where the log of idiosyncratic productivity E evolves according to a Brownian motion with drift μ and volatility σ, as follows:

$$d\log(E_{\omega,t}) = \mu dt + \sigma dW_{\omega,t}, \quad W_{\omega,t} \sim Wiener. \tag{2}$$

A firm chooses capital to maximize its expected stream of profits. For every capital adjustment, a firm pays a random fixed adjustment cost proportional to its productivity $\kappa_t E_{\omega,t}$, where κ_t is described by a compound Poisson process. In a period of length dt, the adjustment cost equals a constant $\kappa > 0$ with probability $1 - \lambda dt$, or it is given by random variable $\xi_{\omega,t}$ with probability λdt, where $\xi_{\omega,t}$ follows the distribution $H(\xi)$ with support $[0, \kappa]$. Letting $\tilde{N_{\omega,t}}$ describe a Poisson process with arrival rate λ, we write the adjustment cost $\kappa_{\omega,t}$ as

$$\kappa_{\omega,t} = \begin{cases} \kappa & \text{if } d\tilde{N}_{\omega,t} = 0 \\ \xi_{\omega,t} & \text{if } d\tilde{N}_{\omega,t} = 1. \end{cases} \tag{3}$$

Profits are discounted at the Arrow–Debreu time-zero price Q_t. Capital between adjustments depreciates at a constant rate ψ. With all the elements above, a firm's problem entering at time t consists in choosing a sequence of adjustment dates $(\tau_{\omega,i})$, and investment rates $(\Delta K_{\tau_{\omega,i}} = K_{\tau_{\omega,i}} - K_{\tau_{\omega,i}^-})$ that solve the following stopping-time problem:

$$\max_{\{\tau_{\omega,i}, \Delta K_{\tau_{\omega,i}}\}_{i=1}^{\infty}} \mathbb{E}\left[\int_0^{\infty} Q_s Y_{\omega,s}\, ds - \sum_{i=1}^{\infty} Q_{\tau_{\omega,i}} \left(\kappa_{\omega,\tau_{\omega,i}} E_{\omega,\tau_{\omega,i}} + \Delta K_{\tau_{\omega,i}}\right)\right], \tag{4}$$

where output, productivity, and fixed costs follow (1), (2), and (3), respectively, and capital follows

$$\log(K_{\omega,s}) = \log(K_{\omega,0}) - \psi(s-t) + \sum_{\tau_{\omega,i}\leq s} \Delta K_{\tau_{\omega,i}}. \tag{5}$$

Household. The household chooses the stochastic process for consumption to maximize its expected utility subject to a budget constraint. The household problem is given by

$$\int_0^\infty e^{-\rho t} C_t dt, \quad \text{subject to} \quad \int_0^\infty Q_t \left(C_t - \Pi_t\right) dt = 0. \tag{6}$$

where $\Pi_t \equiv \mathbb{E}[\pi_{\omega,t}]$ denotes aggregate firm's profits and C_t denotes household's consumption.

Aggregate feasibility. Aggregate output Y_t is used for household's consumption C_t and firms investments I_t, which includes capital adjustments adjustment costs[2]:

$$\underbrace{\mathbb{E}\left[E_{\omega,t}^{1-\alpha} K_{\omega,t}^{\alpha}\right]}_{Y_t} = C_t + \underbrace{\mathbb{E}\left[\mathbb{1}_{\{\tau,t\}}\left[\kappa_{\omega,t} E_{\omega,t} + \Delta K_{\omega,t}\right]\right]}_{I_t}. \tag{7}$$

Equilibrium. Given an initial distribution of $\{K_{\omega,0}, E_{\omega,0}\}$, an equilibrium is a set of stochastic processes for prices $\{Q_t\}$, household's policy $\{C_t\}$, and firms' policies $\{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ such that:

(i) Given $\{Q_t\}$, $\{C_t\}$ solve the household's problem (6).

(ii) Given $\{Q_t\}$, $\{\tau_{\omega,i}, \Delta K_{\omega,i}\}$ solve the firm's investment problem (4).

(iii) Goods market clears (7).

Price system dynamics. In principle, the price system may depend on two types of states variables: endogenous state variables, such as the distribution of capital holdings across firms, or exogenous state variables. The challenge under the first scenario, is that firms' values and policies would depend on the aggregate state (which in turn depends on their distribution) and thus steady state policies would not be adequate to characterize the transitional dynamics. To circumvent this issue in order to use steady state policies, we have specified a general equilibrium structure that generates a price system which is independent from the firm distribution. In this case, due to linearity of preferences, the only price in the economy is that of the Arrow security Q_t, which satisfies $Q_t = Q_0 e^{-\rho t}$ and is *independent* of the firm distribution. This assumption does not imply that the aggregate state is independent of the firm distribution, since for example, aggregate output does depend on the joint distribution of capital and productivity; but prices are.

Discussion of simplifying assumptions. Let us compare our environment with one of the benchmark models in this literature in Khan and Thomas (2008). First, in contrast to that paper, we do not consider labor as a factor of production. Given that we consider a partial equilibrium setting, and the labor decision is static in their model, this assumption is innocuous since adding labor would only affect

[2] Here $\mathbb{1}_{\{\tau,t\}} = \{\omega : \exists i \ s.t. \ \tau_{\omega,i} = t\}$ indicates the set of adjusters.

the value of the output-capital elasticity. Second, all the investments in our model, regardless if these fall within a small range, require the payment of the fixed adjustment cost (in the language of these authors, we do not consider *unconstrained* investments). This assumption is quantitatively irrelevant for transitional dynamics, as in the calibration, most investments are constrained anyways due to the large size of idiosyncratic shocks relative to aggregate shocks.

Lastly, we consider a random-walk process for idiosyncratic productivity instead of mean reversion. This assumption is considered to simplify the exposition at this stage and it is relaxed in Section 5. Moreover, this assumption is also motivated by empirical observation: considering mean-reverting shocks generates a negative autocorrelation in investment rates at the firm level that is not observed in the data. A random-walk process generates *iid* investment rates which are more aligned with the data.

2.2 Characterization of investment policy

Firms' investment policy. The firm's policy is described by three objects. An inaction region, denoted with $\mathcal{C} = (\underline{k}, \overline{k})$, such that the firm adjusts with probability 1 if $k_t \notin \mathcal{C}$. An adjustment hazard, denoted with $\Lambda(k)$, that describes investments within the inaction region $\mathcal{C}$ that happen with a sufficiently low realization of the fixed cost. A reset capital gap, denoted with $\hat{k}$, that describes the new capital gap after investment. Proposition 1 characterizes these three objects. For convenience, let us define the total drift $\nu \equiv -(\psi + \mu)$ and the adjusted discount $\tilde{\rho} \equiv \rho + \lambda - \mu - \frac{\sigma^2}{2}$. Recall that $H(\xi)$ denotes the *cdf* of the adjustment costs.

Proposition 1. *Let $V(K, E)$ be the value of a firm with initial capital K and initial productivity E:*

$$V(K, E) = \mathbb{E}\left[\int_0^\tau e^{-\rho t} E_t^{1-\alpha} K_t^\alpha dt + e^{-\rho\tau}\left(\kappa_\tau E_\tau + \max_{K^*} V(K^*, E_\tau) - (K^* - K_\tau)\right)\right]. \tag{8}$$

Then we can reexpress the value as $V(K, E) = Ev\left(log\left(\frac{K}{E}\right)\right)$, where $v(k)$, the continuation region $\mathcal{C}$ and the reset capital $\hat{k}$ satisfy the Hamilton-Jacobi-Bellman equation

$$\tilde{\rho}v(k) = e^{\alpha k} + \nu v'(k) + \frac{\sigma^2}{2}v''(k) + \lambda \int \max\left\{v(\hat{k}) - \xi - (\hat{k} - k), v(k)\right\} dH(\xi) \quad \forall k \in \mathcal{C}, \tag{9}$$

together with the value matching conditions

$$v(\underline{k}) - e^{\underline{k}} = v(\overline{k}) - e^{\overline{k}} = v(\hat{k}) - e^{\hat{k}} - \kappa, \tag{10}$$

and the optimality for the reset capital and smooth pasting conditions

$$v'(z) = e^z \quad for \quad z \in \{\underline{k}\ \overline{k},\ \hat{k}\}. \tag{11}$$

The adjustment dates $\tau_{\omega,i}$ (when capital is reset to $k_{\tau_{\omega,i}} = \hat{k}$) are given by

$$\tau_{\omega,i+1} = \inf\left\{t \geq \tau_{\omega,i} : k_{\omega,t} \notin \mathcal{C}\ or\ N^k_{\omega,t} - N^k_{\omega,\tau_{\omega,i}} \geq 1\right\}, \qquad \tau_{\omega,0} = 0. \tag{12}$$

where $N^k_{\omega,t}$ is a Poisson process with arrival rate $\Lambda(k) = \lambda H(v(\hat{k}) - v(k) - (\hat{k} - k))$.

Capital gaps and aggregate variables. Given the firms investment policy, we are interested in characterizing the log deviations of aggregate capital from its steady state. To this end, we define three variables. First, we define the capital gap $k_{\omega,t} \equiv \log(K_{\omega,t}/E_{\omega,t})$ as the log of the ratio of a firm's capital to its productivity. Second, we define $k_{ss} \equiv \mathbb{E}[\log(K_\omega/E_\omega)]$ to be the average of capital gaps in the steady state.[3] Lastly, we define the *normalized capital gap* $x_{\omega,t} \equiv k_{\omega,t} - k_{ss}$ as a firm's capital gap minus the steady state average. Redefining the state in this way is convenient to characterize the investment policy, and moreover, it has useful properties to think about aggregate objects.[4]

With these definitions, we compute the aggregate capital log deviation from steady state, denoted with $\hat{K}_t$, which up to a first order approximation, it is equal to the average normalized capital gap $\mathbb{E}[x_{\omega,t}]$:

$$\hat{K}_t \equiv \mathbb{E}[\log(K_{\omega,t})] - \mathbb{E}[\log(K_\omega)] = \mathbb{E}[k_{\omega,t}] - k_{ss} = \mathbb{E}[x_{\omega,t}], \tag{13}$$

where we use the assumption that productivity distribution is in the steady state. Notice that, in this normalization, we first centralize the capital-gap distribution around its steady state average and then we aggregate across firms. By the previous analysis, we may shift the focus from aggregate capital to moments of the normalized capital gaps. Finally, note that the dynamics of other aggregate variables, such as output deviations from steady state $\hat{Y}_t$, can also be expressed in terms of moments of normalized capital gaps:

$$\hat{Y}_t \equiv \mathbb{E}[\log(Y_{\omega,t})] - \mathbb{E}[\log(Y_\omega)] = \alpha\hat{K}_t = \alpha\mathbb{E}[x_{\omega,t}] \tag{14}$$

Law of motion of capital gaps. To derive the law of motion of capital gaps, we use the firm policy and its adjustment hazard from Proposition 1. Given the investment policy, we normalize the state to consider deviations from steady state: $(\underline{x}, \hat{x}, \overline{x}) \equiv (\underline{k} - k_{ss}, \hat{k} - k_{ss}, \overline{k} - k_{ss})$. The *uncontrolled* capital gaps—not considering any investments—follow the process

$$d\tilde{x}_{\omega,t} = \nu dt + \sigma dW_{\omega,t}, \tag{15}$$

where we use tildes to show explicitly that these variables evolve exogenously. The initial conditions $\tilde{x}_{\omega,0}$ are exogenously given. By the discussion above, the initial condition of the uncontrolled capital gap is by $\tilde{x}_{\omega,0} = k_{\omega,0} - k_{ss}$.

In contrast, the *controlled* capital gaps—taking into account investments—evolves as

$$x_{\omega,t} = \tilde{x}_{\omega,t} + \sum_{\tau_{\omega,i} \le t} \Delta x_{\tau_{\omega,i}}, \tag{16}$$

where the adjustment dates are defined in (12) and the investment rates $\Delta x_{\tau_{\omega,i}}$ are defined implicitly by

[3] The notation without time index t refers to moments computed with the steady state distribution.

[4] Note that we are able to transform the state from capital and productivity (K, E) to capital gaps x due to the homothetic production function and the shape of the fixed cost. For all the proofs in this example see Web Appendix C.

the difference in the capital stock between date $\tau_{\omega,i}$ and immediately before adjustment $\tau_{\omega,i}^-$:

$$\Delta x_{\tau_{\omega,i}} = \hat{x} - x_{\tau_{\omega,i}^-}. \tag{17}$$

2.3 Steady state and transitional dynamics

Steady state moments. Consider the steady state distribution of the controlled state $F(x)$. Define $G(a|x)$ the distribution of the time since x's last adjustment, which we refer to it as "age". For any numbers $k, l \in \mathbb{N}$, we define the ergodic cross-sectional moment of capital gaps and age as

$$\mathcal{M}_{k,l}[x,a] \equiv \int_x \int_a x^k a^l \; dG(a|x) \; dF(x) \quad \forall k, l \in \mathbb{N}, \quad \textit{with} \quad \mathcal{M}_{1,0}[x,a] = 0. \tag{18}$$

We use the notation $\mathcal{M}_k[x] \equiv \mathcal{M}_{k,0}[x,a]$ and $\mathcal{M}_l[a] \equiv \mathcal{M}_{0,l}[x,a]$.

Transitional dynamics. Fix an initial distribution of the state $F_0(x) = \mathbb{E}\left[\mathbb{1}_{\{x_{\omega,0} \le x\}}\right]$. We define the impulse-response function for the $m-th$ moment of the capital gap distribution under the initial distribution F_0, denoted by $IRF_{m,t}(F_0)$, as the difference between its time t value and its ergodic value:

$$IRF_{m,t}(F_0, t) \equiv \underbrace{\mathbb{E}\left[x_{\omega,t}^m\right]}_{\text{transition}} - \underbrace{\mathcal{M}_m[x]}_{\text{steady state}}. \tag{19}$$

Following Álvarez, Le Bihan and Lippi (2014), we define the cumulative impulse-response (CIR), denoted by $\mathcal{A}_m(F_0)$, as the area under the $IRF_{m,t}(F_0)$ curve across all dates $t \in (0, \infty)$:

$$\text{(CIR)} \qquad \mathcal{A}_m(F_0) \equiv \int_0^\infty IRF_{m,t}(F_0, t) \; dt. \tag{20}$$

Figure I illustrates these two objects. In the left panel, we plot the initial distribution F_0 and the steady state distribution F, and also highlight the m-th moment of capital gaps $\mathbb{E}[x_0^m]$ which will be tracked in its way towards steady state. In the right panel, the solid line represents the impulse-response of $\mathbb{E}[x_t^m]$, which is a function of time, and the area underneath it is the cumulative impulse-response or CIR.

The CIR is our key measure of the convergence speed towards the steady state. The smaller is the CIR, the faster the convergence. The following Lemma expresses the CIR in a recursive way, and it is a generalization of the result in Álvarez, Le Bihan and Lippi (2014). Although at this stage we are working in a particular example, the result holds true for any moment of the distribution $m > 1$, for an arbitrary Markovian stopping policy, and for any Markovian law of motion of the uncontrolled state.

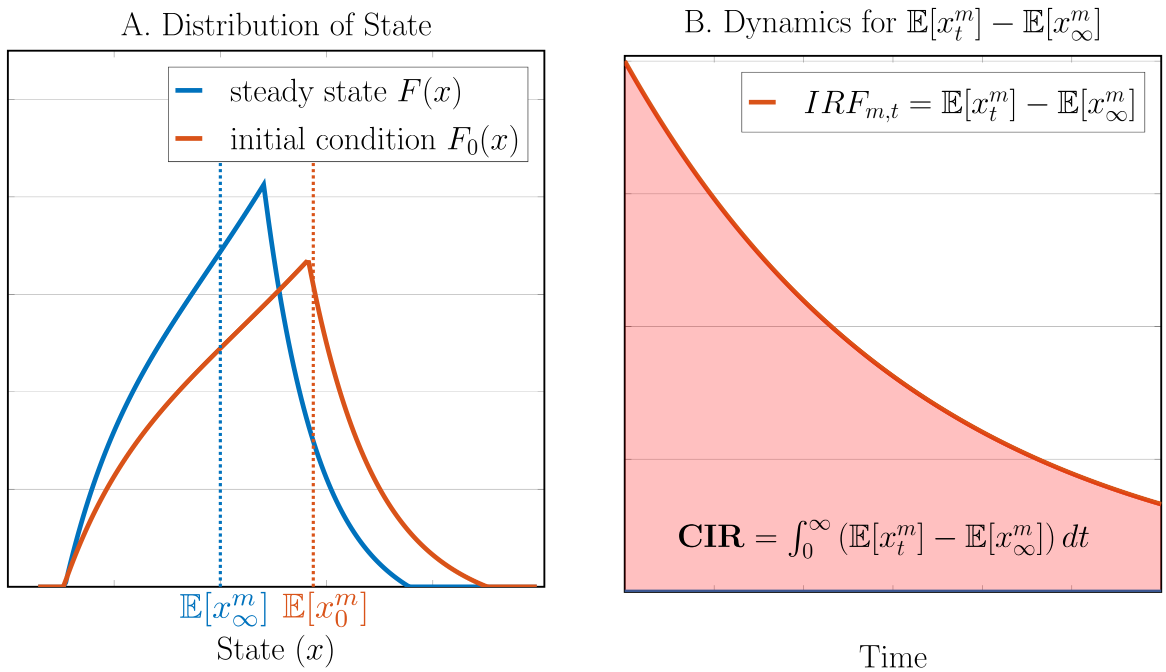
Lemma 1. *The CIR can be written recursively as:*

$$\mathcal{A}_m(F_0) = \int v_m(x) dF_0(x). \tag{21}$$

where the value function for an agent with initial state x is given by:

$$v_m(x) \equiv \mathbb{E}^x \left[\int_0^\tau \left(x_t^m - \mathcal{M}_m[x] \right) dt \right] \tag{22}$$

Figure I – Cumulative Impulse-Response (CIR)



The idea behind Lemma 1 is to exchange the integral across agents (the cross-section) with the infinite time integral (the time-series).[5] Then, it is key to recognize that the first time a firm adjusts its capital it incorporates the deviations into its policy, and thus we only need to keep track until its *first* adjustment; any additional adjustment is driven by idiosyncratic conditions. The average of these additional adjustments equals the ergodic moment $\mathcal{M}_m[x]$, implying that the value function $v_m(x)$ is equal to zero after the first adjustment. For that reason, the infinite time integral gets substituted for an integral between $t = 0$ and the stopping-time $t = \tau$.

3 The Three Properties in a Baseline Environment

This section derives the three properties of the CIR function. The first result—aggregation—approximates the transitional dynamics, measured via the CIR, as the stopping time problem of a representative firm that captures adjustments through the intensive and the extensive margins. The second result—representation—expresses the intensive and extensive margins in terms of moments of the state's ergodic distribution. Finally, the third result—observation—connects the state's ergodic moments with the distribution of policies Δx and τ, which are observable statistics in most micro-data sets.

Our baseline environment focuses on the inaction model of investment presented in Section 2. Here, we study transitional dynamics for average capital gaps, i.e. $m = 1$, when the initial condition consists of a mean translation of the steady state distribution. In Section 5, we extend the results for an arbitrary state space, arbitrary policies, and other additional features.

[5]This can be done due to the ergodic properties of the problem and the fact that moments are finite.

Initial conditions as δ-perturbations around steady state. For ease of exposition, we interpret the initial condition as a perturbation of the steady state distribution, that can be described in terms of one parameter δ. In particular, in the baseline case analyzed here, we consider a perturbation that translates horizontally the distribution of capital gaps. If $f(x-\delta)$ denotes the new density of capital gaps, and we approximate it as $f(x-\delta) \approx f(x) - \delta f'(x)$, we observe that it is equal to a right shift of the steady state density by $\delta f'(x)$. Afterwards, the distribution will evolve according to the agents' policies and will converge back to its steady state. Under this interpretation, we denote $\mathcal{A}_m(F_0)$ with $\mathcal{A}_m(\delta)$.

3.1 Aggregation

Starting from its recursive representation, Proposition 2 computes the cumulative CIR as the sum of intensive margin Γ and extensive margin Θ components, defined below.

Proposition 2. *To a first order, the transitional dynamics towards the steady state, measured through the CIR, are given by*

$$\mathcal{A}_1(\delta) = \delta \times \left(\Gamma_1 + \Theta_1 - \frac{\sigma^2}{2\nu}\Theta_0 \right) + o(\delta^2) \tag{23}$$

where the intensive margin component Γ_1 and the extensive margin components Θ_1 and Θ_0 are:

$$\Gamma_1 \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_1^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \textit{with} \quad \varphi_1^\Gamma(x) \equiv \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau\right] - x\right) \tag{24}$$

$$\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad \textit{with} \quad \varphi_m^\Theta(x) \equiv \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right) \tag{25}$$

Equation (23) measures the total effect of the δ-perturbation as an area, with a height of δ, and a base given by two components: Γ_1, which measure of adjustments through the intensive margin, and Θ_1, Θ_0, that measure adjustments through the extensive margin. In turn, each component is expressed through two nested HJB equations of a representative firm in (24) and (25), what is sometimes known as recursively squared. The inner HJBs denoted by $\varphi_1^\Gamma(x)$, and $\varphi_m^\Theta(x)$ track conditional dynamics for any initial condition x. They captures the evolution of the state from the initial condition towards the new steady state. The outer HJBs measures the mass of firms at each initial condition by computing the occupancy measure or local time spent at such state between any two adjustments. Together, the nested recursive problems capture the total deviations between the initial condition and the final destination, i.e., the steady state.[6]

The result uses four steps. The first step consists in a first order Taylor approximation of $\mathcal{A}_1(\delta)$ together with integration by parts that delivers

$$\mathcal{A}_1(\delta) \;=\; \int_{\underline{x}}^{\overline{x}} v(x) f(x-\delta) dx \;\approx\; \delta \times \int_{\underline{x}}^{\overline{x}} v'(x) f(x) dx. \tag{26}$$

where we have used that there is no mass at the boundary of the inaction region (or the regularity of the boundary of the cotinuation region).

[6]Convergence to the steady state is not needed for the aggregation result. For example, the property goes through in an Ss model without idiosyncratic shocks that features cycles, as in Caplin and Spulber (1987).

The second step consists in decomposing the base into an intensive and an extensive margin. Since $\mathcal{M}_1[x] = 0$ by the normalization, we have that $v(x) = \mathbb{E}^x\left[\int_0^\tau x_t dt\right]$ and its derivative is $v'(x) = d\mathbb{E}^x\left[\int_0^\tau x_t dt\right]/dx$. Substitute this derivative into (26); then add and subtract the expectation of the derivative of the state with respect to the initial conditions, $\mathbb{E}^x\left[\int_0^\tau \frac{dx_t}{dx} dt\right]$, which equals $\mathbb{E}^x[\tau]$ to obtain[7]:

$$\mathcal{A}_1(\delta) \approx \delta \underbrace{\int_{\underline{x}}^{\overline{x}} \mathbb{E}^x[\tau] f(x)dx}_{\Gamma_1 \,=\, \text{intensive margin}} + \delta \underbrace{\int_{\underline{x}}^{\overline{x}} \left(\frac{d\mathbb{E}^x\left[\int_0^\tau x_t dt\right]}{dx} - \mathbb{E}^x[\tau] \right) f(x)dx}_{\Theta_1 - \frac{\sigma^2}{2\nu}\Theta_0 \,=\, \text{extensive margin}}. \tag{27}$$

The first component measures aggregate adjustments through the intensive margin: changes in the path x_t due to the new initial condition x *keeping the duration fixed*. The second component measures the aggregate adjustments through the extensive margin: changes in duration due to the new initial condition *keeping the state's path fixed*. To see this clearly, suppose the state evolves deterministically, then the second term becomes $x_t \cdot d\tau/dx$.

The third step consists in finding an equivalent recursive representation for the conditional dynamics inside the integrals in (27). Let us show the steps for the intensive margin. Using the law of motion of x_t, we have that $dx_t = \nu dt + \sigma dW_t$. Then, integrating from 0 to τ and taking expectations with initial condition x, we have that $\mathbb{E}^x[x_\tau] = x + \nu\mathbb{E}^x[\tau] + \sigma\mathbb{E}^x\left[\int_0^\tau dW_t\right]$. By the Optional Sampling Theorem, the last term is equal to zero because it is a martingale with zero initial condition. Therefore, we can express $\mathbb{E}^x[\tau] = \frac{\mathbb{E}^x[x_\tau]-x}{\nu}$. With similar steps, we find the expression for the extensive margin.

In the final step, we focus on the unconditional dynamics, that is, the measure used to integrate the conditional dynamics outlined above. For this purpose, we use an alternative representation of the steady state distribution as the occupancy measure.[8] Intuitively, the mass of agent in x—given by $f(x)dx$—is equal to the amount of time spent at x. Formally, we have the following equivalence: $F(x) \equiv \Pr[x_t \leq x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau \mathbb{1}_{\{x_t \leq x\}}\, dt\right]/\mathbb{E}^{\hat{x}}[\tau]$. Thus, we rewrite $\frac{\mathcal{A}_1(\delta)}{\delta}$ in (27) as

$$\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\left(\frac{\mathbb{E}^x[x_\tau]-x}{\nu}\right)}^{=\varphi_1^\Gamma(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Gamma_1} + \underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\frac{\partial\mathbb{E}^x\left[x_\tau^2/2\right]/\partial x - \mathbb{E}^x\left[x_\tau\right]}{\nu}}^{=\varphi_1^\Theta(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_1} - \frac{\sigma^2}{2\nu} \underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \overbrace{\frac{\partial\mathbb{E}^x\left[x_\tau\right]/\partial x - 1}{\nu}}^{=\varphi_0^\Theta(x_t)} dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\Theta_0} \tag{28}$$

Usefulness of aggregation. We have shown that transitional dynamics (measured through the CIR) can be written a the solution of a recursive problem for a single agent. The intuition lies in the fact that after adjustment, firms become identical, and then any differences in their dynamics arise due to differences in their initial conditions at the moment of the perturbation. All the information regarding these ex-post differences is neatly summarized in the representative agent problem.

The usefulness of the aggregation result in Proposition 2 is twofold. Within the scope of this paper, it allows us to derive a connection between the intensive and extensive margins and the steady state

[7] $\mathbb{E}^x\left[\int_0^\tau \frac{dx_t}{dx} dt\right] = \mathbb{E}^x\left[\int_0^\tau \frac{d(x+\nu t+\sigma W_t)}{dx} dt\right] = \mathbb{E}^x\left[\int_0^\tau 1 dt\right] = \mathbb{E}^x[\tau]$.

[8] See Stokey (2009) for details.

moments of the distribution as we show next. More generally, its usefulness lies in aiding researchers with an aggregation result in models with inaction in which the cross-sectional distribution is part of the state.

3.2 Representation

With a recursive expression for the transitional dynamics at hand, the second step consists in expressing the three components of the CIR, Γ_1 and Θ_1 and Θ_0, as a function of the steady state cross-sectional moments. We derive a mapping that does not depend on a particular inaction model, therefore different models will produce different aggregate effects if and only if they change the ergodic moments.

Characterization of intensive margin Γ_1. The first component of the CIR, Γ_1, measures the aggregate effects keeping any changes in aggregate duration fixed. Proposition 3 shows that the intensive margin equals the capital's average age across firms.

Proposition 3. *The intensive margin component is given by:*

$$\Gamma_1 = \mathcal{M}_1[a] \tag{29}$$

Technically, the proof consists on finding an equivalence between the average age $\mathcal{M}_1[a]$ and Γ_1 by using the occupancy measure, Itō's lemma, and the law of iterated expectations. First, by the equivalence between the ergodic distribution and the occupancy measure, we have that

$$\Gamma_1 = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^{\tau} (x_\tau - x_t)\, dt\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathbb{E}^{\hat{x}}\left[\int_0^{\tau} x_t\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]}, \tag{30}$$

where we have used that $\frac{\mathbb{E}^{\hat{x}}\left[\int_0^{\tau} x_t\right]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathcal{M}_1[x]}{\nu} = 0$ by the normalization. Lastly, following similar steps as before (using Itō's Lemma and the Optional Sampling Theorem), it is easy to show that $\frac{\mathbb{E}^{\hat{x}}[x_\tau \tau]}{\nu \mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathcal{M}_1[x]}{\nu} + \mathcal{M}_1[a] = 0 + \mathcal{M}_1[a]$. Substituting this equivalence above, we have the result.

How do we understand the connection between the intensive margin and the capital's average age? Average age provides information about the speed at which the average firm adjusts to the perturbation from the steady state. The older is average capital in the economy, the longer the transition. Consider a frictionless limit in which all firms continuously invest to brings capital gaps to zero. Since capital in all firms would have age equal to zero, the economy reaches its steady state immediately. The reason is that any deviation from steady state is immediately absorbed into the representative firm's policy and there are no persistent deviations from steady state.

Characterization of extensive margin Θ_1 and Θ_0. Now, it is the turn to characterize the evolution of the extensive margin Θ_m in terms of ergodic cross-sectional moments. There are two main challenges. First, the extensive margin does not only depend on the *immediate* response of the aggregate adjustment frequency, but it also reflects all current and *future changes* in frequency. Second, even if we had the whole sequence of adjustment frequency that follows a perturbation, the extensive margin also depends on the capital-gaps of the particular set of firms selected to invest. This is clearly seen again in the deterministic case, where the extensive margin becomes $x_t \cdot d\tau/dx_0$, thus the change in frequency is scaled by the state, difficulting its characterization. Next, we develop a theory to discipline these two objects.

Proposition 4 presents a characterization of the extensive margin in terms of two objects: how investment responds to idiosyncratic initial conditions (a micro-elasticity) and the aggregate moments of the distribution. Together, these objects imply a macro-elasticity of the extensive margin with respect to the perturbation. Additionally, as known in the literature, we show that in time-dependent models, the extensive margin does not play a role.

Proposition 4. *Let $g_m(x)$ be a smooth function such that for all m*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m] - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right], \tag{31}$$

and define the micro-elasticities as $\theta_{m,j} \equiv \frac{1}{\nu}\sum_{k\geq j}^{\infty} \frac{\hat{x}^{k-j}}{k!j!}\left[\frac{d^{k+1}g_{m+1}(0)/m+1}{dx^{k+1}} - \frac{d^k g_m(0)}{dx^k}\right]$. Then, the extensive margin is given by

$$\Theta_m = \sum_{j=0}^{\infty} \theta_{m,j}\mathcal{M}_j[x]. \tag{32}$$

Moreover, if τ is independent of x, then $g_m(x) = \theta_{m,j} = 0$ for all m and $\Theta_m = 0$ as well.

First, let us describe each object in equation (31). Recall that the expected capital gap at the moment of adjustment is equal to $x_\tau = \hat{x} - \Delta x$. Now, the first term, given by $\mathbb{E}^{\hat{x}+x}[(\hat{x} - \Delta x)^m]$, equals the expected capital gap at the moment of adjustment when the initial condition is $\hat{x} + x$; while the second term, given by $\mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right]$, equals the expected capital gap at the moment of adjustment plus a deterministic increase of size x when the initial condition is $\hat{x}$. The difference between these two functions of x provides information of how the *stopping time* depends on the initial condition and how it correlates with the state. To see this more clearly, notice that we can re-express $g_m(x)$ in the following way

$$g_m(x) = \mathbb{E}\left[(\hat{x} + x - \nu\tau^{\hat{x}+x} - \sigma W_{\tau^{\hat{x}+x}})^m\right] - \mathbb{E}\left[(\hat{x} + x - \nu\tau^{\hat{x}} - \sigma W_{\tau^{\hat{x}}})^m\right], \tag{33}$$

where τ^z is the stopping time with initial condition z. In equation (33) we observe that if τ is independent of the initial condition, i.e. $\tau^{\hat{x}+x} = \tau^{\hat{x}}$, as in time-dependent models, then $g_m(x) = 0$ for all x. Thus, $g_m(x)$ provides a micro-elasticity of firms idiosyncratic response to the new initial conditions though changes in its stopping-time τ. To construct the aggregate elasticity, we aggregate the micro-elasticities with weights equal to the capital-gap's ergodic moments, which encode information about the state's distribution.

Constructing $g_m(x)$ from the data. To construct the micro-elasticities, we need to ask: what is observable in the data and what is not? The object $\mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right]$ is an observable statistic as it depends on the steady state investment rates.[9] The objects $\mathcal{M}_j[x]$ and $\hat{x}$ can also be recovered from the

[9] If idiosyncratic volatility is large enough with respect to aggregate volatility, then steady state micro-statistics can be recovered with average statistics in a model with business cycles. See Blanco (2015) for a verification of this statement in

data as we show in the next section. Therefore, the only object that might not be directly observable is $\mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right]$, which measures the elasticity of investment with respect to changes in initial conditions. Guided by the theory, we suggest that this elasticity is the key object that future research should focus on computing, both in the data and in the models.

There are two papers that use an adequate methodology and data to construct the micro-elasticities. First, in the pricing literature, Karadi and Reiff (2014) study the immediate monthly price response to a change in the VAT with Hungarian CPI. Since a change in the VAT proxies a cost-push shock, the experiment is equivalent to an increase in firm's markup in the same proportion; this design would allow to compute the micro-elasticity of the expected price change to initial conditions. Second, in the investment literature, Zwick and Mahon (2017) exploit shifts in accelerated depreciation to estimate the effect of temporary tax incentives on equipment investment; such a design would allow to compute the micro-elasticity of expected investment to initial conditions.

The role of micro-elasticities: two examples. Are the micro-elasticities necessary to discipline the extensive margin? Moreover, can we provide an analytic value for the infinite sum in Θ_m, that can be related to ergodic moments? The answer to these questions is model-dependent as we illustrate with one example and one counterexample. First, we show that in the random Ss model presented in Section 2 (known in the pricing literature as the CalvoPlus model, see Nakamura and Steinsson (2008)), the micro-elasticities are not needed to compute the CIR, since there exists a one-to-one mapping from ergodic moments to the CIR. Second, we provide a counterexample where two different models that generate the same ergodic moments have different CIR; thus, in this case, micro-elasticities play a key role in determining the CIR through the extensive margin.

Example 1. *In the random Ss model where $H(\xi)=1$ for all ξ, then CIR is given by*

$$\mathcal{A}_1(\delta)=\delta\left(\frac{\mathcal{M}_2[x]-\nu\ \mathcal{M}_{1,1}[a,x]}{\sigma^2}\right)+o(\delta^2). \tag{34}$$

Equation (34) shows that there exists a one-to-one mapping from the ergodic cross-sectional moments to the CIR, which is determined by the steady state variance of the capital-gaps, normalized by the shock volatility, minus the covariance between capital age (or vintage) and the investment rate. Consequently, micro-elasticities are not needed to characterize transitional dynamics.

To build the intuition for this result, consider the case $\nu=0$ so that the CIR is given exclusively by the normalized dispersion of capital gaps. This dispersion encodes information about agents' *responsiveness* to idiosyncratic shocks (the higher the ratio the less responsive), and in turn, the responsiveness determines the speed of convergence to the steady state. For instance, high levels of capital misallocation or large price dispersion (normalized by the volatility of idiosyncratic shocks) signal little responsiveness to idiosyncratic shocks, from where we infer that there is also slow adjustment to aggregate shocks. In the case $\nu\neq 0$, the covariance between capital vintage and the investment rate (which is negative in this model) appears as a way to correct for the effects of the drift.

From Example 1 a natural question arises: Is this a general result? Are the micro-elasticities presented in Proposition 4 actually not relevant? The answer to both questions is no, as we show with the following

the context of firm pricing decisions.

counterexample.[10]

Example 2. *Let* $T \equiv \mathbb{E}^{\hat{x}}[\tau]$ *denote average duration. Consider an inaction model with adjustments at fixed dates (Taylor-type) and a standard Ss model; assume away idiosyncratic shocks* ($\sigma = 0$) *and allow for a non-zero drift* ($\nu \neq 0$)*. In these two models there exists a steady state with a uniform distribution of capital-gaps and an investment distribution with an atom at* $-\nu T$*; thus they produce the same ergodic moments.*[11] *Now, let us study transitional dynamics for* $\delta < 0$*. As stated by the theory, in both models the intensive margin is equal to the average age:* $\Gamma_1 = T/2$*. Since the Taylor model is time-dependent,* $\Theta_1 = 0$ *and its CIR equals:* $\mathcal{A}^{Taylor}(\delta)/\delta \approx T/2$*. In the Ss model, the extensive margin is equal to* $\Theta_1 = \theta_{1,0}\mathcal{M}_0[x] = -T/2$ *(as* $\theta_{1,j} = 0$ *for* $j > 0$*), and its CIR equals* $\mathcal{A}^{Ss}(\delta)/\delta \approx T/2 - T/2 = 0$*. This result mirrors the classic money non-neutrality outcome in Caplin and Spulber (1987).*

The previous counterexample illustrates that two models may produce the same steady state statistics, but nevertheless, they can exhibit completely different transitional dynamics. Our explanation lies in the differences in micro-elasticities, zero in the Taylor-type model and $-\Gamma_1$ in the Ss model. Therefore, there exist cases for which the micro-elasticities are relevant objects for characterizing the extensive margin, and our theory can guide researchers in finding experiments or exogenous variation to compute them.

3.3 Observation

In the third set of results, we express the ergodic cross-sectional moments of the state distribution and the structural parameters in terms of the investment distribution Δx and adjustment dates τ. The relevance of this result lies in that in many applications, the state x is likely to be unobservable, but the adjustments Δx and τ are. This is the case in our example, as capital gaps are hard to observe but investment rates are readily available in the data.[12] As a consequence of the results in the following Proposition 5, we can track unobservable states using observable statistics. In the next proposition, we use the conditional coefficient of variation squared given by $\mathbb{CV}^2[X] = \frac{\mathbb{V}^{\hat{x}}[X]}{\mathbb{E}^{\hat{x}}[X]^2}$ for any random variable X.

Proposition 5. *Let* $(\Delta x, \tau)$ *denote the adjustment size and the adjustment dates and denote with* $\mathbb{E}^{\hat{x}}[\cdot]$ *the unconditional cross-sectional moments observed in the data.*

1. *The reset capital gap is given by the covariance between adjustment size and duration:*

$$\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2}\left(1 - \mathbb{CV}^2[\tau]\right) + \frac{\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{35}$$

2. *The volatility of idiosyncratic productivity shocks and the drift are recovered as:*

$$\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \sigma^2 = \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]} + 2\nu\hat{x}. \tag{36}$$

[10] See Web Appendix E.4 for the proof.

[11] These two models have several ergodic time-varying distributions that depend on the initial condition. To generate an unique ergodic distribution for any initial condition, we add a small and random probability of free adjustment. Besides generating a unique ergodic distribution, it gives differentiability to the CIR at $\delta = 0$ in the Ss model.

[12] Our formulas require us to compute the change in log capital gaps Δx in the data. Due to the continuity assumption for the idiosyncratic productivity, the changes in the capital-gap equal the observed investment rates: $\Delta x_{\omega,\tau} = \lim_{t\uparrow\tau_i} \log(K_{\omega,\tau_i}/K_{\omega,t}) - \lim_{t\uparrow\tau_i} \log(E_{\omega,\tau_i}/E_{\omega,t}) = \lim_{t\uparrow\tau_i} \log(K_{\omega,\tau_i}/K_{\omega,t})$. Therefore, we can compute the changes in the *capital gap* using changes in the *capital stock*.

3. The steady state moments are given by:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x)^{m+1}\right]}{\mathbb{E}^{\hat{x}}\left[\Delta x\right](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \tag{37}$$

$$\mathcal{M}_1[a] = \frac{1}{2}\mathbb{E}^{\hat{x}}[\tau]\left(1+\mathbb{CV}^2[\tau]\right), \tag{38}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\nu(m+1)\mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a], \tag{39}$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x,a] = \mathcal{M}_1[a]$.

Equation (35) shows the observability property for the reset state $\hat{x}$, which is derived from the cross-equation restriction imposed by the normalization of the ergodic mean $\mathcal{M}_1[x] = 0$. The expression has two components: the first one mainly reflects the effect of the drift in the reset state, while the second one mainly reflects the asymmetry in the policies.

To explain the drift component consider an Ss model without idiosyncratic shocks and a negative drift. In such a model, τ and Δx have a degenerate distribution and the capital-gap distribution is uniform in the domain $[\hat{x}-\Delta x, \hat{x}]$. Our formula implies a reset state of $\hat{x} = \frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2} > 0$, which centers the distribution around zero. By compensating the negative drift, the positive reset state ensures that the ergodic mean is zero. To explain the second component, set the drift to zero and consider an asymmetric inaction region $|\underline{x} - \hat{x}| > |\overline{x} - \hat{x}|$ such that the upper trigger is closer to $\hat{x}$ than the lower trigger. In this case, the capital-gap distribution is skewed towards values lower than $\hat{x}$. Our formula implies a reset state given by $\hat{x} = \frac{\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} > 0$.[13] By reflecting the bias in the policy, the positive reset state shifts the distribution to the right to ensure that the ergodic mean is zero.

Expressions in (36), which extend those in Álvarez, Le Bihan and Lippi (2014) for the case with drift, provide a guide to infer the parameters of the stochastic process. The first expression shows how to infer the drift from the average investment rate in the data, scaled by the adjustment frequency; and the second expression shows how to infer the volatility from the dispersion in investment rates, scaled by the frequency and corrected by the drift (which also generates dispersion, but not due to fundamental volatility).

Equation (38) relates average age to the average and the dispersion in duration, measured through the coefficient of variation. The relationship with the average duration is straightforward. To understand why the dispersion of duration affects average age, it is important to recall a basic property in renewal theory: the probability that a random firm has an expected time between capital changes of τ is increasing in τ, i.e. larger stopping times are more representative in the capital-gap distribution.[14] Therefore, dispersion in duration means there are firms that take a long time to adjust, and on top of that, those firms are more representative in the economy; this rises the average age.

Lastly, equation (37) provides a recursively formula to compute the centralized moments using observed investment rates. By assuming a stochastic process for the uncontrolled capital gaps ($dx_t =$

[13] The covariance is positive since the longer the duration, the higher is the probability to hit the lower trigger and to do an upward adjustment.

[14] This property has been widely studied in labor economics when thinking about long-term unemployment. For example, Mankiw (2014)'s textbook Principles of Macroeconomics mentions that: "[...] many spells of unemployment are short, but most weeks of unemployment are attributable to long-term unemployment".

$\nu dt + \sigma dW_t$), together with Itō's lemma and the Optional Sampling Theorem, we can connect the average *slope* of moment $m+1$ to the *level* of the previous moments m and $m-1$ as follows:

$$\underbrace{\mathbb{E}_t\left[dx_t^{m+1}\right]}_{slope} = \nu(m+1)\underbrace{x_t^m}_{level}dt + \frac{\sigma^2}{2}(m+1)m\underbrace{x_t^{m-1}}_{level}dt + \sigma(m+1)\underbrace{\mathbb{E}_t\left[x_t^m dW_t\right]}_{=0} \tag{40}$$

Therefore, the average investment to the power $m+1$, gives information about the centralized moment m of capital gaps. To see this clearly, set $\hat{x}=0$ and $m=2$, then equation (37) reads $\mathcal{M}_2[x] = \frac{\mathbb{E}^{\hat{x}}\left[(\Delta x)^3\right]}{3\mathbb{E}^{\hat{x}}[\Delta x]}$, relating the dispersion of capital gaps in the LHS to the skewness of investment rates in the RHS. A similar argument holds for (39).

3.4 Relationship to the literature

We conclude this section by explaining the connections and contributions to the literature on pricing and investment.

Pricing literature. The representation property establishes a formal and direct link between the slope of the Phillips curve and the cost of sticky prices, given by the dispersion of relative prices. Additionally, our results complement in two dimensions the analysis in Álvarez, Le Bihan and Lippi (2014). That paper establishes a connection between the first moment's CIR and the kurtosis of price changes Δx in the case of a symmetric menu-cost model with zero drift: $\mathcal{A}_1(\delta)/\delta = \frac{1}{6}\mathbb{E}[\tau]\mathbb{K}ur[\Delta x]$. First, we show how to extend this formula to take into account asymmetric policies and non-zero drift. Second, as explained in Example 1, we derive a connection between the first moment's CIR and the normalized ergodic variance of prices. This connection holds in the cases of symmetric and asymmetric Ss models. To our knowledge, this is the first time such connection is established.

Investment literature. There is a vast theoretical and empirical literature that aims to measure and analyze the consequences of capital misallocation across firms (see Restuccia and Rogerson (2013) for a survey). Misallocation is defined as the dispersion in the marginal product of capital in the cross-section. The problem is that the marginal product cannot be directly observed in the data. The traditional approach to study this phenomenon, as in Hsieh and Klenow (2009), consists in specifying a production function at the micro level that generates an equivalence between misallocation and the average capital-output ratio. However, as argued by Oberfield (2013), this approach suffers from a specification error, as it is hard to test the validity of the technological assumptions.

We propose an alternative way that circumvents this specification problem. Our approach—embedded in the observation property—consists in directly assuming a stochastic process for the unobserved marginal product of capital, and then adding discipline to the parameters of the stochastic process using observable micro data on investment. While this approach clearly depends on the assumptions on the stochastic process (e.g. mean-reversion vs. drift), the theory imposes cross-equation restrictions that allow us to validate such assumptions.

4 Application: Capital Misallocation

In this section, we revisit the investment model from Section 2 and apply our new tools to understand the magnitude of capital misallocation in steady state as well as the dynamics of capital gaps along the business cycle.

4.1 Data description and construction of capital gaps changes

Data description. We use micro data on the cross-section of manufacturing plants in Chile and Colombia.The Chilean data comes from the *Encuesta Nacional Industrial Anual* (Annual National Manufacturing Survey, ENIA) for the period 1995-2007.[15] The ENIA includes approximately 3,500 observations per year. The Colombian data comes from the *Encuesta Anual Manufacturera* (Annual Manufacturers Survey, EAM) for the period 1995-2016.[16] The EAM includes about 6500 observations per year. See Table I for other descriptive statistics.

Data cleaning and other issues. We exclude establishments with less than 10 workers and eliminate outlier observations with investment rates in the 1st and 99th percentiles. Additionally, in order to make our data comparable to previous studies, we consider a balanced panel of establishments that appear throughout the whole sample period.

Constructing capital gaps changes. To construct the capital stock series, we include machinery, equipment, transport, buildings and structures, while excluding land, office equipment and systems, as well as other depreciable and non-depreciable assets. Then, we construct nominal gross investment using information on purchases and sales of capital reported by each establishment

$$I_{\omega,t} = Purchases_{\omega,t} - Sales_{\omega,t}. \tag{41}$$

To construct the investment rate $i_{\omega,t}$, we divide investment by initial capital:

$$i_{\omega,t} = \frac{I_{\omega,t}}{K_{\omega,t}}, \tag{42}$$

where $K_{\omega,t}$ is the nominal value of fixed assets at the *start* of year t (adjusted by depreciation and inflation). Recall that the change in capital gaps is given by $\Delta x_{\omega,t} = \log\left(K_{\tau_{\omega,i}}/K_{\tau^-_{\omega,i}}\right) = \log(1 + i_{\omega,t})$, thus

$$\Delta x_{\tau_{\omega,i}} = \hat{x} - x_{\tau^-_{\omega,i}}. \tag{43}$$

[15] This data has been used by Liu (1993) to examine the role of turnover and learning on productivity growth; by Tybout (2000) to survey the state of the manufacturing sector in developing economies; and more recently, by Oberfield (2013) to study productivity and misallocation during crisis time.

[16] This data has been used by Eslava, Haltiwanger, Kugler and Kugler (2004, 2013) to study the effect of structural reforms and trade liberalization on aggregate productivity.

Using the information on investment rates, we construct log capital gaps as:

$$\Delta x_{\omega,t} = \begin{cases} \log\left(1 + i_{\omega,t}\right) & \text{if} \quad |i_{\omega,t}| > \underline{i} \\ 0 & \text{if} \quad |i_{\omega,t}| < \underline{i}, \end{cases} \tag{44}$$

where $\underline{i} > 0$ is a parameter that captures the idea that small maintenance investments do not incur the fixed cost of investment. Following Cooper and Haltiwanger (2006), we set $\underline{i} = 0.01$, such that all investments smaller than 1% in absolute value are excluded and considered as inaction.

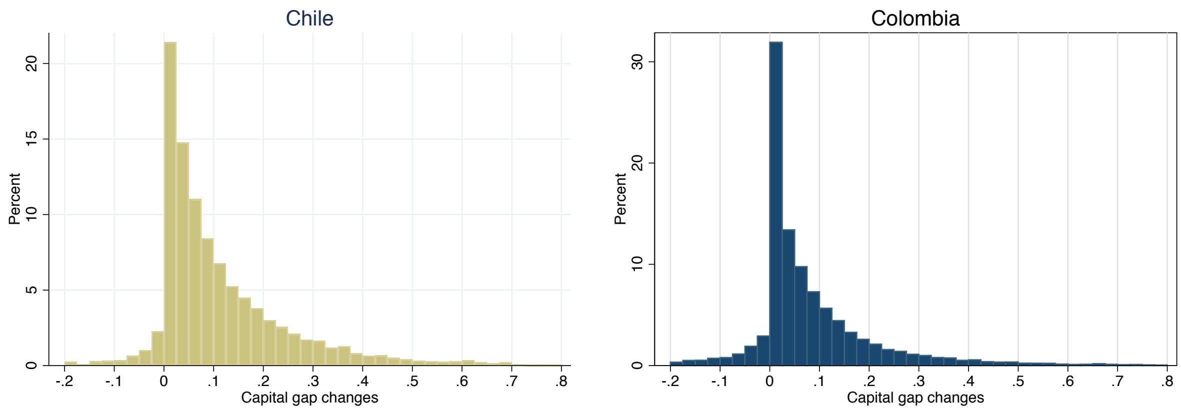
Investment rate distribution. Table I presents some characteristics of the samples we consider as well as the descriptive statistics on investment rates (averages, inaction and spikes). Besides Colombia and Chile, we include numbers for the US from Cooper and Haltiwanger (2006) and from Zwick and Mahon (2017). Inaction is defined as investment below 1% in absolute value; positive spikes are investments above 20% and negative spikes below −20%.

Table I – DESCRIPTIVE STATISTICS

		Chile	**Colombia**	**US I**	**US II**
Sample characteristics					
Period		1995-2007	1995–2016	1972–1988	1993–2010
Establishments per year (yearly avg.)		3,470	5,615	7,000	128,151
Size (avg. number of workers)		87	92		
Investment rates (%)					
Average		9.68	8.8	12.2	10.4
Positive fraction	$i > 1$	65.1	68.7	81.5	
Negative fraction	$i < -1$	3.9	9.2	10.4	
Inaction rate	$\lvert i\rvert < 1$	31.0	22.1	8.1	23.7
Spike rate	$\lvert i\rvert > 20$	17.1	16.0	20.4	14.4
Positive spikes	$i > 20$	16.2	14.4	18.6	
Negative spikes	$i < -20$	0.8	1.6	1.8	
Serial correlation	$corr(i_t, i_{t-1})$	0.15	0.1	0.09	0.40

Sources: Authors' calculations using establishment-level survey data for Chile, Colombia and Mexico. US I shows data from Cooper and Haltiwanger (2006) and US II shows data reported in Zwick and Mahon (2017) for the balanced panel. Following these papers, investment rates reported in this table are computed as Investment divided by Initial Capital. We use the book value of capital instead of perpetual inventories.

Figure II – Histogram of Capital Gap Changes Δx



4.2 Putting the theory to work

The observation results tell us how to use cross-sectional data on investment rates and adjustment frequency to pin down the parameters of the stochastic process and the ergodic moments, which in turn map into the CIR. Table II summarizes the statistics calculated from the micro data which serve as inputs into the formulas, as well as the theory's output. Throughout the discussion, we present the numbers for Colombia and Chile in a vector $(Col, Chile)$. When we refer to the literature, we consider the following abbreviations: Khan and Thomas (2008) [KT], Bachmann, Caballero and Engel (2013) [BCE] and Winberry (2016) [W].

Inputs from Micro Data. Consider first the distribution of expected times τ. We obtain an average expected time to adjustment of $\mathbb{E}^{\hat{x}}[\tau] = (1.4, 1.2)$ years with dispersion $Std^{\hat{x}}[\tau] = (1.2, 0.86)$. This large dispersion suggests substantial heterogeneity in adjustment times across establishments. Now consider the distribution of capital gaps; it has an average of $\mathbb{E}^{\hat{x}}[\Delta x] = (0.108, 0.071)$ with a dispersion of $Std^{\hat{x}}[\Delta x] = (0.146, 0.130)$, and it is right-skewed. The covariance between adjustment size and expected time is almost zero $\mathbb{C}ov^{\hat{x}}[\tau, \Delta x] = (0.001, 0.009)$, which implies almost zero correlation between time of adjustment and their size. This zero covariance is surprising, as one would expect a positive covariance: the longer the inaction period, the stronger the effect of the drift; consequently, upon taking action, the investment rate should be larger. As we show below, this moment is key for computing the CIR.

Output from theory: parameters. The implied drift is $\nu = -\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} = (-0.078, -0.058)$, which includes the sum of the depreciation rate and productivity growth. This number is -0.085 in KT and -0.088 in BCE. Regarding the volatility of idiosyncratic shocks, the formula implies that

$$\sigma = \sqrt{\underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]}}_{(0.024,\ 0.018)} + \underbrace{2\nu\hat{x}}_{(-0.003,\ -0.001)}} = (0.146, 0.122)$$

Note that the dispersion in investment rates primarily drives the volatility estimate and the drift component is negligible. Volatility of the innovations in the literature falls within a very wide range, from 0.052

Table II – INPUTS FROM MICRO DATA AND OUTPUTS FROM THE THEORY

Inputs from Micro Data	**Chile**	**Colombia**	**Model**
Frequency			
$\mathbb{E}^{\hat{x}}[\tau]$	1.392	1.223	1.360
$\mathbb{CV}^2[\tau]$	0.692	0.497	0.895
Capital Gaps			
$\mathbb{E}^{\hat{x}}[\Delta x]$	0.108	0.071	0.106
$\mathbb{E}^{\hat{x}}[\Delta x^2]$	0.033	0.022	0.048
$\mathbb{E}^{\hat{x}}[(\hat{x}-\Delta x)^3]$	-0.009	0.006	-0.008
Covariances			
$\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$	0.001	0.009	0.193
$\mathbb{E}^{\hat{x}}[\tau(\hat{x}-\Delta x)^2]$	0.033	0.027	0.091

Outputs from Theory	**Chile**	**Colombia**	**Model**
Parameters			
ν	-0.078	-0.058	-0.078
σ	0.146	0.122	0.146
$\hat{x}$	0.017	0.025	0.090
Steady State Moments			
$\mathcal{M}_2[x]$	0.027	0.022	0.027
$\mathcal{M}_{0,1}[a]$	1.178	0.915	1.289
$\mathcal{M}_{1,1}[x,a]$	0.132	0.119	-0.082
Transitional Dynamics			
$\mathcal{M}_2[x]/\sigma^2$	1.315	1.463	1.268
$-\nu\mathcal{M}_{1,1}[x,a]/\sigma^2$	0.480	0.461	-0.300
$\mathcal{A}_1(\delta)$	1.796	1.923	0.968

Notes: Capital-gap changes is described above. For the model moments, we use the model in Section 2 with parameters for preferences and technology $(\rho, \alpha, \kappa, \lambda) = (0.04, 0.58, 0.45, 0.71)$ with $H(\xi) = 1$ for all $\xi \in [0, \bar{\xi}]$ and parameters for the stochastic process of the capital-gaps $(\mu, \psi, \sigma) = (0.016, 0.0620, 0.146)$.

in KT to 0.117 in W and 0.202 in BCE.[17] It is worth noting that the calibration of σ in these papers is done jointly with the fixed adjustment cost within a particular inaction model; in contrast, our estimate for the volatility is pinned down directly from the data as our theory generates a mapping between data and parameters that is model independent.

Now recall the observation formula for the reset capital gap in (35):

$$\hat{x} = \underbrace{\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{2}}_{(0.054,\ 0.036)} \underbrace{\left(1 - \mathbb{CV}^2[\tau]\right)}_{(0.308,\ 0.503)} + \underbrace{\frac{\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}}_{(0.0007,\ 0.0074)} = (0.017, 0.025)$$

[17]The original numbers used in the papers are 0.022 and 0.049, respectively. Since we abstract from labor and our productivity is rescaled, we must adjust their volatilities by a factor $1/1-\alpha$ in order to make their numbers comparable to ours. We assuming a labor share of $\alpha = 0.58$ and obtain the numbers above. Additionally, for BCE and W, we convert their quarterly volatilities $\sigma^q = 0.021, 0.047$ to yearly taking into account the persistence as follows: $\sigma^a = \sigma^q\sqrt{1+\rho+\rho^2+\rho^3}$), with $\rho = 0.94, 0.86$. Lastly, for BCE, we only consider the idiosyncratic shocks (excluding the sectorial shocks).

The values we obtain imply that upon adjustment, capital gaps are reset about 2% above the average capital gap. Given a negative drift of minus 8%, this means that the rest of the drift must be accommodated by an asymmetric policy.

Output from theory: ergodic moments. We define misallocation as the ergodic second moment of capital gaps $\mathcal{M}_2[x]$. According to the observation formula, the steady state value of misallocation can be expressed in terms of average investment rates and the reset state as follows:

$$\mathcal{M}_2[x] = \frac{\hat{x}^3 - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^3]}{3\mathbb{E}^{\hat{x}}[\Delta x]} = [0.027, 0.022], \tag{45}$$

where the cubic powers capture asymmetries in the distributions. Note that the ratio $\mathcal{M}_2[x]/\sigma^2$ is above one in both datasets and quantitatively close to the expected duration. This suggests that the ergodic dispersion—a measure of ex-post heterogeneity—is almost the same as fundamental volatility σ^2 times expected duration, even if there is an 8% drift. This suggests that, even though firms are reacting to their structural shocks, they adjust by less than in a standard Ss model (where it is equal to 1/5).

Another moment of interest is the average age $\mathcal{M}_{0,1}[x, a] = \mathcal{M}_1[a]$, recovered using information about the average and the dispersion of adjustment times:

$$\mathcal{M}_1[a] = \frac{1}{2} \underbrace{\mathbb{E}^{\hat{x}}[\tau]}_{(1.392,1.223)} \underbrace{\left(1 + \mathbb{CV}^2[\tau]\right)}_{(1.692,1.497)} = [1.178, 0.915] \tag{46}$$

Following our earlier discussion on renewal theory—larger stopping times are more representative in the sample—the heterogeneity in expected times increases the average age in the economy. Lastly, the covariance between age and adjustment size $\mathcal{M}_{1,1}[x, a]$ is recovered as

$$\mathcal{M}_{1,1}[x, a] = \frac{1}{2\nu} \left(\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\tau\,(\hat{x} - \Delta x)^2\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{(0.024,0.022)} - \underbrace{\mathcal{M}_2[x]}_{(0.027,0.022)} - \underbrace{\sigma^2\mathcal{M}_1[a]}_{(0.025,0.014)} \right) = [0.132, 0.119] \tag{47}$$

The covariance implied by the data is positive. This is counterintuitive as it says that, for the average plant, the older is its capital the more positive is the capital gap. In other words, there is a stronger incentive to disinvest for plants with old capital (contrary to what one would expect).

Misallocation Dynamics. Now we focus on transitional dynamics. Consider an unanticipated permanent aggregate productivity shock that shifts horizontally the distribution of idiosyncratic productivity i.e. a first moment perturbation. From Example 1, we know that the transitional dynamics of capital gaps after a first moment perturbation are given by

$$\frac{\mathcal{A}_1(\delta)}{\delta} \approx \underbrace{\frac{\mathcal{M}_2[x]}{\sigma^2}}_{(1.315,1.463)} - \underbrace{\frac{\nu\ \mathcal{M}_{1,1}[a, x]}{\sigma^2}}_{(-0.480,-0.461)} = (1.796, 1.923). \tag{48}$$

In order to interpret these numbers, assume that the IRF is exponential $\exp(-\lambda t)$. Then the CIR is equal to $CIR = 1/\lambda$. Its half-life is equal to the date T such that $\exp(-T * \lambda) = 0.5$ or $T = -\log(0.5)/\lambda$. Therefore, $T = \log(2) * CIR = (1.25, 1.33)$, i.e. it takes 5 quarters for half of the effect of the shock to vanish.

Can the random Ss model generate the CIR? Now we study the quantitative implications of the random Ss model we presented in Example 1, where $H(\xi) = 1$. We have two parameters to calibrate: the size of the adjustment cost $\kappa = 0.45$ and the arrival rate of free adjustment opportunities $\lambda = 0.71$. We set these parameters in order to match the average expected time to adjustment $\mathbb{E}^{\hat{x}}[\tau]$ and the ergodic second moment of capital gaps $\mathcal{M}_2[x]$. The parameters for the stochastic process of the capital-gaps are taken from the data and our formulas above $(\mu, \psi, \sigma) = (0.016, 0.0620, 0.146)$. The rest of the parameters, the discount $\rho = 0.04$ and the capital share $\alpha = 0.58$, are set externally. The moments produced by the model are reported in the last column of Table II.

We find that the model is able to match most moments from the data, except for two moments: the covariance $\mathbb{C}ov^{\hat{x}}[\tau, \Delta x]$ is equal to 0.193 (almost zero in the data), and the $\mathbb{E}^{\hat{x}}[\tau(\hat{x} - \Delta x)^2]$ equal to 0.091 (and 2/3 smaller in the data). In particular, missing the second covariance has important implications for the CIR through its effect on $\mathcal{M}_{1,1}[x, a]$. While the model is successful in matching the first component of the CIR, equal to the normalized steady state misallocation $\mathcal{M}_2[x]/\sigma^2$, it dramatically misses the second component that includes the covariance between age and capital gap $-\nu\ \mathcal{M}_{1,1}[a, x]/\sigma^2 > 0$. Since this second component is positive (the negative drift times the negative covariance), the implied CIR from the model is substantially below the one implied by our formulas. For illustration, consider a 1% permanent increase in aggregate productivity. The model suggests a CIR of 0.968%; in contrast, the data together with our theory, suggests a CIR between 1.8%–1.9%. These numbers suggest that the model underestimates the effect of aggregate shocks.

5 Extensions and Generalization

In the previous sections we specified parametric restrictions to the inaction model and to the firms' state space. Such assumptions exclude from our analysis models with fixed adjustment dates as in Taylor (1980), models with observation costs as in Álvarez, Lippi and Paciello (2011), and several others. Nevertheless, it is possible to extend our theory to accommodate richer models. In this section, we generalize our results to consider any stopping-time model or state space, explaining the assumptions on policies and processes that are key to apply our tools.

Second, we extend the analysis in three directions, to consider: (i) transitions of higher moments ($m > 1$) of the distribution; (ii) transitions starting from any general initial condition F_0; and (iii) transitions for a mean-reverting process. In each case, we focus on the one property that delivers the most interesting mechanism.[18] We denote conditional distributions as $Z|Y$, conditional expectations with initial condition z as $\mathbb{E}^z[Z]$, and the minimum between two stopping times as $t \wedge s \equiv \min\{t, s\}$.

5.1 Generalization

Let $(\Omega, P, \mathcal{F})$ be a probability space equipped with a filtration $\mathcal{F} = (\mathcal{F}_t; t \geq 0)$. We consider an economy populated by a continuum of agents indexed with $\omega \in \Omega$, where agent ω's information set at time t is the filtration $\mathcal{F}_t$. Each agent's uncontrolled state is given by $\tilde{S}_t(\omega) = [\tilde{x}_t(\omega), S_t^{-x}(\omega)] \in \mathbb{R}^{1+K_{-x}}$. The state is split between a *main state* $\tilde{x}$ and a set of *complementary states* $\tilde{S}_t^{-x}$. The main state follows a Brownian motion $d\tilde{x}_t(\omega) = \sigma dW_t(\omega)$. Agent's policies consist of a sequence of adjustment dates $\{\tau_k\}_{k=1}^{\infty}$ and adjustments sizes $\{\Delta S_{\tau_k}\}_{k=1}^{\infty}$, measurable with respect to $\mathcal{F}_t$. Given these policies $\{\tau_k(\omega), \Delta S_{\tau_k}(\omega)\}_{k=1}^{\infty}$, the controlled state $S_t(\omega)$ evolves as the sum of the uncontrolled state plus the adjustments: $S_t(\omega) = \tilde{S}_t(\omega) + \sum_{\tau_k(\omega) \leq t} \Delta S_{\tau_k(\omega)}$.

The first premise for our theory is a recursive representation of the conditional CIR, both between and within stopping dates. This demands $S_t(\omega)$ to be a sufficient statistic for the conditional CIR, which in turn requires that the policy is history independent. Formally, this mean that

$$\mathbb{E}\left[\int_{\tau_i \wedge t(\omega)}^{\tau_{i+1}} f(x_t)dt|\mathcal{F}_{\tau_i \wedge t(\omega)}\right] = \mathbb{E}\left[\int_0^{\tau} f(x_t)dt|S_{\tau_i \wedge t(\omega)}\right] = v^f(S_{\tau_i \wedge t(\omega)}), \quad \text{for all} \quad t(\omega) \leq \tau_{i+1}.$$

Since the main state follows a Brownian motion, the burden of this requirement falls completely on the complementary state and the policy. Assumption 1 and 2 formalize these requirements.

Assumption 1 (Markovian complementary state). *The complementary state $\tilde{S}_t^{-x}$ follows a Strong Markov process:*

$$\tilde{S}^{-x}_{(t\wedge\tau_k)+h}(\omega)|\mathcal{F}_{t\wedge\tau_k} = \tilde{S}_h^{-x}(\omega)|\tilde{S}_{(t\wedge\tau_k)}(\omega), \qquad \forall k. \tag{49}$$

To understand this assumption, consider a history ω such that $t < \tau_k(\omega)$. In this case, the complementary state's law of motion depends only on its current value; thus it is independent of its own history. Additionally, the complementary state is an homogenous process, since its law of motion at date t is

[18] The Web Appendix presents the full characterization and analysis of the three properties.

equivalent to its law of motion at zero, given an initial condition. In the complementary case $t \geq \tau_k(\omega)$, these properties continue to hold, thus the stopping policy does not reveal new information about the complementary state's law of motion.

Assumption 2 (**Markovian policies**). *Policies satisfy the following conditions:*

$$\tau_{k+1}|\mathcal{F}_{\tau_k+h} \;=\; \tau_1|S_{\tau_k+h} \quad \textit{for all} \quad h \in [0, \tau_{k+1} - \tau_k]. \tag{50}$$

A second premise in our theory is that we can characterize the CIR with the *first* stopping time of every agent. This means that, upon taking action, agents fully adjust to include any deviations from their steady state behavior and come back to the steady state process. This would imply that S_{τ_k} is *iid* across time and independent of the history previous to the adjustment. The challenge with stochastic *iid* resets is that is makes it more difficult to identify the parameters of the stochastic process, e.g. differentiating the fundamental volatility σ from the volatility arising from a random reset state. Therefore, in order for the reset state to be sufficiently informative, we ask that it is a constant $x_{\tau_i} = \hat{x}$.[19]

Assumption 3 (**Constant reset state**). *The reset state is constant: $x_{\tau_k} = \hat{x}$ for all k.*

It is straightforward to check that the previous assumptions hold in the investment example developed in Section 2. For Assumption 1, the complementary state is given by the arrival of free adjustment opportunities N_t, which is assumed to be a Poisson counter process and thus a Strong Markov process. The requirements in Assumption 2 and 3 are also satisfied. We showed that the reset capital gap is constant; and since the stopping policy is an inaction set with respect to the controlled state, the stopping policy is history independent within and between adjustments.

Finally, in order to apply the Optional Sampling Theorem, we require several stopping processes to be well-defined (finite moments at the stopping-time).[20]

Assumption 4 (**Well-defined stopping processes**). *The processes $\left(\left\{\int_0^t s^j x_s^m \, dB_s\right\}_t, \tau\right)$ for all m and $j = 0, 1$, are well-defined stopping processes.*

The previous Markovian requirements are enough in order to characterize the aggregation, representation of the intensive margin, and observation properties; however, in order to apply the representation property to the extensive margin, we must require one additional assumption. There must exist an equivalent representation of the extensive margin as a function exclusively of the main state x. For this, we require that there exists a stopping policy τ^* that only depends on the main state x and can fully describe the extensive margin by itself. For instance, a stopping policy given by a Poisson counter with hazard $\Lambda(x)dt$ satisfies this requirement.

Assumption 5 (**Hazard**). *Assume that there exist a stopping policy τ^* s.t.*

$$\mathbb{E}^{\hat{x}}\left[\int_0^{\tau}\left(\frac{\partial \mathbb{E}^S\left[x_\tau^{m+2}/m+2\right]}{\partial x} - \mathbb{E}^S\left[x_\tau^{m+1}\right]\right)dt\right] = \mathbb{E}^{\hat{x}}\left[\int_0^{\tau}\left(\frac{\partial \mathbb{E}^x\left[x_{\tau^*}^{m+2}/m+2\right]}{\partial x} - \mathbb{E}^x\left[x_{\tau^*}^{m+1}\right]\right)dt\right], \tag{51}$$

[19] In this paper, we ignore ex-ante heterogeneity across agents (that could be reflected in different reset states and policies), but this can be relaxed. Nevertheless, it remains crucial that history is erased at the moment of reseting the state.

[20] See Web Appendix A for a formal definition of a well-defined stopping process.

and there exist a smooth function $g_m(x)$ such that

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x + x)^m\right], \quad \forall m. \tag{52}$$

where Δx is under the policy τ^*.

5.2 Extensions

Now that we have stated the formal requirements needed to apply our theory, we proceed to develop the three extensions. To highlight the new mechanisms, in all the extensions we focus on the driftless case $\nu = 0$, but the proofs are straightforward to extend to consider a non-zero drift.

Extension I: Transitional dynamics for higher moments We first consider the transitional dynamics for higher moments of the distribution ($m \geq 1$). The initial condition remains to be a mean translation of the steady state distribution. In this case, we focus the discussion on the representation of the intensive margin.

Proposition 6. *Assume $d\tilde{x}_t = \sigma dW_t$. To a first order, the transitional dynamics of the m-th moment are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\Gamma_m + \Theta_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{53}$$

where the intensive margin relates to ergodic moments as follows:

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a], \tag{54}$$

$$\mathcal{M}_{m-1,1}[x,a] = \frac{2}{m(m+1)}\left[\frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\mathbb{E}^{\hat{x}}\left[\Delta x^2\right]} - \frac{\mathcal{M}_{m+1}[x]}{\sigma^2}\right]. \tag{55}$$

To focus on the intensive margin, assume $\Theta_m = 0$ for all m and consider the transitional dynamics for the state's first three moments by setting $m = 1, 2, 3$. We have that

$$\mathcal{A}_1(\delta)/\delta = \Gamma_1 = \mathcal{M}_1[a] \tag{56}$$

$$\mathcal{A}_2(\delta)/\delta = \Gamma_2 = 2\mathcal{M}_{1,1}[x,a] \tag{57}$$

$$\mathcal{A}_3(\delta)/\delta = \Gamma_3 = 3\mathcal{M}_{2,1}[x,a] \tag{58}$$

As discussed earlier, the dynamics of the first moment ($m = 1$)—average capital gaps—are fully driven by the state's average age. The dynamics of the second moment ($m = 2$)—dispersion of capital gaps or misallocation—are driven by the covariance between the age and the size of capital gaps. If this covariance is zero, then the distribution's second moment remains constant along the transition path. Asymmetry in in the agents' investment policy, which generates a skewed ergodic distribution, is one way to generate a non-zero covariance. This interaction between the business cycle dynamics of capital misallocation and the asymmetry of the ergodic capital distribution is studied by Ehouarne, Kuehn and Schreindorfer (2016) and Jo and Senga (2014). Finally, the dynamics of the third moment ($m = 3$)—skewness of capital gaps—are driven by the covariance between age and the square of capital gaps. Note that if the ergodic distribution features excess kurtosis, then the skewness of the distribution will change along the transition.

Proposition 6 provides formulas for the CIR of the m-th moment. Additionally, these formulas have two useful applications. They can be used to (i) derive bounds for the dynamics of functions of the m-th moments, and (ii) study transitions of any arbitrary function of the state. To illustrate the first application, let us consider the transitional dynamics for the variance. Using Jensen's inequality, we derive an upper bound on the variance's CIR:[21]

$$\text{CIR}(\mathbb{V}[x]) \equiv \int_0^\infty \left(\mathbb{V}_t\left[x\right] - \mathbb{V}\left[x\right]\right) dt \ \leq \ \mathcal{A}_2(\delta) - \mathcal{A}_1^2(\delta). \tag{59}$$

To illustrate the second application, consider a smooth function of the state $f(x)$. For example, in many models the aggregate welfare criteria can be written in this form. Using a Taylor approximation around zero, we write the CIR of the $f(x)$ function in terms of the state's CIR, weighted by the Taylor factors.

$$\text{CIR}(f(x)) = \int_0^\infty \mathbb{E}_t[f(x)] - \mathbb{E}[f(x)]dt \ = \sum_{j=1}^\infty \frac{df^j(0)}{dx^j}\frac{\mathcal{A}_j(\delta)}{j!}. \tag{60}$$

Extension II: General initial conditions This extension considers transitional dynamics for general initial conditions. For instance, since the work on uncertainty shocks by Bloom (2009), there has been a large literature interested in the macroeconomic consequences of uncertainty in the business cycle. Within our framework, these aggregate uncertainty shocks can be studied by setting the initial distribution as a mean-preserving spread of the steady state distribution. Moreover, the interaction between first and second moment shocks, as studied by Aastveit, Natvik and Sola (2013), Vavra (2014), Caggiano, Castelnuovo and Nodari (2014), Castelnuovo and Pellegrino (2018), and Baley and Blanco (2019), can be accommodated as well.

For simplicity, we consider perturbations that can be expressed via a single parameter δ. The initial distribution is described through a function $\mathcal{G}(x,\delta)$, such that $F_0(x) = F(\mathcal{G}^{-1}(x,\delta))$. To make progress, we impose certain smoothness and differentiability properties to the function $\mathcal{G}$.[22] Additionally, we focus on perturbations to the first and second moments. Since this extension does not affect steady state moments, we omit the characterization of the observation property as it remains as before.

Proposition 7. *Assume $d\tilde{x}_t = \sigma dW_t$ and let $\mathcal{G}(x,\delta)$ be a function that satisfies the following properties:*

1. $\mathcal{G}(x,0) = x$.
2. $\exists z > 0$ *such that* $\forall \epsilon \in (-z,z)$, $\mathcal{G}(\cdot,\epsilon)$ *is bijective.*
3. $\frac{\partial \mathcal{G}(\mathcal{G}^{-1}(y,0),0)}{\partial \delta} = -(\mathcal{G}_0 + \mathcal{G}_1 y)$ *with* $\mathcal{G}_0^2 + \mathcal{G}_1^2 = 1$.

[21] $\text{CIR}(\mathbb{V}[x]) \equiv \int_0^\infty \left(\mathbb{V}_t\left[x\right] - \mathbb{V}\left[x\right]\right) dt = \int_0^\infty \left(\mathbb{E}_t\left[x^2\right] - \mathbb{E}\left[x^2\right]\right) dt - \int_0^\infty \mathbb{E}_t\left[x\right]^2 dt \leq \mathcal{A}_2(\delta) - \mathcal{A}_1^2(\delta)$.

[22] See proof in Web Appendix D.3 for details.

To a first order, the CIR is given by:

$$\begin{aligned}
\mathcal{A}_1(\mathcal{G}) &= \delta \times \Bigg(\underbrace{\mathcal{G}_0 \left(\Gamma_{1,0} + \Theta_{1,0}\right)}_{\textit{1st moment shock}} + \underbrace{\mathcal{G}_1 \left(\Gamma_{1,1} + \Theta_{1,1}\right)}_{\textit{2nd moment shock}} \Bigg) + o(\delta^2) && (61) \\
\Gamma_{1,i} &= (i+1)\mathcal{M}_{i,1}[x,a] && (62) \\
\Theta_{1,i} &= \sum_{j=0}^{\infty} \theta_{1,j} \mathcal{M}_{j+i}[x] && (63)
\end{aligned}$$

with $\theta_{1,j}$ are the micro-elasticities.

Proposition 7 points towards the moments that are crucial to characterize the dynamics for a particular type of initial condition. As long as there exists enough differentiability in the perturbation of the initial condition, we can find ergodic moments that perfectly describe the dynamics of the model. Interestingly, the micro-elasticities needed to compute the extensive margin are independent of the number of moments that are shocked.

As an example, consider $\mathcal{G}$ to be a mean preserving spread of the steady state distribution F_0. This means that $\mathcal{G}(x,\delta) = x(1+\delta)$ and therefore $\mathcal{G}_0 = 0$ and $\mathcal{G}_1 = 1$. Again, let us focus only in the intensive margin by setting $\Theta_{1,i} = 0$ for all i. Then the CIR is approximated as:

$$\frac{\mathcal{A}_1(\delta)}{\delta} \approx \mathcal{G}_1 \Gamma_{1,1} = \mathcal{M}_{1,1}[x,a].$$

Thus mean-preserving perturbations have first order effects if and only if the covariance between age and the state is different from zero. A non-zero covariance is consistent with the data presented in Section 4. Therefore, suggesting that uncertainty shocks (in the form of mean-preserving spreads of the capital gap distribution) would have effects on average investment.

Extension III: Mean-reversion This extension considers a mean-reverting process for the uncontrolled state. This type of process is wildly used due to its empirical relevance and because it ensures the existence of an ergodic distribution. For this application, we focus on the observation properties.

Proposition 8. *Assume the uncontrolled state follows a Ornstein–Uhlenbeck process $d\tilde{x}_t = \rho \tilde{x}_t dt + \sigma dW_t$. Then, the reset state and structural parameters are recovered through a system of equations:*

$$\begin{aligned}
\hat{x} &= \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}\Delta x]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} && (64) \\
\frac{\sigma^2}{\rho} &= 2\frac{\hat{x}^2 - \mathbb{E}^{\hat{x}}\left[e^{-2\rho\tau}(\hat{x}-\Delta x)^2\right]}{\mathbb{E}^{\hat{x}}\left[e^{-2\rho\tau}\right] - 1} && (65) \\
erf\left(\frac{\hat{x}}{\sqrt{\sigma^2/\rho}}\right) &= \mathbb{E}^{\hat{x}}\left[erf\left(\frac{\hat{x}-\Delta x}{\sqrt{\sigma^2/\rho}}\right)\right] && (66)
\end{aligned}$$

where $erf(x) \equiv \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt$ is the Gauss error function.

To gain some intuitions about the observation formulas above, let us consider the limiting case $\rho \to 0$.

Using the approximation $e^{-\rho\tau} \approx 1 - \rho\tau$, it is easy to show that equations (64) and (65) converge to our baseline observations expressions in (35) and (36) with $\nu = 0$ (no mean-reversion):

$$\hat{x} \to_{\rho \to 0} \frac{\mathbb{E}^{\hat{x}}[\tau \Delta x]}{\mathbb{E}^{\hat{x}}[\tau]}, \qquad \sigma^2 \to_{\rho \to 0} \frac{\mathbb{E}^{\hat{x}}\left[\Delta x^2\right]}{\mathbb{E}^{\hat{x}}[\tau]}. \tag{67}$$

Therefore, as long mean reversion is "sufficiently small", the mappings between the data and the reset state, and between the data and idiosyncratic volatility do not change.

Let us make a deeper comparison of how $\hat{x}$ is determined with and without mean-reversion. With *iid* shocks, we can write (35) as a weighted sum of investment rates across firms:

$$\hat{x}^{iid} = \mathbb{E}^{\hat{x}}[\eta(\tau)\Delta x], \quad \textit{with} \quad \eta(\tau) \equiv \frac{\tau}{\mathbb{E}^{\hat{x}}[\tau]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta(\tau)] = 1,$$

where the weights $\eta(\tau)$ are increasing in τ, i.e. more weight is given to the investment rate of firms with large periods of inaction (with "old" capital). In order to understand this result, note that conditional of surviving, the distribution of the state is more centered around the reset state for "young" capital vintages, which cannot reflect policy asymmetries. The opposite happens for firms with "old" vintages, as the distribution of the state is more centered around *the domain's middle point*, reflecting the policy asymmetries. Thus investment rates associated with large stopping times are more informative about these asymmetries.

The opposite happens when we consider a mean-reverting process. An analogous decomposition yields

$$\hat{x}^{mr} = \mathcal{R}\mathbb{E}^{\hat{x}}[\eta'(\tau)\Delta x], \quad \textit{with} \quad \eta'(\tau) \equiv \frac{e^{-\rho\tau}}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]} > 0, \quad \mathbb{E}^{\hat{x}}[\eta'(\tau)] = 1, \quad \mathcal{R} \equiv \frac{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}]}{\mathbb{E}^{\hat{x}}[e^{-\rho\tau}] - 1} < 0,$$

where now the weights are decreasing in duration and it is preceded by a negative number. As the inaction period of increases, the mean-reverting productivity process goes back to its zero long-run mean, and the distribution gets centered *around zero* on its own, so there is no need to correct for policy asymmetries with the initial condition.

6 Conclusion

This paper provides a structural relation in model of inaction between the CIR (a measure of persistence for aggregate dynamics) and micro-data. This relation holds for any moment of the distribution, any inaction model, and any initial condition. In the same way we apply our tools to a model of lumpy investment, we foresee applications in models with labor adjustment costs, inventory models, portfolio management, government debt management, among others.

For developing our theory, we assume that upon taking action, agents fully adjust to include any deviations from their steady state behavior. Thus our results do not accommodate partial adjustments which are due, for instance, to imperfect information or convex adjustment costs. One example of these frameworks is the menu cost model with information frictions in Baley and Blanco (2019). We leave for future research the application of the tools developed here to that type of frameworks.

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A Auxiliary Theorems

The following three theorems will be heavily used in the proofs. We only prove the third theorem, and provide references for the others.

Theorem 1. ***[Optional Sampling Theorem (OST)]*** *Let Z be a (sub) martingale on the filtered space $(\Omega, \mathcal{P}, \mathcal{F})$ and τ an stopping time. If $(\{Z_t\}_t, \tau)$ is a well-define stopping process, then*

$$\mathbb{E}[Z_\tau](\geq) = \mathbb{E}[Z_0] \tag{A.1}$$

Proof. See Theorem 4.4 in Stokey (2009). □

Theorem 2. ***[Ergodic distribution and occupancy measure]*** *Let S be a strong Markov process and g a function of S. Denote with F is the ergodic distribution of S and with R the renewal distribution (the distribution conditional on adjustment). If $\int g(S)dF(S) = \lim_{T\to\infty} \frac{\int_0^T g(S_t)dt}{T}$ for all initial conditions S_0, then the following relationships hold:*

$$\int g(S)dF(S) = \frac{\int \mathbb{E}^S\left[\int_0^\tau g(S_t)dt\right] dR(S)}{\int \mathbb{E}^S\left[\tau\right] dR(S)} \tag{A.2}$$

If $\Pr[S = \hat{S}] = 1$ under the renewal distribution R, then

$$\int g(S)dF(S) = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau g(S_t)dt\right]}{\mathbb{E}^{\hat{S}}\left[\tau\right]} \tag{A.3}$$

Proof. (1) Start from the ergodicity assumption and (2) write T as the sum of n stopping times. (3) Take conditional expectations with respect to the filtration at time t. Since S is a strong Markov process, there is history independence across stopping times, and thus (4) we can write $\mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt|\mathcal{F}_{\tau_i}\right] = \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]$ for each τ_i. (5) Exchange the order between the outer expectation and the limit and divide by n; then use the definition of the renewal distribution to (6) substitute the infinite sum $\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]}{n}$ with $\int \mathbb{E}^S\left[\int_0^\tau g(S_t)dt\right] dR(S)$ and we reach the first result.

$$\begin{aligned}
\int g(S)dF(S) \quad &=^{(1)} \quad \lim_{T\to\infty} \frac{\int_0^T g(S_t)dt}{T} =^{(2)} \lim_{n\to\infty} \frac{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} g(S_t)dt}{\sum_{i=1}^n \int_{\tau_i}^{\tau_{i+1}} 1dt} =^{(3)} \lim_{n\to\infty} \frac{\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} g(S_t)dt|\mathcal{F}_{\tau_i}\right]\right]}{\mathbb{E}\left[\sum_{i=1}^n \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}} 1dt|\mathcal{F}_{\tau_i}\right]\right]} \\
&=^{(4)} \quad \frac{\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]\right]}{\lim_{n\to\infty} \mathbb{E}\left[\sum_{i=1}^n \mathbb{E}^{S_\tau}\left[(\tau_1)\right]\right]} =^{(5)} \frac{\mathbb{E}\left[\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_{\tau_i}}\left[\int_0^{\tau_1} g(S_t)dt\right]}{n}\right]}{\mathbb{E}\left[\lim_{n\to\infty} \frac{\sum_{i=1}^n \mathbb{E}^{S_\tau}\left[(\tau_1)\right]}{n}\right]} \\
&=^{(6)} \quad \frac{\mathbb{E}\left[\int \mathbb{E}^S\left[\int_0^\tau g(S_t)dt\right] dR(S)\right]}{\mathbb{E}\left[\mathbb{E}^S\left[\tau\right] dR(S)\right]}
\end{aligned}$$

If the renewal distribution has a mass point at $\hat{S}$, then the last expression simplifies to:

$$\frac{\mathbb{E}\left[\int \mathbb{E}^S\left[\int_0^\tau g(S_t)dt\right] dR(S)\right]}{\mathbb{E}\left[\mathbb{E}^S\left[\tau\right] dR(S)\right]} = \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau g(S_t)dt\right]}{\mathbb{E}^{\hat{S}}\left[\tau\right]}$$

□

B Appendix: Proofs

Proposition 1. *Assume that:*

- *The uncontrolled state follows $d\tilde{x}_t = \nu dt + \sigma dW_t$, with W_t a Wiener process;*
- $\left(\left\{\int_0^t x_s^m s^n \ dW_s\right\}_t, \tau\right)$ *are a well-defined stopping processes for any m and $n = 0, 1$; and*
- *The moments of adjustment size can be decomposed as follows:*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x} - \Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x} - \Delta x + x)^m\right],$$

1. ***Aggregation:*** *To a first order, the CIR is given by*

$$\mathcal{A}_m(\delta) = \delta \times \left(\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0\right) + o(\delta^2) \tag{B.4}$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{B.5}$$

$$\Gamma_m = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) \tag{B.6}$$

$$\Theta_m = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right] \tag{B.7}$$

2. ***Representation for the intensive margin:***

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a] \tag{B.8}$$

3. ***Representation for the extensive margin:***

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad \text{with} \quad \theta_{m,j} \equiv \frac{2}{\sigma^2(m+1)}\sum_{k\geq j}^{\infty}\frac{\hat{x}^{k-j}}{k!j!}\left[\frac{d^{k+1}g_{m+2}(0)}{dx^{k+1}}/m + 2 - \frac{d^k g_{m+1}(0)}{dx^k}\right]. \tag{B.9}$$

   - *If $\tau|x_t \sim \tau$, $g_m(x) = \theta(m,j) = 0$ for all m, i.*

4. ***Observation:*** *The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1 - \mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau, \Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x} \tag{B.10}$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \tag{B.11}$$

$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\,(\hat{x} - \Delta x)^{m+1}\right] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{B.12}$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

The proof is divided into 4 Lemmas for clarity.

Lemma 1. *[Aggregation] To a first order, the transitional dynamics of the m-th moment are given by*

$$\mathcal{A}_m(\delta) = \delta \times (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0) + o(\delta^2) \tag{B.13}$$

where the intensive and extensive margin are given by

$$\mathcal{Z}_m = \Theta_m + \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1} \tag{B.14}$$

$$\Gamma_m = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Gamma(x_t) = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) \tag{B.15}$$

$$\Theta_m = \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]} \quad ; \quad \varphi_m^\Theta(S_t) = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right] \tag{B.16}$$

Proof. The proof consists of 6 steps.

Characterization of CIR as the recursive problem of a representative agent. Fix an $m \in \mathbb{N}$. Start from the CIR's definition:

$$\mathcal{A}_m(\delta) = \mathbb{E}\left[\int_0^\infty \left(x_t(\omega|\delta)^m - \mathcal{M}_m[x]\right)dt\right], \tag{B.17}$$

where the expectation is taken across agents ω. Let $\{\tau_i\}_{i=1}^\infty$ be the sequence of stopping times after the arrival of the perturbation. In (1), we write the CIR as the cumulative deviations between time $t = 0$ and the first stopping time τ_1 plus the sum of deviations between all future stopping times. In (2), we use the Law of Iterated Expectations to condition on the information set $\mathcal{F}_{\tau_i}$. In (3), we use the Strong Markov Property of the Brownian motion, the assumption of homogenous resets and that $\hat{x}$ is independent of δ for $i \geq 1$ to change the conditioning from $x_{\tau_i+h}|\mathcal{F}_{\tau_i}$ to $x_h|\hat{x}$ and write the problem recursively. In (4), we show that *every element* inside the infinite sum is equal to zero. For this purpose, recall the relationship between ergodic moments and expected duration derived in Auxiliary Theorem 2, $\mathcal{M}_m[x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t(\delta|\omega)^m\right]/\mathbb{E}^{\hat{x}}[\tau]$, and thus we are left with the simple expression in the fourth line (we also relabel τ_1 as τ):

$$\begin{aligned}
\mathcal{A}_m(\delta) \quad &=^{(1)} \quad \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt + \sum_{i=1}^\infty \int_{\tau_i}^{\tau_{i+1}}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\right] \\
&=^{(2)} \quad \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt + \sum_{i=1}^\infty \mathbb{E}\left[\int_{\tau_i}^{\tau_{i+1}}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\Big|\mathcal{F}_{\tau_i}\right]\right] \\
&=^{(3)} \quad \mathbb{E}\left[\int_0^{\tau_1}\left(x_t(\delta|\omega)^m - \mathcal{M}_m[x]\right)dt\right] + \mathbb{E}\left[\sum_{i=1}^\infty \underbrace{\mathbb{E}\left[\int_0^\tau x_t(\delta|\omega)^m\, dt\Big|\hat{x}\right] - \mathcal{M}_m[x]\mathbb{E}[\tau|\hat{x}]}_{=0}\right] \\
&=^{(4)} \quad \mathbb{E}\left[\int_0^\tau \left(x_t(\delta|\omega)^m\right)dt\right] - \mathcal{M}_m[x]\mathbb{E}^x[\tau].
\end{aligned}$$

As a final step, define the following value function conditional on a particular initial condition x:

$$v^m(x) \equiv \mathbb{E}^x\left[\int_0^\tau x_t(\omega)^m dt\right] - \mathcal{M}_m[x]\mathbb{E}^x[\tau], \tag{B.18}$$

and notice that $\mathcal{A}_m(\delta)$ is equal to the average of $v^m(x)$ across all initial conditions after the perturbation, given by the shift in the ergodic distribution ($F_0(x) = F(x - \delta)$):

$$\mathcal{A}_m(\delta) = \int v^m(x)dF(x - \delta). \tag{B.19}$$

2. State's support. Since Brownian motions are continuous in t, and initial conditions are identical across agents (by the assumption of homogeneous resets), the ergodic set is connected. Thus, the support of x is given by an interval $[\underline{x}, \overline{x}]$.

3. Taylor approximation to $\mathcal{A}_m(\delta)$ and decomposition into two terms. We do a first order Taylor approximation of $\mathcal{A}_m(\delta)$ around zero: $\mathcal{A}_m(\delta) = \mathcal{A}_m(0) + \mathcal{A}_m'(0)\delta$. Since $\mathcal{A}_m(0) = 0$ by definition, we have that: $\mathcal{A}_m(\delta) = \delta\mathcal{A}_m'(0)$, which we now characterize. Start from the representation in (B.19), expressed in terms of the marginal density of x:

$$\mathcal{A}_m(\delta) = \int \tilde{v}^m(x)f(x - \delta)dx. \tag{B.20}$$

The derivative with respect to δ, at $\delta = 0$, is given by:

$$\begin{aligned}\mathcal{A}_m'(0) &= \frac{\partial}{\partial \delta}\int \tilde{v}^m(x) f(x-\delta)dx\bigg|_{\delta=0} = -\int \tilde{v}^m(x) f'(x)dx = -\,\tilde{v}^m(x)f(x)|_{\underline{x}}^{\overline{x}} + \int \frac{d}{dx}\tilde{v}^m(x)f(x)dx \\ &= \int \frac{d}{dx}\tilde{v}^m(x)f(x)dx\end{aligned}$$

where in the third equality we do integration by parts, and in the fourth equality we use the result that there are is no mass at the endpoints (or $Pr^{x=\underline{x}}[\tau = 0] = Pr^{x=\overline{x}}[\tau = 0] = 1$). The previous expression says that the effect of the perturbation is equivalent to the changes in the stopping time problem of one agent when her initial conditions change (derivative of v^m with respect to x), averaged across all the possible initial conditions (the steady state distribution). In turn, as we show next, changes in the stopping time problem are reflected by alterations in the state paths and by shifts in duration.

From v^m's definition in (B.18), take its derivative with respect to initial conditions and substitute it back into $\mathcal{A}_m'(0)$

$$\mathcal{A}_m'(0) = \int \frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t(\omega)^m dt\right] dF(x) - \mathcal{M}_m[x]\int \frac{\partial \mathbb{E}^S[\tau]}{\partial x} dF(x)$$

Lastly, by adding and subtracting the term $\int \mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] dF(S)$, we re-express $\mathcal{A}_m'(0)$ as the sum of three terms Γ_m, Θ_m, and Θ_0 defined in the brackets.

$$\mathcal{A}_m'(0) = \underbrace{\int \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] dF(x)}_{\mathcal{B}_m} - \mathcal{M}_m[x]\underbrace{\int \frac{\partial \mathbb{E}^x[\tau]}{\partial x} dF(x)}_{\Theta_0} + \underbrace{\int\left(\frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]\right) dF(x)}_{\mathcal{C}_m}. \quad \text{(B.21)}$$

Now we further characterize each of these terms. Note that for various extensions, the proof up to this point is exactly the same. The results change from this point forward as we make use of the particular stochastic process for the uncontrolled state.

4. Characterize Γ_m. Since $x_t = x + \nu t + \sigma W_t$, for all $t \le \tau$ we have that

$$\Gamma_m \equiv \int \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] dF(x) = \int \mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right] dF(x).$$

Applying Itō's Lemma to x_t^m we have $dx_t^{m+1} = \nu m x_t^{m-1}dt\sigma m x_t^{m-1}dW_t + \frac{\sigma^2}{2}m(m-1)x_t^{m-1}dt$, and integrating both sides from 0 to τ and taking expectations with initial condition S we get

$$\mathbb{E}^x\left[x_\tau^m\right] - x^m = m\Big\{\sigma\underbrace{\mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dW_t\right]}_{=0 \text{ by OST}} + \nu\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right] + \frac{\sigma^2(m-1)}{2}\mathbb{E}^x\left[\int_0^\tau m x_t^{m-2}dt\right]\Big\}.$$

Given that $\int_0^t x_s^m dW_t$ is a martingale with zero initial condition and it is well-defined by assumption, we apply the Optional Sampling Theorem (OST) to conclude that $\mathbb{E}^x\left[\int_0^\tau x_t^m dW_t\right] = 0$. Solve for $\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right]$ and recognizing φ_m^Γ:

$$\mathbb{E}^x\left[\int_0^\tau m x_t^{m-1}dt\right] = \underbrace{\frac{\mathbb{E}^x\left[x_\tau^m\right] - x_t^m}{\nu}}_{\varphi_m^\Gamma(x_t)} - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau (m-1)x_t^{m-2}dt\right].$$

Integrating both sides across all initial conditions, using the definition of Γ_m in (B.6), and recognizing $\mathcal{B}_m$ and $\mathcal{B}_{m-1}$ we get:

$$\mathcal{B}_m = \Gamma_m - \frac{\sigma^2 m}{2\nu}\mathcal{B}_{m-1}, \qquad \Gamma_0 = 0, \quad \text{(B.22)}$$

where we used the Auxiliary Theorem 2, exchanging the ergodic distribution for the local occupancy measure.

5. Characterize Θ_m. With similar steps as in the previous point, we characterize Θ_m as follows.

$$\Theta_m \equiv \int \underbrace{\frac{\partial}{\partial x}\mathbb{E}^S\left[\int_0^\tau x_t^m dt\right]}_{A} - \underbrace{\mathbb{E}^S\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right]}_{B} dF(S).$$

First we get an expression for the term A. Applying Itō's Lemma to x_t^{m+1} we have $dx_t^{m+1} = (m+1)\nu x_t^m dt + \sigma(m+1)x_t^{m+1}dW_t + \frac{\sigma^2}{2}(m+1)m)x_t^m dt$. Integrating both sides from 0 to τ, taking expectations with initial condition S, using the OST, and rearranging we get: $\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu}\left(\mathbb{E}^S\left[x_\tau^{m+1}\right]/(m+1) - x_t^{m+1}\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]$, and its derivative with respect to initial condition:

$$A \equiv \frac{\partial}{\partial x}\mathbb{E}^x\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x}$$

Now, for the term B, recall from the characterization of Γ_m that

$$B \equiv \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right) - \frac{\sigma^2 m}{2\nu}\mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right].$$

Subtract the equations for A and B and simplify to obtain:

$$\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^m dt\right]}{\partial x} - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^m}{\partial x}dt\right] = \underbrace{\frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}\right]/(m+1)}{\partial x} - \mathbb{E}^S\left[x_\tau^m\right]\right)}_{\varphi_m^\Theta(x_t)} - \frac{\sigma^2 m}{2\nu}\left\{\frac{\partial \mathbb{E}^x\left[\int_0^\tau x_t^{m-1}dt\right]}{\partial x} - \mathbb{E}^x\left[\int_0^\tau \frac{\partial x_t^{m-1}}{\partial x}dt\right]\right\}$$

Integrating with the ergodic distribution and using the definition of Θ_m in (D.10) and recognizing $\mathcal{C}_m$ and $\mathcal{C}_{m-1}$ we get:

$$\mathcal{C}_m = \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{C}_{m-1}, \qquad \mathcal{C}_{-1} = 0. \tag{B.23}$$

Define $\mathcal{Z}_m \equiv \mathcal{B}_m + \mathcal{C}_m$, which implies $\mathcal{Z}_m = \Gamma_m + \Theta_m - \frac{\sigma^2 m}{2\nu}\mathcal{Z}_{m-1}$. Combine the results in (D.16), (D.17) and (D.20) to obtain (D.7): $\mathcal{A}_m'(0) = (\mathcal{Z}_m - \mathcal{M}_m[x]\Theta_0)$.

6. Characterize Θ_0. We corroborate that the expression $\int \frac{\partial \mathbb{E}^x[\tau]}{\partial x}dF(x)$ is equal to Θ_0. By the OST, we have $\mathbb{E}^x[x_\tau] - x = \nu\mathbb{E}^x[\tau]$. Thus $\frac{\partial \mathbb{E}^x[\tau]}{\partial x} = \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1\right]$. Substituting and using Auxiliary Theorem 2 we recover the expression for Θ_0 in the definition of Θ_m:

$$\Theta_0 \equiv \int \frac{\partial \mathbb{E}^x[\tau]}{\partial x}dF(x) = \int \frac{1}{\nu}\left[\frac{\partial \mathbb{E}^x[x_\tau]}{\partial x} - 1\right]dF(x)$$

□

Lemma 2. *[Representation for intensive margin] The intensive margin Γ_m defined as*

$$\Gamma_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi^\Gamma(x_t)dt\right]}{\mathbb{E}^{\hat{x}}\left[\tau\right]}, \qquad \textit{with} \qquad \varphi_m^\Gamma(x_t) = \frac{1}{\nu}\left(\mathbb{E}^x\left[x_\tau^m\right] - x_t^m\right).$$

can be represented as a function of steady state moments as:

$$\Gamma_m = m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a].$$

Proof. Start in (1) from the definition of Γ_m and $\varphi_m^\Gamma(S)$, then (2) exchange the time integral with the expectation conditional on adjustment $\mathbb{E}^{\hat{x}}[\cdot]$, which introduces an indicator $\mathbb{1}_{\{t\leq\tau\}}$. Use the law of iterated expectations in (3) to condition on the set $\{t \leq \tau\}$.

$$\begin{aligned}
\nu\mathbb{E}^{\hat{x}}[\tau]\Gamma_m &=^{(1)} \mathbb{E}^{\hat{x}}\left[\int_0^\tau \mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m dt\right] =^{(2)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\left(\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\right]dt \\
&=^{(3)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\left(\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\middle| t\leq\tau\right\}\right]dt =^{(4)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\mathbb{E}^{x_t}\left[x_\tau^m\right] - x_t^m\middle| t\leq\tau\right\}\right]dt \\
&=^{(5)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left[x_\tau^m - x_t^m\middle| t\leq\tau\right]\right]dt =^{(6)} \int_0^\infty \mathbb{E}^{\hat{x}}\left[\mathbb{E}\left\{\left(x_\tau^m - x_t^m\right)\mathbb{1}_{\{t\leq\tau\}}\middle| t\leq\tau\right\}\right]dt \\
&=^{(7)} \mathbb{E}^{\hat{x}}\left[\int_0^\tau \left(x_\tau^m - x_t^m\right)dt\right] =^{(8)} \mathbb{E}^{\hat{x}}\left[x_\tau^m\int_0^\tau dt\right] - \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]
\end{aligned} \tag{B.24}$$

We now characterize $\mathbb{E}^{\hat{x}}\left[x_\tau^{m+1}\int_0^\tau dt\right]$. Applying Ito's lemma followed by the OST to $Y_t^m \equiv x_t^m\int_0^t ds$

$$dY_t^m = x_t^m dt + \mathbb{1}_{\{m\geq 1\}}\nu m x_t^{m-1}\int_0^t ds dt + \mathbb{1}_{\{m\geq 1\}}\sigma m x_t^{m-1}\int_0^t ds dW_t + \mathbb{1}_{\{m\geq 2\}}\frac{m(m-1)\sigma^2}{2}x_t^{m-2}\int_0^t ds dt \tag{B.25}$$

$$\mathbb{E}^{\hat{x}}\left[Y_\tau^m\right] = \mathbb{E}^{\hat{x}}\left[x_\tau^m\int_0^\tau dt\right] = \underbrace{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]}_{\mathcal{M}_m[x]\mathbb{E}^{\hat{x}}[\tau]} + \mathbb{1}_{\{m\geq 1\}}\nu m\underbrace{\frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}t dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}}_{\mathcal{M}_{m-1,1}[x,a]} + \mathbb{1}_{\{m\geq 2\}}\frac{m(m-1)\sigma^2}{2}\underbrace{\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-2}t dt\right]}_{\mathcal{M}_{m-2,1}[x,a]} \tag{B.26}$$

Using equations (B.24) and (B.26), we have that

$$\Gamma_m = \mathbb{1}_{\{m\geq 1\}} m\mathcal{M}_{m-1,1}[x,a] + \frac{\mathbb{1}_{\{m\geq 2\}}\sigma^2 m(m-1)}{2\nu}\mathcal{M}_{m-2,1}[x,a]$$

□

Lemma 3. *[Representation for extensive margin] Assume the moments of the adjustment size can be written as:*

$$g_m(x) = \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x + x)^m\right] \tag{B.27}$$

Then the extensive margin given by $\Theta_m \equiv \frac{\mathbb{E}^{\hat{x}}\left[\int_0^\tau \varphi_m^\Theta(x_t)dt\right]}{\mathbb{E}^{\hat{x}}[\tau]}$ *with* $\varphi_m^\Theta(x) \equiv \frac{1}{\nu}\left(\frac{\partial \mathbb{E}^x\left[x_\tau^{m+1}/(m+1)\right]}{\partial x} - \mathbb{E}^x\left[x_\tau^m\right]\right)$ *can be represented as a function of steady state moments as follows:*

$$\Theta_m = \sum_{j=0}^{\infty}\theta_{m,j}\mathcal{M}_j[x] \quad with \quad \theta_{m,j} = \sum_{k\geq j}^{\infty}\frac{1}{\nu}\frac{\left[\frac{d^{k+1}g_{m+1}(x)}{dx^{k+1}}/m+1 - \frac{d^k g_m(x)}{dx^k}\right]\Big|_{x=0}\hat{x}^{k-j}}{k!j!}. \tag{B.28}$$

- *If* $\tau|x_t \sim \tau$, $g_m(x) = \theta(m,j) = 0$ *for all* m, i.

Proof. Using a change of variable in assumption (B.27), we have that :

$$\mathbb{E}^y\left[x_\tau^m\right] = g_m(y-\hat{x}) + \mathbb{E}^{\hat{x}}\left[(y-\hat{x}+x_\tau)^m\right]. \tag{B.29}$$

Using the previous equation we have that

$$\begin{aligned}
\Theta_m &= \frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left(\frac{\partial \mathbb{E}^S[x_\tau^{m+1}/(m+1)]}{\partial x} - \mathbb{E}^S[x_\tau^m]\right)dt\right]}{\nu\mathbb{E}^{\hat{S}}[\tau]} && \text{(B.30)}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^{m+1}\right]\right] - \left[g_m(y-\hat{x}) + \mathbb{E}^{\hat{x},S^{-x}}\left[(y-\hat{x}+x_\tau)^m\right]\right]\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\frac{dg_{m+1}(y-\hat{x})}{dy}/(m+1) - g_m(y-\hat{x})\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau \left[\sum_{j=0}^\infty \frac{d^j}{j!dx^j}\left[\frac{dg_{m+1}(x)}{dx}/(m+1) - g_m(x)\right]_{x=0}(y-\hat{x})^j\right]dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^{\infty}\sum_{z=0}^{j}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\frac{\mathbb{E}^{\hat{S}}\left[\int_0^\tau x_t^{j-z}dt\right]}{\mathbb{E}^{\hat{S}}[\tau]}\\
&= \frac{1}{\nu}\sum_{j=0}^{\infty}\sum_{z=0}^{j}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{j!}\begin{pmatrix} j \\ z\end{pmatrix}\mathcal{M}_{j-z}[x]\\
&= \sum_{j=0}^{\infty}\sum_{z=0}^{j}\underbrace{\frac{1}{\nu}\frac{\left[\frac{d^{j+1}g_{m+1}(0)}{dx^{j+1}}/m+1 - \frac{d^j g_m(0)}{dx^j}\right]\hat{x}^z}{z!(j-z)!}}_{=\mathcal{H}_{m,j,z}}\mathcal{M}_{j-z}[x]\\
&= \sum_{h=0}^{\infty}\theta_{m,h}\mathcal{M}_h[x], \; with \; \theta_{m,h} = \sum_{k\geq h}^{\infty}\mathcal{H}_{m,k,k-h}. && \text{(B.31)}
\end{aligned}$$

If $\tau|x_t \sim \tau$, then we have that

$$\begin{aligned}
g_m(x) &= \mathbb{E}^{\hat{x}+x}\left[(\hat{x}-\Delta x)^m\right] - \mathbb{E}^{\hat{x}}\left[(\hat{x}-\Delta x+x)^m\right]\\
&= \mathbb{E}\left[(\hat{x}+x+\nu\tau^{x+\hat{x}}+\sigma W_{\tau^{x+\hat{x}}})^m\right] - \mathbb{E}\left[(\hat{x}+x+\nu\tau^{\hat{x}}+\sigma W_{\tau^{\hat{x}}})^m\right]\\
&= \mathbb{E}\left[(\hat{x}+x+\nu\tau+\sigma W_\tau)^m\right] - \mathbb{E}\left[(\hat{x}+x+\nu\tau+\sigma W_\tau)^m\right]\\
&= 0 && \text{(B.32)}
\end{aligned}$$

□

Lemma 4. *[Observation] The reset state $\hat{x}$ and structural parameters (ν, σ) are recovered as*

$$\hat{x} = \mathbb{E}[\Delta x]\left(\frac{1 - \mathbb{CV}^2[\tau]}{2}\right) + \frac{Cov[\tau, \Delta x]}{\mathbb{E}[\tau]}, \qquad \nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}, \qquad \sigma^2 = \frac{\mathbb{E}[\Delta x^2]}{\mathbb{E}[\tau]} + 2\nu\hat{x}$$

and the ergodic moments are recovered as:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x],$$
$$\mathcal{M}_{m,1}[x,a] = \frac{\mathbb{E}\left[\tau/\mathbb{E}[\tau]\,(\hat{x} - \Delta x)^{m+1}\right] - \mathcal{M}_{m+1}[x]}{\nu(m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a]$$

with initial conditions $\mathcal{M}_1[x] = 0$ and $\mathcal{M}_{0,1}[x,a] = \frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

Proof. The basis of the proof is the application of Itō's lemma and the OST.

- **Average adjustment size.** We show that $\frac{\mathbb{E}^{\hat{x}}[\Delta x]}{\mathbb{E}^{\hat{x}}[\tau]} = -\nu$. From the law of motion $x_t = \hat{x} + \nu t + \sigma W_t$, we find the following equalities: $\sigma W_\tau = -\nu\tau + x_\tau - \hat{x} = -\nu\tau - \Delta x$. Taking expectations on both sides, we have $\sigma\mathbb{E}[W_\tau] = -\nu\mathbb{E}^{\hat{x}}[\tau] - \mathbb{E}[\Delta x]$. Since W_τ is a martingale, $\mathbb{E}[W_\tau] = W_0 = 0$ by the OST. Therefore, $\nu = -\frac{\mathbb{E}[\Delta x]}{\mathbb{E}[\tau]}$ as well.
- **Observation of fundamental volatility:** For characterizing σ define $Y_t = x_t - \nu t$ with initial condition $Y_0 = \hat{x}$. With similar steps as before we have that

$$\sigma^2 = \frac{\mathbb{E}^{\hat{x}}\left[\Delta Y_\tau^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}\left[(x_\tau - \nu\tau - x_\tau + \hat{x})^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} = \frac{\mathbb{E}^{\hat{x}}\left[(\nu\tau + \Delta x)^2\right]}{\mathbb{E}^{\hat{x}}[\tau]} \tag{B.33}$$

  or equivalently

$$\sigma^2 = \frac{\mathbb{E}^{\hat{x}}[\Delta x^2]}{\mathbb{E}^{\hat{x}}[\tau]} + 2\nu\left(\frac{\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} + \nu\frac{\mathbb{E}^{\hat{x}}[\tau^2]}{\mathbb{E}^{\hat{x}}[\tau]}\right)$$

  Applying the formula (B.35) we have the result.
- **Observation of reset state:** For the reset state $\hat{x}$, we apply Itō's lemma to x_t^2 to obtain $d(x_t^2) = 2x_t dx_t + (dx_t)^2 = \left(2\nu x_t + \sigma^2\right)dt + 2\sigma x_t dB_t$. Using the OST $\mathbb{E}^{\hat{x}}[\int_0^\tau x_s dB_s] = 0$. Moreover, given that $\mathbb{E}^{\hat{x}}[\int_0^\tau x_s ds] = \mathcal{M}_1[x]\mathbb{E}^{\hat{x}}[\tau] = 0$, we have that

$$\mathbb{E}^{\hat{x}}[x_\tau^2] = \hat{x}^2 + \sigma^2\mathbb{E}^{\hat{x}}[\tau] \tag{B.34}$$

  Completing squares

$$\mathbb{E}^{\hat{x}}[x_\tau^2] = \mathbb{E}^{\hat{x}}[(\hat{x} - (\hat{x} - x_\tau))^2] = \mathbb{E}^{\hat{x}}[\Delta x^2] - 2\hat{x}\mathbb{E}^{\hat{x}}[\Delta x] + (\hat{x})^2$$

  Therefore

$$\begin{aligned}\hat{x} &= \frac{1}{2\mathbb{E}^{\hat{x}}[\Delta x]}\left[\mathbb{E}^{\hat{x}}[\Delta x^2] - \sigma^2\mathbb{E}^{\hat{x}}[\tau]\right] \\ &= \frac{1}{2\mathbb{E}^{\hat{x}}[\Delta x]}\left[\mathbb{E}^{\hat{x}}[\Delta x^2] - \left(\mathbb{E}^{\hat{x}}[\Delta x^2] + 2\frac{\mathbb{E}^{\hat{x}}[\Delta x]\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} + \frac{\mathbb{E}^{\hat{x}}[\Delta x]^2\mathbb{E}^{\hat{x}}[\tau^2]}{\mathbb{E}^{\hat{x}}[\tau]^2}\right)\right] \\ &= \frac{\mathbb{E}^{\hat{x}}[\Delta x\tau]}{\mathbb{E}^{\hat{x}}[\tau]} - \frac{\mathbb{E}^{\hat{x}}[\Delta x]\mathbb{E}^{\hat{x}}[\tau^2]}{2\mathbb{E}^{\hat{x}}[\tau]^2}.\end{aligned} \tag{B.35}$$

  Applying the formula for the covariance $\mathbb{E}^{\hat{x}}[\tau\Delta x] + \mathbb{E}^{\hat{x}}[\tau]\mathbb{E}^{\hat{x}}[\Delta x] = Cov[\tau, \Delta x]$ and coefficient of variation square $\mathbb{CV}^2[\tau] = \frac{\mathbb{V}^{\hat{x}}[\tau]}{\mathbb{E}^{\hat{x}}[\tau]^2}$, we have the result.
- **Observation of ergodic moments with respect to the state:** For observability of ergodic moments of x, apply Itō's lemma to x^{m+1} and get $dx_t^{m+1} = (m+1)x_t^m\nu dt + (m+1)x_t^m\sigma dW_t + \frac{\sigma^2}{2}m(m+1)x_t^{m-1}dt$. Integrating from 0 to τ, using the OST to eliminate martingales, and rearranging:

$$\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right] = \frac{1}{\nu(m+1)}\left(\mathbb{E}^{\hat{x}}[x_\tau^{m+1}] - \hat{x}^{m+1}\right) - \frac{\sigma^2}{2\nu}m\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}dt\right] \tag{B.36}$$

  Substituting the equivalences $\mathcal{M}_m[x] = \mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m dt\right]/\mathbb{E}^{\hat{x}}[\tau]$ and $\mathbb{E}^{\hat{x}}[\Delta x] = -\nu\mathbb{E}^{\hat{x}}[\tau]$ yields:

$$\mathcal{M}_m[x] = \frac{\hat{x}^{m+1} - \mathbb{E}^{\hat{x}}[(\hat{x} - \Delta x)^{m+1}]}{\mathbb{E}^{\hat{x}}[\Delta x](m+1)} - \frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1}[x], \qquad \mathcal{M}_1[x] = 0 \tag{B.37}$$

- **Observation of ergodic moments with respect to the joint moments of state and age:** For observability of ergodic moments of $x^m a$, where a stand for the duration of the last action, we use Itō's lemma and the OST on

$x_t^{m+1}t$:

$$\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]=\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m+1}dt\right]+(m+1)\nu\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^m t dt\right]+\frac{\sigma^2 m(m+1)}{2}\mathbb{E}^{\hat{x}}\left[\int_0^\tau x_t^{m-1}tdt\right] \tag{B.38}$$

and therefore

$$\mathcal{M}_{m,1}[x,a]=\frac{\mathbb{E}^{\hat{x}}\left[\tau\left(\hat{x}-\Delta x\right)^{m+1}\right]}{\nu(m+1)\mathbb{E}^{\hat{x}}[\tau]}-\frac{\mathcal{M}_{m+1}[x]}{\nu(m+1)}-\frac{\sigma^2 m}{2\nu}\mathcal{M}_{m-1,1}[x,a] \tag{B.39}$$

with initial condition $\mathcal{M}_{0,1}[x,a]=\frac{\mathbb{E}[\tau^2]}{2\mathbb{E}[\tau]}$.

□