

## **Stationary Rational Bubbles in Non-Linear Business Cycle Models**

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October 16, 2019

This paper shows that multiple stationary equilibria can exist in standard *non-linear* DSGE models, even when the *linearized* versions of those models have a unique solution. Thus, the non-linear models can exhibit stationary fluctuations, even if there are no shocks to productivity, preferences or other ‘fundamentals’. In the equilibria considered here, the economy may temporarily diverge from the steady state, before abruptly reverting towards the steady state. In contrast to rational bubbles in linear models (Blanchard (1979)), the rational bubbles in non-linear models considered here are stationary--their path does not explode. Numerical simulations suggest that non-linear DSGE models driven solely by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts.

Applications to both closed and open economies are analyzed.

JEL codes: E1,E3,F3,F4, C6

Keywords: stationary bubbles, non-linear dynamic general equilibrium models, business cycles in closed and open economies.

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I thank Sumru Altug for an insightful discussion. Thanks for useful comments are also due to Guido Ascari, Mick Devereux, Jordi Gali, Matthew Knowles, Cédric Tille, Gauthier Vermandel, Raf Wouters, and to workshop participants at National Bank of Belgium, Honk Kong University of Science and Technology, St. Andrews University, University of Navarra, Swiss National Bank and at CEPR-MMCN and DYNARE conferences.

## 1. Introduction

Linearized dynamic stochastic general equilibrium (DSGE) models with a unique non-explosive solution are the workhorses of modern quantitative macroeconomics (e.g., King and Rebelo, 1999; Kollmann et al., 2011a,b). This paper shows that multiple stationary equilibria may exist in standard non-linear DSGE models, even when the linearized versions of those models have a unique solution. Thus, the non-linear models can exhibit stationary fluctuations of endogenous variables, even if there are no shocks to productivity, preferences or other ‘fundamentals’. The Blanchard and Kahn (1980) conditions for the uniqueness of stable solutions to linear rational expectations models are, hence, irrelevant for non-linear models. In the equilibria considered here, the economy may temporarily diverge from the steady state, but with some (exogenous) probability the economy reverts towards the steady state later. These dynamics are consistent with rational expectations. Importantly, the ‘rational bubbles’ studied here are stationary.

The multiple equilibria identified here have similarities and important differences, compared to the rational bubbles in linearized models analyzed by Blanchard (1979) (see also Blanchard and Watson (1982)). Like these paper, the study here focuses on models whose linearized versions have a unique non-explosive equilibrium. Like the Blanchard bubbles, the rational bubbles in non-linear models discussed here imply that endogenous variables can diverge from the steady state, before abruptly reverting towards the steady state.

The key difference is that the bubbles in non-linear models considered here are stationary, while Blanchard’s bubbles in linearized models exhibit explosive *expected* trajectories that tend to  $\pm\infty$ . This feature greatly limits the appeal of the Blanchard bubbles for DSGE models. In standard DSGE models, the numerical accuracy of linear approximations deteriorates sharply when the state variables deviate substantially from the point of approximation—in particular, non-negativity constraints on endogenous variables and other technological feasibility restrictions may be violated. For example, in a standard closed economy Real Business Cycle (RBC) model with decreasing returns to capital and capital depreciation, an explosive trajectory of the capital stock and output is infeasible. A linear model approximation is, thus, not suitable for studying rational bubbles.

By contrast, the non-linear model analysis here takes non-negativity constraints and decreasing returns into account. Decreasing returns and risk aversion generate stabilizing forces that prevent explosive trajectories. While rational bubbles in linearized models can be positive or

negative, I find that rational bubbles in standard non-linear models are generally one-sided; e.g., they tend to predict over-accumulation of capital, but not under-accumulation. I show that rational bubbles in non-linear models can induce fluctuations that remain close to deterministic steady state most of the time, and the unconditional mean of endogenous variables can thus be close to the deterministic steady state. Numerical simulations suggest that non-linear DSGE models driven solely by stationary bubbles can generate persistent fluctuations of real activity and capture key business cycle stylized facts.

A large literature that studied linearized DSGE models with stationary sunspot equilibria (i.e. multiple stationary equilibria). These equilibria exist (in linearized models) if the number of eigenvalues outside the unit circle is less than the number of non-predetermined variables (Blanchard and Kahn (1980), Prop. 3).<sup>1</sup> By contrast, the paper here focuses on models for which the number of eigenvalues (of the linearized state-space form) equals the number of non-predetermined variables, so that the linearized structure has a unique non-explosive solution (Blanchard and Kahn (1980), Prop. 1). Linearized models may exhibit stationary sunspot equilibria if increasing returns or externalities (e.g., Schmitt-Grohé (1997), Benhabib and Farmer (1999)), financial frictions (e.g., Martin and Ventura (2018)) or overlapping generations (OLG) population structures (e.g., Woodford (1986), Galí (2018)) are assumed. The specific assumptions and calibrations that deliver sunspot equilibria in linearized models can be debatable.<sup>2</sup> By contrast, the paper here shows that very standard DSGE model (without the features that were just mentioned) whose *linearized* versions have a *unique* stationary solution can have multiple equilibria, if non-linear effects are considered.

Holden (2016a,b) shows that multiple stationary equilibria can emerge when occasionally binding constraints (such as borrowing constraints or non-negativity constraints on endogenous variables) are integrated into an otherwise linear model (where that linear model has a unique stable solution when the occasionally binding constraints are ignored). By contrast, the analysis here considers *fully* non-linear models: all model equations are non-linear, and all relevant occasionally binding constraints are imposed. The model solutions considered here are globally accurate (up to machine accuracy). The multiple equilibria describe here have a ‘bubbly’ dynamics that differs from the dynamics highlighted by Holden (2016a,b).

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<sup>1</sup> See Taylor (1977) for an early example of a model with sunspots, due to the presence of ‘too many’ stable roots.

<sup>2</sup> E.g., increasing returns/externalities need to be sufficiently strong, in OLG models the steady state interest rate has to be smaller than the trend growth rate ( $r < g$ ) etc.

The standard DSGE models discussed in this paper are usually presented as structures with an optimizing infinitely-lived representative household. The set of optimality conditions of that household's decision problem include a transversality condition (TVC) that stipulates that the value of capital has to be zero, at infinity. The TVC (in conjunction with Euler equations and static efficiency conditions) implies a unique equilibrium, in standard DSGE models. When TVCs do not hold, the economy is 'dynamically inefficient' (e.g., Abel et al. (1989)).

I do not impose the TVC in this paper. My goal is to show that stationary rational bubbles can exist in standard non-linear DSGE models. Note that explosive bubbles in linear models (Blanchard (1979)) likewise violate the TVC. A possible justification for disregarding the TVC is that there is not TVC because agents are finitely-lived. I show that there exists an overlapping generations (OLG) structure with finitely-lived households that deliver the same set Euler equations and static efficiency conditions as the standard DSGE models discussed here. However, the TVC does not apply in that OLG structure. The key features of this OLG structure are complete risk sharing among generations (alive in periods  $t$  and  $t+1$ ) and the assumption that newborn agents receive a wealth endowment such that consumption by newborns represents a time-invariant share of aggregate consumption (under log utility, this requires that the wealth endowments of newborns is a time-invariant fraction of aggregate wealth). The linearized version of the OLG structure presented here has a unique non-explosive solution, but the non-linear structure has multiple bubble equilibria. Another motivation for disregarding the TVC is that detecting TVC violations may be very difficult in non-linear stochastic economies. Households may thus lack the cognitive/computing power to detect deviations from TVC (see discussion in Blanchard and Watson (1982), Lansing (2010) and Ascari et al. (2019)).

The next Section discusses stationary rational bubbles in a one-sector version of the Long and Plosser (1983) model, i.e. closed-economy RBC model with log utility and full capital depreciation per period. I provide a detailed discussion of that model, as closed form solutions with bubbles can be derived for that model. Section 3 considers a more realistic non-linear RBC model with incomplete capital depreciation. Section 4 studies stationary rational bubbles in a non-linear two-country RBC model. Section 5 analyzes stationary rational bubbles in a non-linear New-Keynesian DSGE model (nominal rigidities).

## 2. Rational bubbles in the Long-Plosser RBC model

Following Long and Plosser (1986), this Section considers a closed economy inhabited by a household with time-separable life-time utility. The period utility function is logarithmic:  $u(C_t)=\ln(C_t)$ , where  $C_t$  is consumption in period  $t$ . The production function is Cobb-Douglas:

$$Y_t=\theta_t K_t^\alpha, \quad 0<\alpha<1, \quad (1)$$

where  $Y_t, K_t, \theta_t > 0$  are output, capital and exogenous total factor productivity (TFP). For simplicity, I assume that labor hours are constant and normalized at 1 (the next Sections allow for variable hours).<sup>3</sup> The resource constraint is

$$C_t + I_t = Y_t, \quad (2)$$

where  $I_t$  is gross investment. The capital depreciation rate is 100%, so that gross investment equals next period's capital stock:  $I_t = K_{t+1}$ . The household's Euler equation is

$$E_t \beta (C_t / C_{t+1})^\alpha Y_{t+1} / K_{t+1} = 1, \quad (3)$$

where  $0 < \beta < 1$  is the household's subjective discount factor. Substitution of the resource constraint into the Euler equation gives an expectational difference equation in the investment/output ratio  $Z_t \equiv K_{t+1} / Y_t$ :

$$E_t H(Z_{t+1}, Z_t) \equiv E_t \alpha \beta [(1 - Z_t) / (1 - Z_{t+1})] / Z_t = 1. \quad (4)$$

$Z_t = Z_{t+1} = \alpha \beta$  solves (4). This corresponds to the textbook solution of the Long-Plosser model (see, e.g., Blanchard and Fischer (1989)). Under this solution, consumption and investment are constant shares of output:  $C_t = (1 - \alpha \beta) Y_t$ ,  $K_{t+1} = \alpha \beta Y_t$ .

### 2.1. Bubbles in the linearized Long-Plosser model

Linearization of (4) around  $Z = \alpha \beta$  gives:

$$E_t z_{t+1} = \lambda z_t, \quad \text{with } z_t \equiv Z_t - Z \text{ and } \lambda \equiv 1 / (\alpha \beta) > 1. \quad (5)$$

$\lambda$ , the eigenvalue of (5) exceeds unity. The model has one non-predetermined variable ( $z_t$ ).

Thus, the linearized model has a unique non-explosive solution (Blanchard and Kahn (1980), Proposition 1). This solution is given by  $z_t = 0$ , i.e.  $Z_t = \alpha \beta \forall t$ .

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<sup>3</sup> With endogenous hours, and a period utility function that is additively separable in consumption and hours, hours are constant in the no-bubbles solution of the Long-Plosser model, while hours fluctuate in the bubble equilibrium.

Blanchard (1979) pointed out that a linear expectational difference equation of form (5) is also solved by a process  $\{z_t\}$  such that

$$z_{t+1} = [\lambda/(1-\pi)] \cdot z_t \text{ with probability } 1-\pi \text{ and } z_{t+1} = 0 \text{ with probability } \pi \quad (0 < \pi < 1). \quad (6)$$

If there is a **bubble** at date  $t$ , i.e.  $z_t \neq 0$ , then next period the bubble grows with probability  $1-\pi$ ; the bubble bursts with probability  $\pi$ . The larger the bubble, the greater the magnitude of the subsequent ‘correction’. The bubble process (6) implies that after a bubble has burst, a new bubble never arises again (the bubble is ‘self-ending’). As noted by Blanchard (1979), recurrent bubbles obtain if a bursting bubble reverts to a value  $\mu \neq 0$ :  $z_{t+1} = (\lambda z_t - \mu \pi)/(1-\pi)$  with probability  $1-\pi$  and  $z_{t+1} = \mu$  with probability  $\pi$ .

An important feature of bubbles in the linearized model (5) is that the expected path of the bubbles explodes:  $\lim_{s \rightarrow \infty} E_t z_{t+s} = \infty$  when  $z_t > 0$  and  $\lim_{s \rightarrow \infty} E_t z_{t+s} = -\infty$  when  $z_t < 0$ . As discussed in Sect. 1, this feature greatly limits the appeal of the Blanchard (1979) type bubble. Note that, in the Long-Plosser model, the investment/output ratio is bounded by 0 and 1: an infinite investment ratio is not feasible. The linear approximation (on which (5) is based) neglects this constraint. A linear approximation is thus not suitable for studying rational bubbles.

## 2.2. Stationary bubbles in the non-linear Long-Plosser model

I now show that, by contrast to the linearized model, the **non-linear** Long-Plosser model can produce **stationary** bubbles. Note that (4) holds for any process  $\{Z_t\}$  such that

$$\alpha\beta[(1-Z_t)/(1-Z_{t+1})]/Z_t = 1 + \varepsilon_{t+1}, \quad (7)$$

where  $\varepsilon_{t+1}$  is a forecast error with zero conditional mean:  $E_t \varepsilon_{t+1} = 0$ .  $\varepsilon_{t+1}$  reflects unanticipated changes in  $Z_{t+1}$  that are driven by changes in households’ expectations about the future path  $\{Z_{t+s}\}_{s>1}$ . (7) can be written as:

$$Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1}) \equiv 1 - \alpha\beta(1/Z_t - 1)/(1 + \varepsilon_{t+1}). \quad (8)$$

$Z_{t+1}$  is strictly increasing and strictly concave in both  $Z_t$  and in  $\varepsilon_{t+1}$ , for  $\varepsilon_{t+1} > -1$ . The strict concavity reflects decreasing returns and risk aversion. Fig.1 plots  $Z_{t+1}$  as a function of  $Z_t$ , and that for three values of  $\varepsilon_{t+1}$ :  $\varepsilon_{t+1} = 0$  (thick black line),  $\varepsilon_{t+1} = 0.5$  and  $\varepsilon_{t+1} = -0.5$  (thin dashed

lines). Throughout this Section, I set  $\alpha=0.35$  and  $\beta=0.99$ , so that  $\alpha\beta=0.3465$ ; these parameter values are standard in quarterly business cycle models.

In a deterministic economy,  $\varepsilon_t=0$  holds  $\forall t$ , and the dynamics of the investment/output ratio obeys thus  $Z_{t+1}=\Lambda(Z_t,0)$  (see (8)). Fig. 1 shows that the function  $Z_{t+1}=\Lambda(Z_t,0)$  cuts the 45-degree line at two point:  $Z_t=Z_{t+1}=\alpha\beta$  and  $Z_t=Z_{t+1}=1$ . In a deterministic economy, the slope of the mapping from  $Z_t$  to  $Z_{t+1}$  is  $1/(\alpha\beta)$ , at the steady state  $Z=\alpha\beta$ . In a deterministic economy, a realization  $Z_t<\alpha\beta$  puts the investment ratio on a trajectory that reaches  $Z=0$  in finite time; after  $Z=0$  has been reached, output and consumption are zero indefinitely. By contrast, a realization  $Z_t>\alpha\beta$  initiates a path that converges asymptotically to  $Z=1$  (without ever reaching  $Z=1$ ), in a deterministic economy.

The main contribution of this paper is to show that there exist stationary bubble equilibria. These bubble equilibria do not converge to  $Z=0$  or  $Z=1$ . Thus, consumption and capital are strictly positive in all periods. In what follows, I focus on these stationary (interior) model solutions. When  $Z_t<\alpha\beta$ , then the law of motion (8) implies that the economy can hit a zero-capital corner solution in subsequent periods. I thus restrict attention to solutions for which  $\{Z_t\}$  stays forever in the interval  $[\alpha\beta,1)$ . It is apparent from Fig. 1 that this requires that the support of the distribution of  $\varepsilon_{t+1}$  has to be bounded from below. (8) implies that when  $Z_t \in [\alpha\beta,1)$  holds, then  $Z_{t+1} \in [\alpha\beta,1)$  requires  $\varepsilon_{t+1} \geq -1 + [\alpha\beta/(1-\alpha\beta)] \cdot [1/Z_t - 1] \geq -1$ .<sup>4</sup>

For simplicity, I assume that  $\varepsilon_{t+1}$  only takes two values:  $-\bar{\varepsilon}_t$  and  $\bar{\varepsilon}_t \cdot \pi/(1-\pi)$  with exogenous probabilities  $\pi$  and  $1-\pi$ , respectively, where  $\bar{\varepsilon}_t \in [0,1)$ .  $Z_{t+1}$  then takes these two values with probabilities  $\pi$  and  $1-\pi$ :

$$Z_{t+1}^L \equiv \Lambda(Z_t, -\bar{\varepsilon}_t) \text{ and } Z_{t+1}^H \equiv \Lambda(Z_t, \bar{\varepsilon}_t \pi / (1-\pi)) \text{ with } Z_{t+1}^L \leq Z_{t+1}^H \leq 1. \quad (9)$$

In the spirit of Blanchard (1979), I assume that when an investment ‘crash’ occurs in period  $t+1$ , then the investment/output ratio takes a value that is close the no-bubble investment/output ratio,  $\alpha\beta$ . Specifically, I postulates that  $Z_{t+1}^L = \alpha\beta + \Delta$ , where  $\Delta > 0$  is a small

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<sup>4</sup> The lower bound of  $\varepsilon_{t+1}$  is strictly negative if  $Z_t > \alpha\beta$ , and it is strictly decreasing in  $Z_t$ .

positive constant. A strictly positive value of  $\Delta$  is needed to generate recurrent bubbles.<sup>5</sup> When we set  $Z_{t+1}^L = \alpha\beta + \Delta$ , the first equation in (9) pins down  $-\bar{\varepsilon}_t$ ; substitution into the second equation shown in (9) then determines  $Z_{t+1}^H$ .

Alternatively, note that under the assumed bubble process with  $Z_{t+1}^L = \alpha\beta + \Delta$ , the Euler equation (4) can be expressed as

$$\pi H(\alpha\beta + \Delta, Z_t) + (1 - \pi)H(Z_{t+1}^H, Z_t) = 1. \quad (10)$$

For any  $Z_t \in [\alpha\beta + \Delta, 1)$  there exists a unique value  $Z_{t+1}^H \in [\alpha\beta + \Delta, 1)$  that solves (10).

Consider an economy that starts in period  $t=0$ , with an initial capital stock  $K_0$ . Let  $u_t \in \{0, 1\}$  be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities  $\pi$  and  $1 - \pi$ , respectively. An (interior) **bubble equilibrium** is a sequence of investment/output ratios  $\{Z_t\}_{t \geq 0}$  defined by  $Z_0 \in [\alpha\beta + \Delta, 1)$  and  $Z_{t+1} = Z_{t+1}^L \equiv \alpha\beta + \Delta$  if  $u_{t+1} = 0$  and  $Z_{t+1} = Z_{t+1}^H$  if  $u_{t+1} = 1$ , for  $t \geq 0$ , where  $Z_{t+1}^H$  solves (10).

Note that the investment/output ratio in the initial period,  $Z_0$ , does not obey the recursion that governs investment ratios in subsequent periods. However,  $Z_0 \in [\alpha\beta + \Delta, 1)$  has to hold to ensure that investment/output ratios in all subsequent periods lie in the interval  $[\alpha\beta + \Delta, 1)$ . Given a sequence  $\{Z_t\}_{t \geq 0}$ , the path of capital  $\{K_{t+1}\}_{t \geq 0}$  can be generated recursively (for the given initial capital stock  $K_0$ ) using  $K_{t+1} = Z_{t+1} \theta_t(K_t)^\alpha$  for  $t \geq 0$ .

I now discuss numerical simulations in which I set  $\Delta = 0.01$  and  $\pi = 0.5$ . Panel (a) of Fig. 2 plots  $Z_{t+1}^L, Z_{t+1}^H$  and  $E_t Z_{t+1} = \pi_t Z_{t+1}^L + (1 - \pi_t) Z_{t+1}^H$ , as functions of  $Z_t$ . Also shown in Panel (a) is the value of  $Z_{t+1}$  that would obtain in a deterministic economy ( $\varepsilon_{t+1} = 0$ ):  $Z_{t+1} = \Lambda(Z_t, 0)$ . In the stochastic bubble equilibrium, the investment/output ratio grows between  $t$  and  $t+1$  ( $Z_{t+1} > Z_t$ ) when  $\varepsilon_{t+1} = \bar{\varepsilon}_t \cdot \pi / (1 - \pi) > 0$ ; when  $\varepsilon_{t+1} = -\bar{\varepsilon}_t$ , the investment rate either remains unchanged at  $\alpha\beta + \Delta$  (if  $Z_t = \alpha\beta + \Delta$ ), or it drops to  $Z_{t+1} = \alpha\beta + \Delta$  (if  $Z_t > \alpha\beta + \Delta$ ).

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<sup>5</sup> Assume that  $\Delta = 0$  (so that  $Z_{t+1}^L = \alpha\beta$ ) and consider what happens when  $Z_t = \alpha\beta$ . The first equation shown in (9) then becomes  $\alpha\beta \equiv \Lambda(\alpha\beta, -\bar{\varepsilon}_t)$  which implies  $\bar{\varepsilon}_t = 0$ , so that  $Z_{t+1}^H = Z_{t+1}^L = \alpha\beta$ , i.e.  $Z$  is (forever) stuck at  $\alpha\beta$ . Setting  $\Delta > 0$  rules out that absorbing state.



Fig. 2 shows that  $Z_{t+1}^H$  is a steeply increasing function of  $Z_t$ . A sequence of positive draws of the forecast error  $\varepsilon$  thus generates a run of rapid increases in the investment ratio, that is followed by an abrupt contraction in the investment ratio once a negative draw of  $\varepsilon$  is realized. A sequence of negative forecast errors keeps the investment ratio at the lower bound  $\alpha\beta + \Delta$ .

The strict concavity of the recursion  $Z_{t+1} = \Lambda(Z_t, \varepsilon_{t+1})$  with respect to  $\varepsilon_{t+1}$  (which reflects household risk aversion, as mentioned above) implies that  $E_t Z_{t+1} < \Lambda(Z_t, 0)$ . For any given  $Z_t$ , the conditional mean of the date t+1 investment ratio  $E_t Z_{t+1}$  is thus strictly below the value of  $Z_{t+1}$  that would obtain in a deterministic economy ( $\Lambda(Z_t, 0)$ ).

$E_t Z_{t+1}$  is an increasing and strictly concave function of  $Z_t$ :  $E_t Z_{t+1} = \zeta(Z_t)$ ,  $\zeta' > 0$ ,  $\zeta'' < 0$ . The graph of  $E_t Z_{t+1}$  intersects the 45-degree line at  $Z_t = 0.62$ . Strict concavity of  $\zeta$  implies that the unconditional mean of the investment/output ratio  $E(Z)$  is smaller than 0.62 (as  $E(Z) = E(\zeta(Z)) < \zeta(E(Z))$ ). The unconditional mean of the investment ratio is  $E(Z) = 0.45$ . While in a deterministic economy, the investment-output ratio would rise steadily and converge to 1, if a value  $Z_t > \alpha\beta$  is realized, we see that, with stochastic bubbles, the investment ratio fluctuates around a mean value that is close to the stationary no-bubbles investment ratio ( $\alpha\beta$ ).

A stochastic bubble implies that the absolute value of the forecast error  $\varepsilon_{t+1}$  is larger the greater  $Z_t$ . Thus, the variance of the forecast error  $\varepsilon_{t+1}$  is an increasing function of  $Z_t$ . Figure 1 shows that the conditional variance of  $Z_{t+1}$  is likewise increasing in  $Z_t$ . Furthermore, the conditional distribution of  $Z_{t+1}$  is left skewed. The left-skewness is likewise increasing in  $Z_t$ : the greater the bubble at date t, the bigger the (negative) ‘correction’ if the bubble bursts in t+1.<sup>6</sup>

Panel (b) of Fig. 2 shows representative simulated sample paths of output, consumption, gross investment ( $I$ ) and of the investment/output ratio ( $Z$ ). In order to assess whether the bubble alone can generate a realistic business cycle, I assume that TFP is constant. The Figure shows

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<sup>6</sup>Bacchetta et al. (2012) study a stylized asset pricing model with two-period lived agents in which stationary stock price bubbles can arise if the sunspot shock is heteroscedastic. The work here highlights the importance of heteroscedasticity, for generating stationary bubbles, in a non-linear DSGE business cycle models.

that the model generates massive swings in investment and output. During an expanding bubble, the rapid rise in investment is accompanied by a contraction in consumption.

Table 1 reports moments of HP filtered logged time series generated by the model. Line 1 of Panel (a) shows moments for specification I, with probability  $\pi_t=0.5$ . Predicted moments are based on a simulation run of 10000 periods.<sup>7</sup> The predicted standard deviation of output is 11.7% which is about five times larger than the historical standard deviation of quarterly GDP in advanced economies. The model-predicted volatility of consumption and investment too is excessive, when compared to the data. The model predicts that output, consumption and investment are serially correlated. However, consumption is predicted to be countercyclical, which is inconsistent with the data.

The model variant above assumes a constant 50% probability that the bubble grows next period. The model predicts smaller, more realistic, fluctuations in real activity occur if we assume that the probability of growth in the bubble falls once the investment/output ratio exceeds a threshold. As an illustration, assume that  $\pi_t$  is very close to unity, for values of the investment/output ratio greater than 0.36.<sup>8</sup> This threshold is chosen as it generates (more) realistic output volatility. It implies that the investment/output ratio oscillates between these two values: 0.3565 and 0.3916 (see below). Note that the ‘High’ investment ratio exceeds the ‘Low’ ratio by about 10%. When the investment/output ratio at date  $t$  takes the ‘Low’ value  $Z^L \equiv \alpha\beta + \Delta = 0.3565$ , then next period’s investment ratio is either ‘Low’ ( $Z^L$ ) or ‘High’ ( $Z^H(\alpha\beta + \Delta, 0.5) = 0.3916$ ) with 50% probability. If the date  $t$  investment ratio is ‘High’, then the investment ratio falls to the ‘Low’ value in the next period almost surely. Panel (c) of Figure 2 shows simulated sample paths generated for this model version, and the second Line in Panel (a) of Table 1 reports the corresponding model-predicted business cycle statistics. This model variant produces output fluctuations that are more in line with the data (predicted standard deviation of GDP: 1.33%), however now output, consumption and investment are negatively serially correlated.

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<sup>7</sup> The initial investment/output ratio is set at  $Z_0 = \alpha\beta + \Delta$ . Due to stationarity,  $Z_0$  does not affect simulated moments over a long simulation run. The effect of the initial  $Z_0$  on subsequent simulated values vanishes fast.

<sup>8</sup> I set  $\pi_t = 0.5$  when  $Z_t \in [\alpha\beta + \Delta, 0.36]$  and  $\pi_t = 1 - 10^{-100}$  when  $Z_t > 0.36$ .

### 2.3. Transversality condition

Long and Plosser (1983) assume an infinitely-lived representative household. The competitive equilibrium of the Long-Plosser economy corresponds to the maximum of the household's decision problem. As that decision problem is a well-behaved concave programming problem, its solution is unique. The necessary and sufficient optimality condition of that decision problem are the resource constraint (2), the Euler equation (3) and a transversality condition (TVC) that requires that the value of the capital stock is zero, at infinity:  $\lim_{\tau \rightarrow \infty} \beta^\tau E_t u'(C_{t+\tau}) K_{t+\tau+1} = 0$ . Note that, under the assumptions of the Long-Plosser model,  $u'(C_t) K_{t+1} = K_{t+1}/C_t = Z_t/(1-Z_t)$ . When  $Z_t = \alpha\beta$  holds, then  $u'(C_t) K_{t+1} = \alpha\beta/(1-\alpha\beta)$ , which shows that the textbook solution  $Z_t = \alpha\beta \forall t$  satisfies the TVC. Uniqueness of the infinitely-lived household's decision problem implies that any other processes  $\{Z_t\}$  consistent with (2) and (3) violates the TVC. This implies that the bubble equilibrium discussed above violates the TVC.<sup>9</sup>

This paper focuses on *stationary* model solutions consistent with the resource constraint and the Euler equation, but it disregards the TVC. The purpose of the paper is to show that stationary rational bubbles can exist in standard non-linear DSGE models. Note that explosive bubbles in linear models (Blanchard (1979)) likewise violate the TVC.

One possible justification for disregarding the TVC is to assume an OLG structure with finitely-lived households. The Appendix presents an OLG structure that has the *same* aggregate resource constraint and the *same* aggregate Euler equation as the Long-Plosser model. Thus equations (1)-(4) continue to hold in that OLG structure, but there is no TVC in the OLG structure. Such an OLG structure provides a motivation for exploring rational bubbles in standard DSGE models. The two key features of the OLG structure are:<sup>10</sup> (I) Efficient risk sharing between periods  $t$  and  $t+1$ , among all agents who are alive in both periods. (II) Newborn agents receive a wealth endowment such that consumption by newborns represents a time-

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<sup>9</sup> Under the bubble process (9),  $Z_t$  approaches 1 if a long uninterrupted string of positive draws of the sunspot  $\varepsilon$  is realized, which entails large positive values of  $Z_t/(1-Z_t)$ . Although this only happens with a very small probability, it causes the TVC to be violated.

<sup>10</sup> Assumption I is also used by Galí (2018). Assumption II is novel (to the best of my knowledge). Assumptions I and II allow to derive simple non-linear dynamic relations among aggregate variables for the economy. Without these two assumptions, approximate aggregation across generations may be possible, based on linear approximations. The focus of the paper here is on stationary rational bubbles induced by non-linearity. Thus, aggregation based on linear approximations is not useful here.

invariant share of aggregate consumption; under log utility, this requires that the wealth endowments of newborns is a time-invariant fraction of total wealth across all generations.<sup>11</sup>

Another motivation for disregarding the TVC is that detecting TVC violations may be very difficult, in more complicated stochastic models (than the Long-Plosser economy), for which no closed form solution exists (see below). TVC violations may be caused by low-probability events in a distant future. Agents may thus lack the cognitive/computing power to detect deviations from TVC; see discussion in Blanchard and Watson (1982), Lansing (2010) and Ascari et al. (2019).<sup>12</sup>

### 3. Stationary rational bubbles in an RBC model with incomplete capital depreciation

I next construct an equilibrium with stationary rational bubbles for an RBC model with incomplete capital depreciation and variable labor. It is now assumed that the period utility function is  $U(C_t, L_t) = \ln(C_t) + \Psi \cdot \ln(1 - L_t)$ ,  $\Psi > 0$ , where  $0 \leq L_t \leq 1$  are aggregate hours worked. The household's total time endowment (per period) is normalized to one, so that  $1 - L_t$  is household leisure.<sup>13</sup> The resource constraint and the production technology are

$$C_t + K_{t+1} = Y_t + (1 - \delta)K_t \text{ with } Y_t = \theta_t (K_t)^\alpha (L_t)^{1-\alpha}, \quad (10)$$

where  $0 < \delta \leq 1$  is the depreciation rate of capital. TFP is exogenous and follows a stationary Markov process. The economy has these efficiency conditions:

$$C_t \Psi / (1 - L_t) = (1 - \alpha) \theta_t (K_t)^\alpha (L_t)^{-\alpha} \text{ and} \quad (11)$$

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<sup>11</sup> Assume that the young can appropriate a constant share of total wealth. The wealth endowment of newborn may be provided by bequest, or by a government transfer financed by a (lump sum) tax levied on older generations. In reality, all societies make significant transfers to young generations (as bequest and through spending on their health and education). Wealthy countries make bigger transfers to the young than poor countries. It seems reasonable to assume that the wealth endowment of the young is a (roughly) constant share of total wealth.

<sup>12</sup> Blanchard and Watson (1982) and Ascari et al. (2019) analyze explosive bubbles in linearized models without TVC. Lansing (2010) disregards the TVC in a non-linear Lucas-style asset pricing models with bubbles, arguing that “agents are forward-looking but not to the extreme degree implied by the transversality condition” (p.1157); Lansing documents the existence of stationary asset price bubbles, when the TVC is dropped. The present paper considers fully-fledged DSGE macro models with endogenous output and capital accumulation. The method for constructing stationary bubble equilibria here is different and more general than that proposed by Lansing (it can be applied to a wide range of models).

<sup>13</sup> Due to decreasing returns to capital and a positive capital depreciation rate, the assumed upper bound on hours worked implies that the support of the distribution of capital and output is bounded, in equilibrium, which greatly simplifies the analysis. Some widely used preference specifications (such as  $U(C_t, L_t) = \ln(C_t) - \Psi \cdot (L_t)^\mu$ ,  $L_t \geq 0$ ,  $\mu > 1$ ) do not impose an upper bound on hours worked. Then the support of the distribution of hours, capital and output may be unbounded, in stationary bubble equilibria, which makes it much harder to analyze (and compute) those equilibria.

$$E_t \beta \{C_t / C_{t+1}\} (\alpha \theta_{t+1} (K_{t+1})^{\alpha-1} (L_{t+1})^{1-\alpha} + 1 - \delta) = 1 \quad (12)$$

where (11) indicates that the household's marginal rate of substitution between consumption and leisure is equated to the marginal product of labor, while (12) is the Euler equation.

(11) shows that hours worked  $L_t$  are a decreasing function of consumption  $C_t$ . Maximum hours worked  $L_t=1$  are chosen when consumption is zero. Provided date  $t$  gross investment  $I_t \equiv K_{t+1} - (1-\delta)K_t$  does not exceed maximum output (i.e. output with  $L_t=1$ )

$$I_t \leq \theta_t (K_t)^\alpha, \quad (13)$$

equations (10) and (11) uniquely pin down consumption and hours worked as functions of  $K_{t+1}, K_t, \theta_t$ :

$$C_t = \gamma(K_{t+1}, K_t, \theta_t) \text{ and } L_t = \eta(K_{t+1}, K_t, \theta_t). \quad (14)$$

Using these expressions to substitute out consumption and labor in the Euler equation gives:

$$E_t [\beta \{ \gamma(K_{t+1}, K_t, \theta_t) / \gamma(K_{t+2}, K_{t+1}, \theta_{t+1}) \} (\alpha \theta_{t+1} (K_{t+1})^{\alpha-1} (\eta(K_{t+2}, K_{t+1}, \theta_{t+1}))^{1-\alpha} + 1 - \delta)] = 1, \quad (15)$$

which can be written as

$$E_t H(K_{t+2}, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1, \quad (16)$$

where the function  $H$  maps  $R_+^5$  into  $R$ . (The function 'H' in (16) differs from the H function used to denote the Euler equation (4) in Sect. 2).

The model thus boils down to an expectational difference equation in capital. Once a process for capital has been found that is consistent with (16) in all periods, one can use (14) to generate sequences for consumption, hours and output that are consistent with the resource constraint (10) and with the intra-temporal efficiency condition (11). Solving the model amounts, thus, to finding a stochastic process for capital that solves (16).

The conventional no-bubbles solution that imposes a TVC can be described by a unique policy function  $K_{t+1} = \lambda(K_t, \theta_t)$  (e.g., Schmitt-Grohé and Uribe (2004)). Disregarding the TVC allows to generate stationary model solutions in which agents deviate from that 'no-bubbles' policy function.

By analogy to the bubble process in the Long-Plosser model (see Sect. 2), I consider equilibria with the property that, in any period  $t$ , the capital stock  $K_{t+1}$  takes one of two values:

$$K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\} \text{ with exogenous probabilities } \pi \text{ and } 1-\pi, \text{ respectively, with } K_{t+1}^L = \lambda(K_t, \theta_t) e^\Delta,$$

where  $\Delta$  is a constant ( $\Delta$  is set to a positive value close to zero in the simulations reported below). Whether  $K_{t+1}^L$  or  $K_{t+1}^H$  is realized depends on an exogenous i.i.d. sunspot that is assumed independent of TFP (see below). At date  $t$ , agents anticipate that the capital stock set in  $t+1$ ,  $K_{t+2}$  likewise takes two values  $K_{t+2} \in \{K_{t+2}^L, K_{t+2}^H\}$  with probabilities  $\pi$  and  $1-\pi$ , respectively, where  $K_{t+2}^L = \lambda(K_{t+1}, \theta_{t+1})e^\Delta$ . The Euler equation between periods  $t$  and  $t+1$  (see (16)) can then be written as:

$$\pi E_t H(\lambda(K_{t+1}, \theta_{t+1})e^\Delta, K_{t+1}, K_t, \theta_{t+1}, \theta_t) + (1-\pi) \cdot H(K_{t+2}^H, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1 \quad (17)$$

for  $K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}$ .

Consider an economy that starts in period  $t=0$ , with an initial capital stock  $K_0$ . Let  $u_t \in \{0,1\}$  be an exogenous i.i.d. **sunspot** that takes values 0 and 1 with probabilities  $\pi$  and  $1-\pi$ , respectively. A sequence of capital stocks  $\{K_t\}_{t \geq 0}$  such that, for all  $t \geq 0$ ,  $K_{t+2} = K_{t+2}^L \equiv \lambda(K_{t+1}, \theta_{t+1})e^\Delta$  if  $u_{t+1}=0$  and  $K_{t+2} = K_{t+2}^H$  if  $u_{t+1}=1$ , where  $K_{t+2}^H$  solves (17), is a **'bubble equilibrium'**.

$K_1$  (the capital stock set in period 0) is not pinned down by the conditions of the bubble equilibrium. Henceforth, I set  $K_1 = \lambda(K_0, \theta_1)e^\Delta$ . (The effect of  $K_0$  and  $K_1$  on endogenous variables in later periods vanishes as time progresses, due to the stationarity of the process).

Note that the trajectory of the capital stock is determined sequentially: Given  $K_0, K_1$  the Euler equation (17) for period  $t=0$  pins down  $K_2^H$ .  $K_2^L$  is determined by  $K_2^L = \lambda(K_1, \theta_1)e^\Delta$ . In  $t=1$ , the random sunspot  $u_1$  determines whether  $K_2$  equals  $K_2^L$  or  $K_2^H$ . Given  $K_1, K_2$  the Euler equation (18) for period  $t=1$  determines  $K_3^H$ , while  $K_3^L$  is determined by  $K_3^L = \lambda(K_2, \theta_2)e^\Delta$ . Etc in all subsequent periods. Note that although  $K_{t+2}$  is only realized in  $t+1$ , agents know already at date  $t$  that  $K_{t+2}$  will equal  $K_{t+2}^L$  or  $K_{t+2}^H$ . In period  $t+1$ , agents are free to select a value of  $K_{t+2}$  that differs from  $K_{t+2}^L$  or  $K_{t+2}^H$ , but they chose not to do so, in equilibrium, because a choice  $K_{t+2} \in \{K_{t+2}^L, K_{t+2}^H\}$  is validated by their expectations about  $K_{t+3} \in \{K_{t+3}^L, K_{t+3}^H\}$ .

### 3.1. Economy with constant TFP

To build intuition, consider first a model variant with constant TFP  $\theta_t = \theta \forall t$ , so that the sunspot is the only source of fluctuations. In the constant TFP economy, I write the no-bubbles policy rule for capital as  $K_{t+1} = \lambda(K_t)$ , and the Euler equation (16) as  $E_t H(K_{t+2}, K_{t+1}, K_t) = 1$ .

In a deterministic economy, any deviation from the no-bubbles policy function puts the economy on a trajectory that converges to a zero-consumption and/or zero-capital corner (e.g., Blanchard and Fischer (1989)). The present paper shows that there exist stationary stochastic bubble equilibria that do not converge to zero consumption/capital. With constant TFP, the Euler equation (17) between periods  $t$  and  $t+1$  becomes:

$$\pi H(\lambda(K_{t+1})e^\Delta, K_{t+1}, K_t) + (1-\pi) \cdot H(K_{t+2}^H, K_{t+1}, K_t) = 1. \quad (18)$$

This equation determines  $K_{t+2}^H$  as a function of  $K_t$  and  $K_{t+1}$ .

As discussed in the Appendix,  $\Delta > 0$  is needed to generate stationary bubbles. When  $\Delta = 0$  then bubbles are self-ending.<sup>14</sup>  $\Delta < 0$  implies that the capital stock can be put on a downward trajectory that leads to a zero capital corner. Throughout the subsequent analysis, I will thus assume  $\Delta > 0$ . As discussed in the Appendix,  $\Delta > 0$  implies that  $K_{t+1}^L < K_{t+1}^H$  holds, i.e. we can interpret  $u_t = 0$  and  $u_t = 1$  as investment ‘bust’ and ‘boom’ states, respectively, while  $\pi$  represents the ‘bust’ probability.

Let  $K^{\max}$  be the maximum feasible constant capital stock,  $K^{\max} = \theta(K^{\max})^\alpha + (1-\delta)K^{\max}$ , and let  $K_\Delta^{\min}$  be the steady state capital stock that would hold if  $K_{t+1} = \lambda(K_t)e^\Delta$  held each period:  $K_\Delta^{\min} = \lambda(K_\Delta^{\min})e^\Delta$ . Clearly,  $K_\Delta^{\min} < K^{\max}$  (for values of  $\Delta$  close to zero). If the initial capital stock is in the range  $K_0 \in (K_\Delta^{\min}, K^{\max})$ , then the capital stock stays in that range, in all subsequent periods, when  $\Delta \geq 0$  is assumed. An *uninterrupted* infinite sequence of investment booms (driven by an uninterrupted string of  $u=1$  sunspot realizations) would asymptotically drive the capital to the upper bound  $K^{\max}$ . An *uninterrupted* infinite sequence of investment busts (i.e. a string of  $u=0$

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<sup>14</sup> Let  $\Delta = 0$ . Consider a situation with  $u_t = 0$ , so that  $K_{t+1} = K_{t+1}^L \equiv \lambda(K_t)$ . Then (18) is solved by  $K_{t+2}^H \equiv \lambda(\lambda(K_t)) = \lambda(K_{t+1})$ , because  $H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1$ . Thus,  $K_{v+1} = K_{v+1}^L = K_{v+1}^H \equiv \lambda(K_v)$  holds  $\forall v > t$ . Thus, if  $K_{t+1}$  equals the value defined by the no-bubbles decision rule, then the agent has to continue sticking to the no-bubbles decision rule in all subsequent periods, and thus the trajectory of the capital stock becomes deterministic.

realizations) would asymptotically drive the capital stock to lower bound  $K^{\min}$ . Of course, such infinite boom or bust runs have zero probability.

### 3.2. Economy with stochastic TFP

In the economy with a stochastic TFP process, the capital stock picked in a ‘bust’ sunspot state at date  $t+1$  (if  $u_{t+1}=0$ ) is:  $K_{t+2}^L=\lambda(K_{t+1},\theta_{t+1})e^\Delta$ . Thus,  $K_{t+2}^L$  depends on  $\theta_{t+1}$ . I assume that, conditional on the inherited capital stock  $K_{t+1}$ , an unanticipated productivity shock at  $t+1$  has the same proportionate effect on  $K_{t+2}^L$  and  $K_{t+2}^H$ . Specifically:  $K_{t+2}^H=s_t^H \cdot K_{t+2}^L$ , where  $s_t^H \geq 0$  is in the date  $t$  information set. Across possible realizations of  $\theta_{t+1}$ , the sunspot ‘boom’ capital stock set at  $t+1$  is, thus proportional to the ‘bust’ capital stock. This greatly simplifies the analysis. Substituting the above formulae for  $K_{t+2}^L$  and  $K_{t+2}^H$  into the Euler equation (17) gives:

$$\pi E_t H(\lambda(K_{t+1},\theta_{t+1})e^\Delta, K_{t+1}, K_t, \theta_{t+1}, \theta_t) + (1-\pi) \cdot H(s_t^H \cdot \lambda(K_{t+1},\theta_{t+1})e^\Delta, K_{t+1}, K_t, \theta_{t+1}, \theta_t) = 1. \quad (19)$$

In the economy with stochastic TFP, the task of computing a bubble equilibrium thus boils down to determining, in each period  $t$ , the scalar  $s_t^H$  that solves the Euler equation (19) between  $t$  and  $t+1$ . As in the case with constant TFP, we have to set  $\Delta > 0$  to ensure existence of a stationary bubble equilibrium.

Simulations of the RBC model with stochastic TFP discussed below assume an AR(1) process for TFP:  $\ln(\theta_{t+1})=\rho \ln(\theta_t)+\varepsilon_{t+1}^\theta$ ,  $0 \leq \rho < 1$ , where  $\varepsilon_{t+1}^\theta$  is a white noise with standard deviation  $\sigma_\theta > 0$ . To simplify computations, it is assumed that  $\varepsilon_{t+1}^\theta$  only takes 2 values with equal probability:  $\varepsilon_{t+1}^\theta \in \{-\sigma_\theta, \sigma_\theta\}$ .

### 3.3. Simulation results

I set  $\alpha=1/3$ ,  $\beta=0.99$ . The capital depreciation rate is set at  $\delta=0.025$ . The preference parameter  $\Psi$  (utility weight on leisure) is set so that the Frisch labor supply elasticity is unity, at the steady state.<sup>15</sup> In model variants with stochastic TFP, I set the autocorrelation of TFP at  $\rho=0.979$ ,

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<sup>15</sup> (11) implies that the Frisch labor supply elasticity (LSE) with respect to the real wage rate (marginal product of labor) is  $LSE=(1-L)/L$  at the steady state, where  $L$  are steady state hours worked.  $\Psi$  is set such that  $L=0.5$ , which implies  $LSE=1$ .



while the standard deviation of TFP innovations is set at  $\sigma_\theta=0.0072$ . Parameters in this range are conventional in quarterly macro models (King and Rebelo, 1999). The no-sunspot policy rule for capital,  $\lambda(K_{t+1}, \theta_t)$ , is approximated using a second-order Taylor expansion.

Table 2 reports simulated business cycle statistics for several model variants. Standard deviations (in %) of GDP (Y), consumption (C), investment (I) and hours worked (L) are reported, as well as cross-correlations with GDP, autocorrelations and mean values of these variables. The statistics are based on a simulation run of T=10000 periods.<sup>16</sup> The reported standard deviations and correlations are medians moments computed across rolling windows of 200 periods.<sup>17</sup> By contrast, mean values (of Y,C,I,L) are computed for the whole simulation run (T periods) and expressed as % deviations from the deterministic steady state. The Table also reports the sample mean of the difference between capital income and investment spending (where this difference is normalized by GDP), as well as fraction of the T periods in which this difference is positive.

### 3.3.1. Model versions with just bubble shocks

Cols. (1)-(4) of Table 2 pertain to model variants in which fluctuations are just driven by bubbles (constant TFP is assumed). Cols. (5)-(8) assume bubbles and stochastic TFP shocks. Cols. (9)-(10) assume just TFP shocks, without bubbles (the no-bubbles equilibrium is computed using a linear model approximation). Col. (11) reports historical statistics for the US.<sup>18</sup> Cols. labelled ‘Unit Risk Aversion’ (or ‘Unit RA’) assume the log utility function described above. Columns labelled ‘High RA’ assume greater risk aversion for consumption:  $U(C_t, L_t) = \ln(C_t - \bar{C}) + \Psi \cdot \ln(1 - L_t)$ , where  $\bar{C}$  is a constant that is set at 0.8 times steady state consumption. The ‘High RA’ case implies that consumption has a lower bound:  $C_t \geq \bar{C}$ . In the

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<sup>16</sup> For several of the model variants, I also considered simulation runs with T=1000000 periods. The predicted statistics are virtually unchanged when the much longer runs are used.

<sup>17</sup> For each 200-periods window of artificial data, I computed standard deviations and correlation, using logged series that were HP filtered on the respective window. The Table reports median values (across all windows) of these standard deviations and correlations. 200 periods windows of simulated series are used as the historical business cycle statistics reported in Col. 11 of Table 2 pertain to an empirical sample of 200 quarters (see below).

<sup>18</sup> Historical standard deviations and correlations (HP filtered logged quarterly series) are taken from King and Rebelo (1999) and pertain to the period 1947Q1-1996Q4. Statistics for capital income minus investment were computed using annual data (1929-1985) reported in Abel et al. (1989).

‘High RA’ case, the coefficient of relative risk aversion is 5, in steady state; risk aversion is higher for consumption levels below steady state consumption.

All numerical simulations in Table 2 assume  $\Delta=0.001$ . That value generates standard deviations of real activity that are roughly in line with empirical statistics (higher values of  $\Delta$  induce greater volatility and a greater unconditional mean of real activity variables). Cols. (1), (3), (5) and (7) assume a bust probability  $\pi=0.5$ , while Cols, (2),(4),(6) and (8) assume  $\pi=0.2$ .

Simulated paths of GDP (continuous black line), consumption (red dashed line), investment (blue dash-dotted line) and hours worked (blue dotted line) are shown in Figure 3. Panel (i) ( $i=1,\dots,10$ ) of the Figure assumes the model variant considered in Col. (i) of Table 2. GDP, C and I series shown in Fig. 2 are normalized by steady state GDP; hours worked are normalized by steady state hours.

Cols. (1) of Table 2 assumes unit risk aversion and a bust probability  $\pi=0.5$ . Constant TFP is postulated, so that economic fluctuations are purely driven by the bubble shocks. The predicted standard deviations of output, consumption, investment and hours are 0.49%, 1.08%, 4.29% and 0.74%, respectively. The model-predicted output volatility is about 1/3 of the empirical GDP volatility. Consistent with the data, investment is predicted to be more volatile than output. However, the model predicts that consumption is more volatile than output, which is counterfactual. As in other models driven by investment shocks, the model here predicts that consumption is negatively correlated with output; however, the model predicts that investment and hours worked are strongly procyclical, as is consistent with the data. In the model, output, consumption, investment and hours worked are serially correlated, but the predicted autocorrelations (about 0.35) are smaller than the empirical autocorrelations (about 0.85).

As pointed out above, the bubble equilibrium implies capital over-accumulation (compared to a no-bubbles equilibrium), i.e. the economy is ‘dynamically inefficient’ (the TVC is violated). Abel et al. (1989) propose an empirical method for assessing dynamic efficiency. Their key insight is that an economy is dynamically efficient if income accruing to capital (i.e. output minus the wage bill) exceeds investment. Table 2 shows that, for all variants of the bubbles model here, the average (capital income – investment)/GDP ratio is positive and large (the average ratio is only slightly smaller than the value of that ratio in steady state, 9.59%).<sup>19</sup> In

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<sup>19</sup> The steady state (capital income – investment)/GDP ratio is  $\alpha r / (\delta + r)$  where  $r = (1 - \beta) / \beta$  is the steady state interest rate

fact, capital income also exceeds investment in close to 100% of all periods. This highlights the difficulty of detecting dynamic inefficiency (as discussed above).

Panel (1) of Figure 3 shows that the bubble equilibrium, with unit risk aversion and  $\pi=0.5$ , generates output, labor hours and investment booms that are relatively infrequent and short-lived. Periods of high investment are also periods of low consumption: in the model, a sudden fall in consumption triggers a rise in labor hours and output. However, in most periods, real activity remains close to (but slightly above) its steady state level. This explains the low predicted autocorrelation of real activity.

A lower bust probability  $\pi$  generates bigger and more persistent ‘spikes’ in real activity, and thus real activity becomes more volatile and more serially correlated. This is illustrated in Col. (2) of Table 2, where unit risk aversion and  $\pi=0.2$  are assumed (see also Panel (2) of Figure 3). However, output, consumption, investment and hours worked are now excessively volatile, when compared to the data. Consumption, again, is predicted to be more volatile than output.

Model variants with the ‘high risk aversion (RA)’ utility function (that assumes a lower bound for consumption  $C_t \geq \bar{C} > 0$ ) generates less consumption volatility—those variants capture the fact that consumption is less volatile than output; see Cols. (3) and (4) of Table 2 (and Panels (3) and (4) of Fig. 3), where  $\pi=0.5$  and  $\pi=0.2$  are assumed, respectively.

In summary, the model versions with just bubbles shocks considered so far can generate realistic volatility of real activity and of aggregate demand components. Real activity in the model is serially correlated, but less than in the historical data.

Setting the bust probability at even lower values (e.g.,  $\pi=0.05$ ) generates higher, realistic serial correlation in real activity but the predicted volatility of real activity becomes too large. A lower labor supply elasticity and higher consumption risk aversion are then needed to produce realistic volatilities (results available on request).

### **3.3.2. Model versions with TFP shocks**

The no-bubbles model driven by stochastic TFP shocks underpredicts the volatility of real activity, but it captures the fact that consumption is less volatile than output, while investment is more volatile (see Cols. (9),(10)). In the no-bubbles model version, consumption and investment are pro-cyclical; furthermore, real activity is highly serially correlated, which reflects the high assumed autocorrelation (0.979) of TFP.

The bubble equilibrium with TFP shocks generates fluctuations in real activity that are more volatile than the fluctuations exhibited by the no-bubbles equilibrium (see Fig. 3, Cols. (5)-(8)). In this sense, the bubble equilibrium (with TFP shocks) is closer to the historical data.

Panels (5)-(10) show that the effect of bubbles on the simulated series is clearly noticeable (compared to the ‘no-bubbles’ series with just TFP shocks): the bubbles induce rapid, but short-lived increases in investment, hours worked and output.

#### **4. Two country model**

[TBA]

#### **5. New Keynesian model**

[TBA]

#### **6. Conclusion**

Linearized Dynamic Stochastic General Equilibrium (DSGE) models with a unique stable solution are the workhorses of modern macroeconomics. This paper shows that stationary sunspot equilibria exist in standard *non-linear* DSGE models, even when the linearized versions of those models have unique solutions. In the sunspot equilibria considered here, the economy may temporarily diverge from the no-sunspots allocation, before abruptly reverting towards that allocation. In contrast to rational bubbles in linear models (Blanchard (1979)), the bubbles considered here are stationary--their expected path does not explode to infinity. The quantitative results presented in this paper suggest that simple non-linear DSGE models driven just by stationary bubbles can capture key business cycle stylized facts.

## References

- Abel, Andrew, Gregory Mankiw, Lawrence Summers and Richard Zeckhauser, 1989. Assessing Dynamic Efficiency: Theory and Evidence. *Review of Economic Studies* 56, 1-20.
- Ascari, Guido, Paolo Bonomolo and Hedibert Lopes, 2019. Walk on the Wild Side: Temporarily Unstable Paths and Multiplicative Sunspots. *American Economic Review* 109, 1805–1842.
- Bacchetta, Philippe, Cédric Tille and Eric van Wincoop, 2012. Self-Fulfilling Risk Panics. *American Economic Review* 102, 3674-3700.
- Benhabib, Jess and Roger Farmer, 1999. Indeterminacy and Sunspots in Macroeconomics. In: *Handbook of Macroeconomics* (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1A, 387-448.
- Blanchard, Olivier, 1979. Speculative Bubbles, Crashes and Rational Expectations. *Economics Letters* 3, 387-398.
- Blanchard, Olivier and Charles Kahn, 1980. The Solution of Linear Difference Models under Rational Expectations. *Econometrica* 48, 1305-1311.
- Blanchard, Olivier and Mark Watson, 1982. Bubbles, Rational Expectations and Financial Markets. NBER Working Paper 945.
- Blanchard, Olivier and Stanley Fischer, 1989. *Lectures on Macroeconomics*. MIT Press.
- Galí, Jordi, 2018. Monetary Policy and Bubbles in a New Keynesian Model with Overlapping Generations. Working Paper, CREI.
- Holden, Tom, 2016a. Existence and Uniqueness of Solutions to Dynamic Models with Occasionally Binding Constraints. Working Paper, University of Surrey.
- Holden, Tom, 2016b. Computation of Solutions to Dynamic Models with Occasionally Binding Constraints. Working Paper, University of Surrey.
- King, Robert and Sergio Rebelo, 1999. Resuscitating Real Business Cycles. In: *Handbook of Macroeconomics* (J. Taylor and M. Woodford, eds.), Elsevier, Vol. 1B, 927-1007.
- Kollmann, Robert, Serguei Maliar, Benjamin Malin and Paul Pichler, 2011a. Comparison of Numerical Solutions to a Suite of Multi-Country Models. *Journal of Economic Dynamics and Control* 35, pp.186-202.
- Kollmann, Robert, Jinill Kim and Sunghyun Kim, 2011b. Solving the Multi-Country Real Business Cycle Model Using a Perturbation Method. *Journal of Economic Dynamics and Control* 35, 203-206.

- Lansing, Kevin, 2010. Rational and Near-Rational Bubbles Without Drift. *Economic Journal* 120, 1149-1174.
- Long, John and Charles Plosser, 1983. Real Business Cycles. *Journal of Political Economy* 91, 39-69.
- Martin, Alberto and Jaume Ventura, 2018. The Macroeconomics of Rational Bubbles: A User's Guide. *Annual Review of Economics* 10, 505-539.
- Schmitt-Grohé, Stephanie, 1997. Comparing Four Models of Aggregate Fluctuations Due to Self-Fulfilling Expectations. *Journal of Economic Theory* 72, 96-47.
- Schmitt-Grohé, Stephanie and Martin Uribe, 2004. Solving Dynamic General Equilibrium Models Using a Second-Order Approximation to the Policy Function. *Journal of Economic Dynamics and Control* 28, 755 – 775.
- Taylor, John, 1977. Conditions for Unique Solutions in Stochastic Macroeconomic Models with Rational Expectations. *Econometrica* 45, 1377-1385.

## Appendix 1: OLG model with same aggregate Euler equation as a model with an infinitely-lived representative agent

This Appendix shows that an economy inhabited by overlapping generations (OLG) of finitely-lived agents can have the same aggregate equations—with the exception of the transversality condition (TVC)—as an economy with an infinitely lived representative agent. Here, this point is made for the Long-Plosser model discussed in Sect.2. It is assumed that the economy has the same aggregate production function and the same aggregate resource constraint as the corresponding representative agent economy.

The two key assumptions that deliver this result are: I. Efficient risk sharing between periods  $t$  and  $t+1$ , among all agents who are alive in both periods. II. Newborn agents receive a wealth endowment such that consumption by newborns represents a time-invariant share of aggregate consumption. Under log utility, this requires that newborn agents receive a wealth endowment that is a time-invariant share of total wealth.

Assume that agents live  $N < \infty$  periods. A measure 1 of agents is born each period. Thus, a fraction  $1/N$  of the population is aged  $n=1, \dots, N$ . All members of the same age cohort are identical. All agents have log utility and the same subjective discount factor,  $\beta$ . Let  $c_{i,t}$  denote the date  $t$  consumption of agents who are in the  $i$ -th period of their life ('generation  $i$ ') at date  $t$ . The expected life-time utility of the generation born at date  $t$  is, thus,  $E_t \sum_{s=0}^{N-1} \beta^s \ln(c_{i+s,t+s})$ . Aggregate consumption at date  $t$  is  $C_t = \sum_{i=1}^N c_{i,t}$ . Assume that there exists a market at date  $t$  in which a complete set of one-period claims with state-contingent date  $t+1$  payouts is traded. This implies that, in equilibrium, the consumption growth rate between  $t$  and  $t+1$  is equated across all agents who are alive in both periods (risk sharing):

$$c_{i+1,t+1}/c_{i,t} = c_{2,t+1}/c_{1,t} \text{ for } i=1, \dots, N-1. \quad (\text{A.1})$$

Let  $\lambda_{i,t} \equiv c_{i,t}/C_t$  denote the ratio of generation  $i$ 's consumption divided by aggregate consumption, in period  $t$ . I refer to  $\lambda_{i,t}$  as the 'consumption share' of generation  $i$ , in period  $t$ . (A.1) implies

$$\lambda_{i+1,t+1}/\lambda_{i,t} = \lambda_{2,t+1}/\lambda_{1,t} \text{ for } i=1, \dots, N-1. \quad (\text{A.2})$$

(A.2) and the adding up constraint  $\sum_{i=1}^N \lambda_{i,t+1} = 1$  provide a system of N equations that pin down the date t+1 consumption shares  $\{\lambda_{i,t+1}\}_{i=1,\dots,N}$  for given date t shares  $\{\lambda_{i,t}\}_{i=1,\dots,N}$ :  $\lambda_{i+1,t+1} = \lambda_{i,t}(1-\lambda_{1,t})/(1-\lambda_{N,t})$  for  $i=1,\dots,N-1$ .

Assume that the consumption share of newborn agents, during the first period of their life, is time-invariant:  $\lambda_{1,t} = \lambda_1 \forall t$ . A constant newborn consumption share can be sustained by allocating to newborns a suitable time-invariant wealth share (see below). When  $\lambda_{1,t} = \lambda_1$ , then (A.1) is a stable difference equation in the consumption shares, and the consumption shares of generations  $i=2,\dots,N$  converge asymptotically to a constant consumption shares  $\lambda_i$  (numerical experiments show that convergence to the steady state shares is fast). The N steady state consumption shares obey

$$\lambda_{i+1} = \lambda_i(1-\lambda_1)/(1-\lambda_N) \text{ for } i=1,\dots,N-1. \quad (\text{A.3})$$

Given any newborn's consumption share  $0 < \lambda_1 \leq 1$ , these equations pin down unique consumption shares of generations  $i=2,\dots,N$  that are consistent with the adding up constraint  $\sum_{i=1}^N \lambda_i = 1$ . The following discussion assumes that the consumption shares equal their steady state values, so that all generational consumption shares are time-invariant:  $\lambda_{i,t} = \lambda_i \forall t, \forall i = 1,\dots,N$ .

The Euler equation for capital of generation  $i=1,\dots,N-1$  between periods t and t+1 is  $E_t \rho_{i,t+1} r_{K,t+1} = 1$ , where  $r_{K,t+1}$  is the gross rate of return (between t and t+1) on capital investment, while  $\rho_{i,t+1} = \beta c_{i,t} / c_{i,t+1}$  is the common intertemporal marginal rate of substitution (IMRS) of these generations. Full risk sharing implies that the IMRS is equated across generations  $i=1,\dots,N-1$  (see (A.1)). Thus

$$\rho_{i,t+1} = \beta \sum_{i=1}^{N-1} c_{i,t} / \sum_{i=1}^{N-1} c_{i,t+1} \text{ and} \quad (\text{A.4})$$

$$\rho_{i,t+1} = \beta (C_t - c_{N,t}) / (C_{t+1} - c_{1,t}) = \beta [(1-\lambda_N)/(1-\lambda_1)] \cdot C_t / C_{t+1}. \quad (\text{A.5})$$

The capital Euler equation can thus be expressed as

$$E_t \tilde{\beta} C_t / C_{t+1} r_{K,t+1} = 1, \text{ with } \tilde{\beta} \equiv \beta \times (1-\lambda_N)/(1-\lambda_1). \quad (\text{A.6})$$

We thus see that, up to a rescaling of the subjective discount factor when  $\lambda_1 \neq \lambda_N$ , this OLG model implies that the same 'aggregate' Euler equation (in terms of aggregate



consumption) holds as in a model with an infinitely-lived representative agent. If the initial wealth endowment of newborns is such that  $\lambda_1=1/N$ , then  $\lambda_i=1/N$  holds for  $i=1,\dots,N$ , which implies  $\tilde{\beta}\equiv\beta$ . In the special case where  $\lambda_1=1/N$ , the aggregate Euler equation of the OLG economy is thus *identical* to the Euler equation of an economy with an infinitely-lived agent. The only difference between the two economies is that the transversality condition  $\lim_{\tau\rightarrow\infty}\beta^\tau E_t u'(C_{t+\tau})K_{t+\tau+1}=0$  does not hold in the OLG economy, as there is no infinitely-lived agent in the OLG economy. This OLG structure thus provides a motivation for considering a business cycle models that lack a TVC, but whose other equilibrium conditions (aggregate resource constraint, aggregate Euler equation) are identical to those of a standard business cycle model with an infinitely-lived representative agent.

### *Wealth shares*

A time-invariant consumption share  $\lambda_i$  of the new-born cohort is sustained by allocating to newborn agents a time-invariant share of the aggregate wealth of all cohorts. To see this, let  $\omega_{i,t}$  denote the wealth of generation  $i$  in period  $t$ .  $\omega_{i,t}$  equals the present value of generation  $i$ 's consumption stream:  $\omega_{i,t}=E_t \sum_{s=0}^{N-i} \rho_{t,t+s} c_{i+s,t+s}$ , where the stochastic discount factor  $\rho_{t,t+s}$  is a product of the one-period-ahead discount factors defined in (A.5):  $\rho_{t,t}=1$  and  $\rho_{t,t+s}=\prod_{\tau=1}^{s-1} \rho_{t+\tau,t+\tau+1}$  for  $s>1$ . Note that  $\rho_{t,t+s}=\beta^s c_{i,t}/c_{i+s,t+s}$  for  $0<s\leq N-i$ . Therefore  $\omega_{i,t}=c_{i,t} \sum_{s=0}^{N-i} \beta^s$  and hence

$$c_{i,t}=\phi_i \cdot \omega_{i,t}, \text{ with } \phi_i\equiv(1-\beta)/(1-\beta^{N-i+1}) \text{ for } i=1,\dots,N. \quad (\text{A.7})$$

Thus, in each period, generation  $i$  consumes a fraction  $\phi_i$  of her wealth that is generation-specific, but time invariant. In an equilibrium with time-invariant generational consumption shares, the period  $t$  wealth of generation  $i$  equals thus  $\omega_{i,t}=(\lambda_i/\phi_i)C_t$ , and the wealth share generation  $i$  is

$$\omega_{i,t}/\sum_{s=1}^N \omega_{s,t} = (\lambda_i/\phi_i)/\sum_{s=1}^N (\lambda_s/\phi_s) \equiv \kappa_i. \quad (\text{A.8})$$

Note that this wealth share is time-invariant. Thus, an equilibrium with time-invariant generational consumption shares exhibits time-invariant generational wealth shares. As pointed out above, the consumption share of newborn generations,  $\lambda_1$ , pins down uniquely the

consumption shares of older generations, i.e.  $\lambda_t$  is a function of  $\lambda_1$ :  $\lambda_t = \Lambda_t(\lambda_1)$ . There is, hence, a unique mapping from  $\lambda_1$  to the wealth shares of all generations (see (A.8) for definition of  $\kappa_i$ ):

$$\kappa_i = K_i(\lambda_1) = (\Lambda_i(\lambda_1)/\phi_i) / \sum_{s=1}^N (\Lambda_s(\lambda_1)/\phi_s). \quad (\text{A.9})$$

If the new-born generation is allocated a wealth share  $\kappa_1 = (\lambda_1/\phi_1) / \sum_{s=1}^N (\Lambda_s(\lambda_1)/\phi_s)$ , then this sustains an equilibrium in which the consumption share of the new-born generation is  $\lambda_1$ . A consumption allocation in which all generations have consumption share  $\lambda_t = 1/N$  is sustained by allocating to the newborn generation a wealth share  $\kappa_1 = (1/\phi_1) / \sum_{s=1}^N 1/\phi_s$ . As an example, assume that life lasts 80 years, i.e.  $N=320$  quarters, and that the quarterly subjective discount factor is  $\beta=0.99$ ; then the consumption allocation with equal consumption shares  $\lambda_t = 1/N = 0.3125\%$  requires a newborn wealth share of  $\kappa_1 = 0.4267\%$ .

## Appendix 2: Bubble equilibrium in RBC model with incomplete capital depreciation (constant TFP)

This Appendix discusses the role of  $\Delta$  for the bubble equilibrium, in the RBC model with constant TFP. Recall that  $\Delta$  denotes the deviation of the capital stock selected in the bust state, from the no-bubbles decision rule,  $\lambda$ :  $K_{t+1}^L = \lambda(K_t)e^\Delta$ .

Consider first a decision economy with  $\Delta=0$ , so that  $K_{t+1} = \lambda(K_t) \forall t$ . Then the agent's Euler equation holds between  $t$  and  $t+1$ :  $H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1$ . If  $K_{t+1} = \lambda(K_t)e^\Delta \forall t$  the Euler equation fails to hold if  $\Delta \neq 0$ . Specifically:

$$H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) < 1 \text{ when } \Delta > 0, \quad (\text{A.10})$$

$$\text{while } H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) > 1 \text{ when } \Delta < 0. \quad (\text{A.11})$$

(Intuitively,  $\Delta > 0$  implies overinvestment in capital, and thus the intertemporal marginal rate of transformation is smaller than the intertemporal marginal rate of transformation, IMRS, which implies  $H < 1$ ;  $\Delta < 0$  implies underinvestment in capital, and thus the intertemporal marginal rate of transformation is greater than the IMRS and hence  $H > 1$ .)

I now discuss bubble equilibria. Recall that the bubble equilibria considered here are such that the capital stock set at date  $t$  takes two possible values:  $K_{t+1} \in \{K_{t+1}^L, K_{t+1}^H\}$  with exogenous probabilities  $\pi$  and  $1-\pi$ , respectively, where  $K_{t+1}^L = \lambda(K_t)e^\Delta$ . I now show that a bounded equilibrium with recurrent bubbles exists if  $\Delta > 0$ . When  $\Delta = 0$ , the bubble equilibrium is self-ending. When  $\Delta < 0$  the Euler equation between  $t$  and  $t+1$  fails to have a solution for  $K_{t+2}^H$ , for certain values of  $K_t, K_{t+1}$ . Thus, there is not bubble equilibrium if  $\Delta < 0$ .

**I) Consider first a situation in which the date  $t+1$  capital stock equals  $K_{t+1}^L$ :  $K_{t+1} = \lambda(K_t)e^\Delta$ .**

Then  $K_{t+2}^L = \lambda(\lambda(K_t)e^\Delta)e^\Delta$  and the Euler equation (17) between periods  $t$  and  $t+1$  becomes:

$$\pi H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) + (1-\pi) \cdot H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t) = 1. \quad (\text{A.12})$$

To establish the existence of a ‘bubble equilibrium’, one needs to show that there exists a  $K_{t+2}^H \in (K_\Delta^{\min}, K^{\max})$  that solves (A.12).

- Consider first the case where  $\Delta=0$ . Recall that  $H(\lambda(\lambda(K_t)), \lambda(K_t), K_t) = 1$ . Thus, for  $\Delta=0$ , the Euler equation of the bubbly economy (A.12) requires that  $H(K_{t+2}^H, \lambda(K_t), K_t) = 1$  holds. This implies  $K_{t+2}^H = \lambda(\lambda(K_t))$ , and thus  $K_{t+2} = K_{t+2}^H = K_{t+2}^L = \lambda(K_{t+1})$ . By the same logic,  $K_{s+1} = \lambda(K_s)$  has to hold  $\forall s \geq t+1$ . Thus, if  $K_{s+1} = \lambda(K_s)$ , then the agent has to continue sticking to the no-bubbles decision rule in all subsequent periods. Hence, the bubble is self-ending when  $\Delta=0$ .

- Consider next the case  $\Delta > 0$ . Because  $H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) < 1$  when  $\Delta > 0$ , the Euler equation (A.12) can only hold when  $H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t) > 1$ . Note that  $H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t) < 1$  when  $K_{t+2}^H = \lambda(\lambda(K_t)e^\Delta)e^\Delta$ . It can be verified that  $H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t)$  is an increasing function of  $K_{t+2}^H$  (as a rise in  $K_{t+2}^H$  lowers  $C_{t+1}$  and raises hours worked  $L_{t+1}$  which raises the marginal utility of consumption at  $t+1$ , and raises the marginal product of capital at  $t+1$ ). Setting  $K_{t+2}^H$  arbitrarily close to (but below) the maximum feasible value  $\theta(\lambda(K_t)e^\Delta)^\alpha + (1-\delta)\lambda(K_t)e^\Delta < K^{\max}$  makes  $C_{t+1}$  very close to zero (which implies that  $L_{t+1}$  is very close to 1), which makes

$H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t)$  very big. This implies that there exists a unique value of  $K_{t+2}^H$  that solves the Euler equation (A.12). Note that  $K_{t+2}^H \in (\lambda(\lambda(K_t)e^\Delta)e^\Delta, K^{\max})$ . Thus,  $K_{t+2}^L < K_{t+2}^H$ . When  $K_t \in (K_\Delta^{\min}, K^{\max})$ , then  $\lambda(K_t)e^\Delta \in (K_\Delta^{\min}, K^{\max})$  and  $\lambda(\lambda(K_t)e^\Delta)e^\Delta \in (K_\Delta^{\min}, K^{\max})$  for values of  $\Delta > 0$  sufficiently close to 0. If  $K_t \in (K_\Delta^{\min}, K^{\max})$  we thus have that  $K_{t+1}^L, K_{t+2}^L, K_{t+2}^H \in (K_\Delta^{\min}, K^{\max})$ .

- Finally, consider the case  $\Delta < 0$ . It follows from (A.11) that then the Euler equation (A.12) requires that  $H(K_{t+2}^H, \lambda(K_t)e^\Delta, K_t) < 1$  holds. If there exists a value of  $K_{t+2}^H$  that solves (A.12), then that value must be smaller than  $K_{t+2}^L$ :  $K_{t+2}^H < K_{t+2}^L = \lambda(\lambda(K_t)e^\Delta)e^\Delta$  when  $\Delta < 0$ . There is no solution for  $K_{t+2}^L$  when  $\pi H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) + (1-\pi) \cdot H(0, \lambda(K_t)e^\Delta, K_t) > 1$ . When  $\Delta < 0$ , then a succession of positive (!) draws of the sunspot ( $u > 0$ ) puts the capital stock on a downward trajectory until (A.12) cannot be solved anymore for  $K_{t+2}^H \geq 0$ .

## II) Consider next a situation in which the date $t+1$ capital stock equals $K_{t+1}^H$ : $K_{t+1} = K_{t+1}^H$

Assume  $\Delta > 0$ . As shown above,  $K_{t+1}^H \geq \lambda(K_t)e^\Delta$  holds when  $\Delta > 0$ . When  $K_{t+1} = K_{t+1}^H$  (which is triggered by a positive realization of the date  $t$  sunspot,  $u_t > 0$ ), then  $K_{t+2}^L = \lambda(K_{t+1}^H)e^\Delta$  and the Euler equation between periods  $t$  and  $t+1$  is given by:

$$\pi H(\lambda(K_{t+1}^H)e^\Delta, K_{t+1}^H, K_t) + (1-\pi) \cdot H(K_{t+2}^H, K_{t+1}^H, K_t) = 1. \quad (\text{A.13})$$

$H(\lambda(K_{t+1}^H)e^\Delta, K_{t+1}^H, K_t)$  is a decreasing function of  $K_{t+1}^H$  for  $K_{t+1}^H \geq \lambda(K_t)e^\Delta$ . Recall that  $H(\lambda(\lambda(K_t)e^\Delta)e^\Delta, \lambda(K_t)e^\Delta, K_t) < 1$  when  $\Delta > 0$ . Therefore  $H(\lambda(K_{t+1}^H)e^\Delta, K_{t+1}^H, K_t) < 1$  for any  $K_{t+1}^H \geq \lambda(K_t)e^\Delta$ . Thus  $H(K_{t+2}^H, K_{t+1}^H, K_t) > 1$ . It follows from the discussion above that  $H(K_{t+2}^H, K_{t+1}^H, K_t)$  is increasing in  $K_{t+2}^H$  and that  $H(K_{t+2}^H, K_{t+1}^H, K_t)$  can be made arbitrarily big by setting  $K_{t+2}^H$  close to  $\theta(K_{t+1}^H)^\alpha + (1-\delta)K_{t+1}^H$ . Thus, there exists a unique  $K_{t+2}^H$  that solves (A.13). Furthermore,  $K_{t+2}^H > K_{t+2}^L \equiv \lambda(K_{t+1}^H)e^\Delta$ .

**Table 1. Long-Plosser model with bubbles: predicted business cycle statistics**

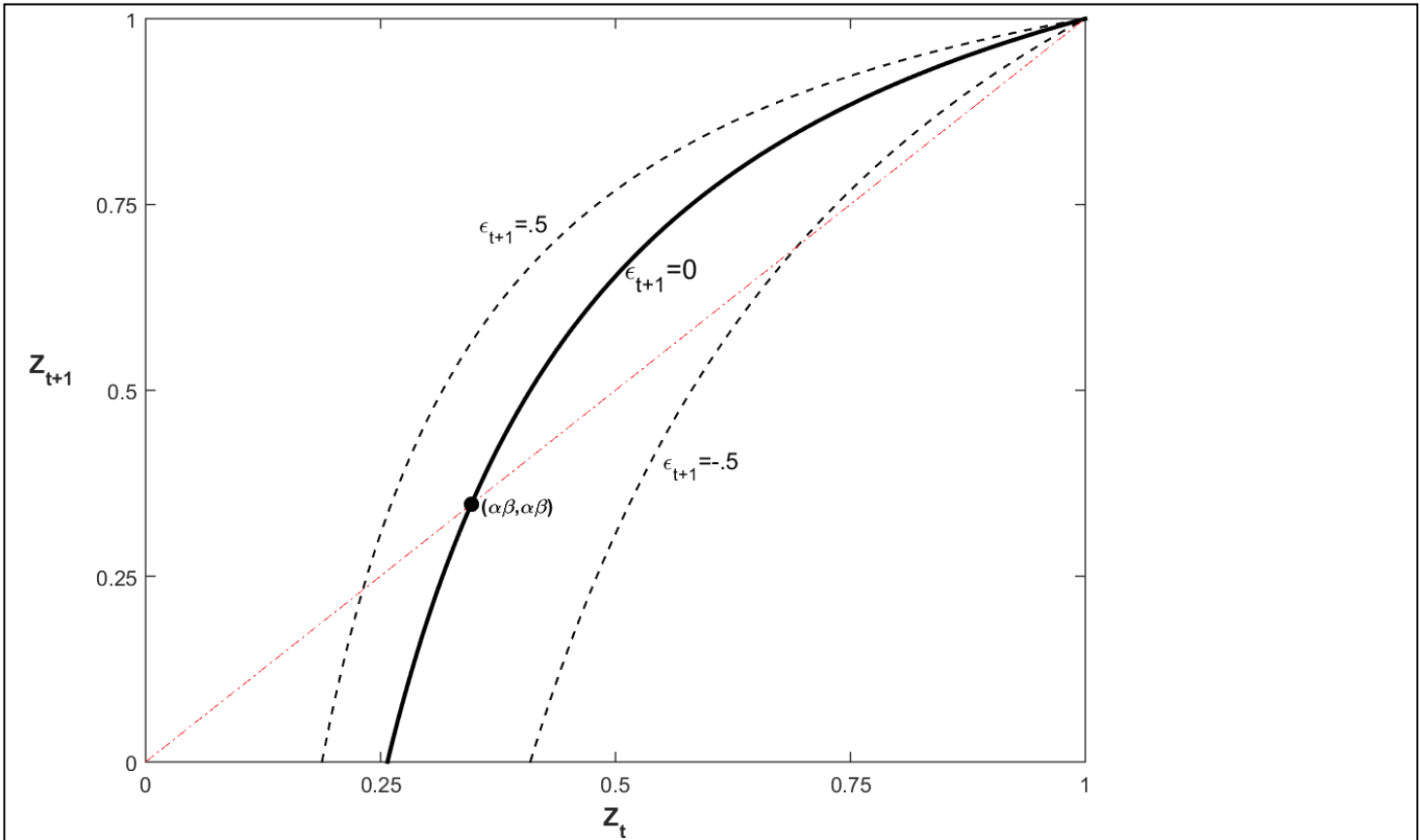
	<u>Standard dev. %</u>			<u>Corr. with Y</u>		<u>Autocorr.</u>			<u>Mean (% deviation from SS)</u>			
	Y	C	I	C	I	Y	C	I	Y	C	I	Z
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)	(12)
<b>(a) Specification I: <math>Z_t^L = \alpha\beta + \Delta</math></b>												
$\pi_t = 0.5$	11.72	100.19	33.48	-0.42	0.62	0.62	0.47	0.62	13.49	-7.62	53.31	31.15
$\pi_t \approx 1$ for $z_t > 0.36$	1.33	3.51	3.82	0.77	-0.26	-0.26	-0.66	-0.26	3.27	-0.13	9.71	6.25
<b>(b) US Data (from King and Rebelo (1999))</b>												
	1.81	1.35	5.30	0.88	0.80	0.88	0.80	0.87				

Note: all business statistics pertain to HP-filtered logged variables.

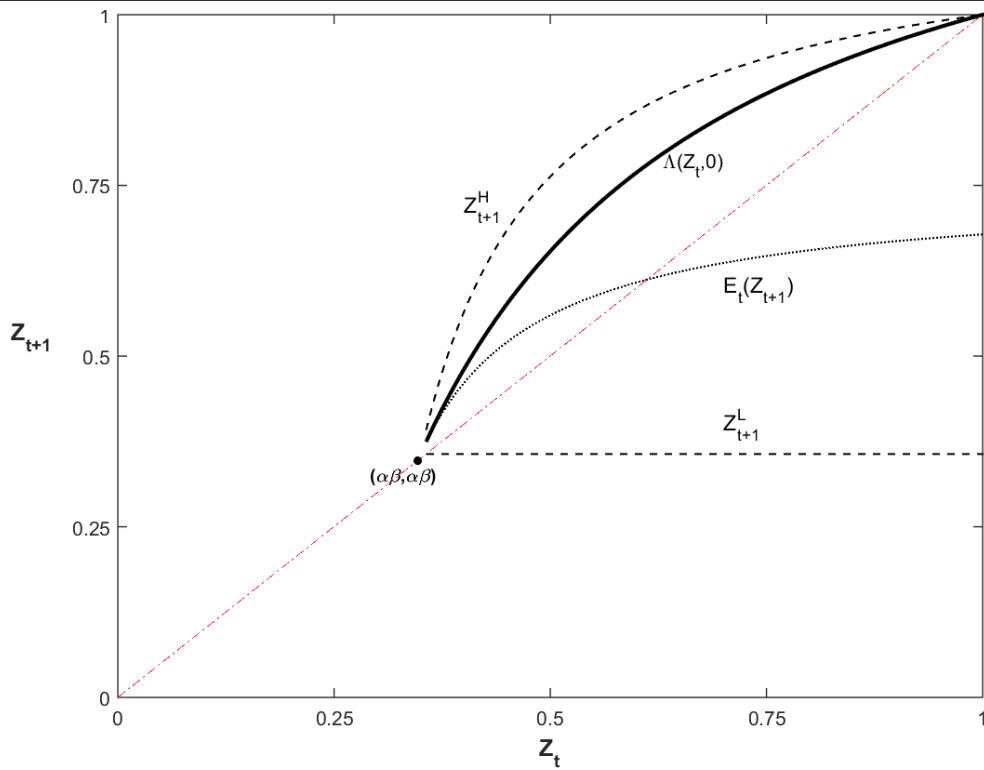
**Table 2. RBC model (incomplete capital deprec.) with bubbles: predicted business cycle statistics**

	Bubbles, no TFP shocks				Bubbles & TFP shocks				Just TFP shocks		Data
	Unit Risk aversion		High RA		Unit RA		High RA		Unit RA	High RA	
	$\pi=0.5$	$\pi=0.2$	$\pi=0.5$	$\pi=0.2$	$\pi=0.5$	$\pi=0.2$	$\pi=0.5$	$\pi=0.2$	(9)	(10)	
	(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)	(9)	(10)	(11)
<b>Standard deviations [in %]</b>											
Y	0.49	1.16	0.68	1.43	1.27	1.60	0.98	1.57	1.14	0.72	1.81
C	1.08	2.63	0.29	0.61	1.16	2.71	0.38	0.72	0.49	0.26	1.35
I	4.29	9.38	3.22	6.51	5.38	9.85	3.86	6.72	3.33	2.20	5.30
L	0.74	1.73	1.04	2.18	0.82	1.70	1.05	2.22	0.34	0.30	1.79
<b>Correlations with GDP</b>											
C	-0.97	-0.95	-0.99	-0.98	0.04	-0.54	0.01	-0.62	0.95	0.99	0.88
I	0.98	0.96	0.99	0.99	0.89	0.86	0.97	0.98	0.99	0.99	0.80
L	0.99	0.97	0.99	0.99	0.79	0.81	0.45	0.82	0.98	-0.96	0.88
<b>Autocorrelations</b>											
Y	0.36	0.63	0.35	0.62	0.65	0.68	0.57	0.66	0.71	0.70	0.84
C	0.33	0.60	0.35	0.62	0.43	0.62	0.53	0.65	0.76	0.72	0.80
I	0.36	0.63	0.37	0.64	0.53	0.65	0.51	0.65	0.70	0.70	0.87
L	0.34	0.61	0.35	0.62	0.45	0.62	0.41	0.63	0.70	0.74	0.88
<b>Means [% deviation from steady state]</b>											
Y	1.41	2.80	1.25	2.12	1.37	2.75	1.31	2.17	0.00	0.00	--
C	0.73	1.39	0.33	0.55	0.68	1.34	0.33	0.55	0.00	0.00	--
I	3.62	7.33	4.22	7.19	3.61	7.28	4.44	7.40	0.00	0.00	--
L	0.36	0.74	-0.02	-0.02	0.34	0.73	0.01	-0.03	0.00	0.00	--
<b>Mean (capital income – investment)/GDP [in %]</b>											
	9.12	8.75	8.93	8.54	9.16	8.78	8.92	8.53	9.58	9.58	13.42
<b>Fraction of periods with (capital income &gt; investment) [in %]</b>											
	99.20	96.31	99.55	97.72	99.20	96.43	99.37	97.74	100	100	100

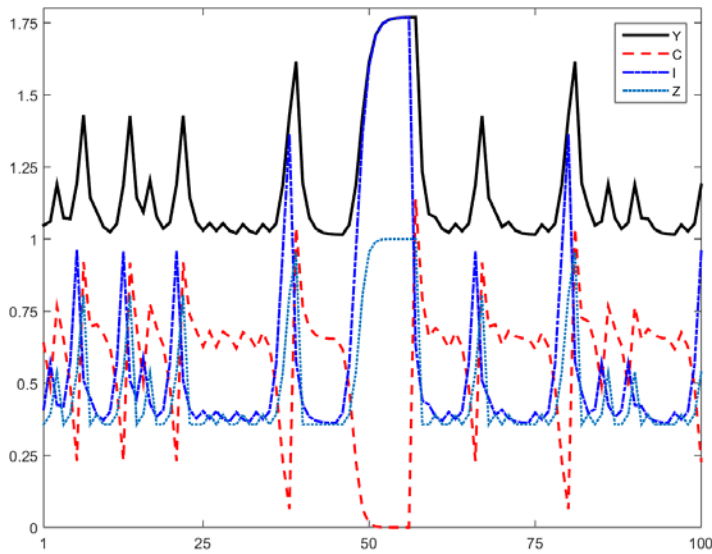
Notes: Business cycle statistics reported here are based on simulations runs of T=10000 periods. Standard deviations, correlations of GDP and autocorrelations pertain to medians of statistics across rolling windows of 200 periods. These moments pertain to logged series that were HP filtered (for each window of 200 periods). “Means” are sample averages over the total sample of T periods. The “Fraction of periods with (capital income > investment)” likewise pertains to the whole simulation run of T periods. Cols. (1)-(4) pertain to model variants in which fluctuations are just driven by bubbles (constant TFP). Cols. (5)-(8) pertain to variants with bubbles and TFP shocks. Cols. (9)-(10) assume just TFP shocks (without bubbles). Col. (11) reports empirical statistics (for the US).



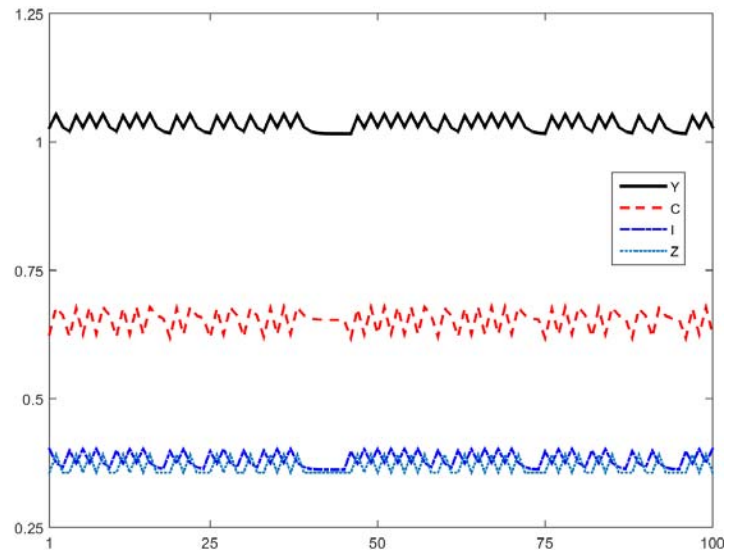
**Fig.1. Long & Plosser model:** investment/output ratio at t+1,  $Z_{t+1}$ , as a function of  $Z_t$  for  $\epsilon_{t+1} \in \{-0.5; 0; 0.5\}$



**(a)** ‘Low’ and ‘High’ values of date  $t+1$  investment/output ratio ( $Z_{t+1}^L, Z_{t+1}^H$ ) and expected value ( $E_t Z_{t+1}$ ) shown as function of  $Z_t \in [\alpha\beta + \Delta, 1)$ .  $\Lambda(Z_t, 0)$  is value of  $Z_{t+1}$  without random sunspot. Probability of ‘Low’ value  $Z_{t+1}^L$ :  $\pi_t = 0.5 \quad \forall t \quad Z_t \in [\alpha\beta + \Delta, 1)$



**(b)** Simulated series with constant probability:  $\pi_t = 0.5$ .

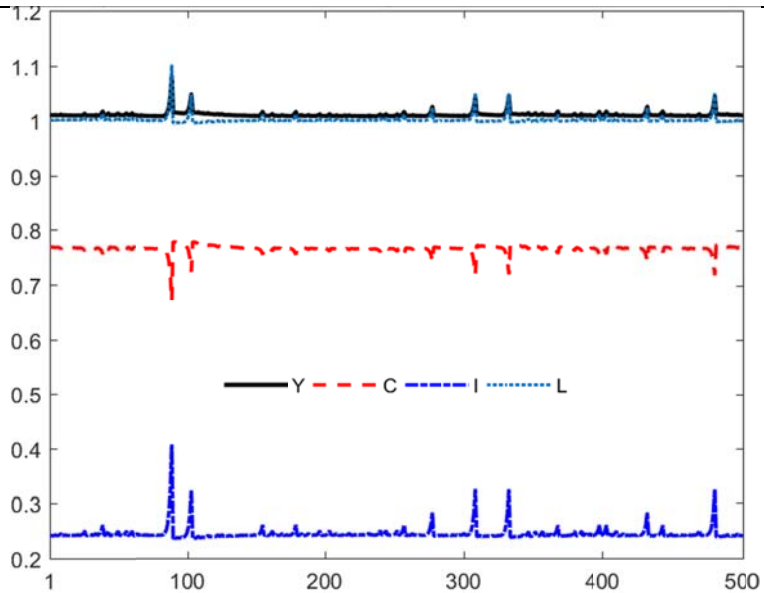


**(c)** Simulated series with  $\pi_t = 0.5$  for  $Z_t \leq 0.36$  and  $\pi_t \approx 1$  for  $Z_t > 0.36$

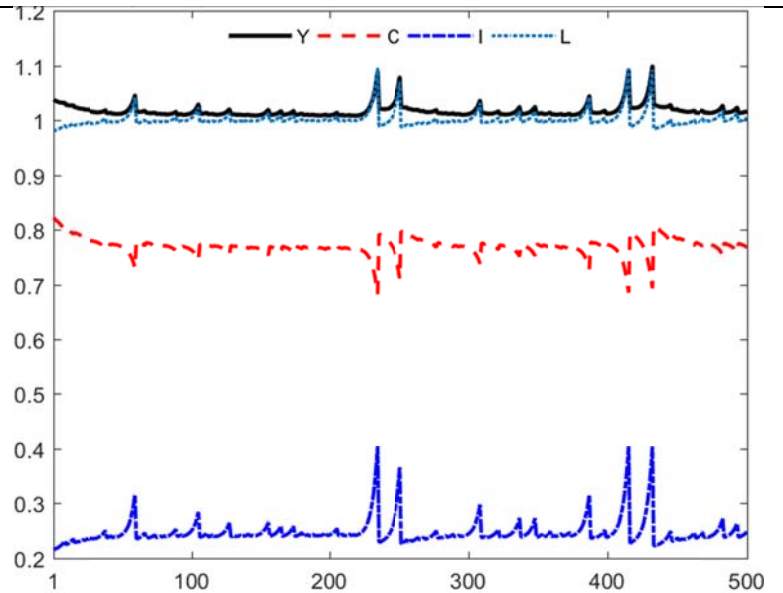
**Fig.2. Long & Plosser model with bubbles.**

Simulated series of output (Y), consumption (C) and investment are normalized by steady state output.

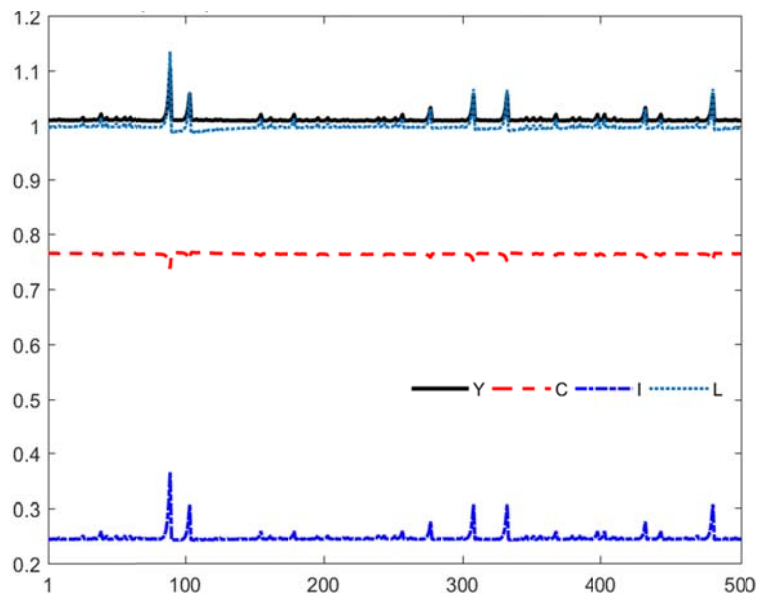




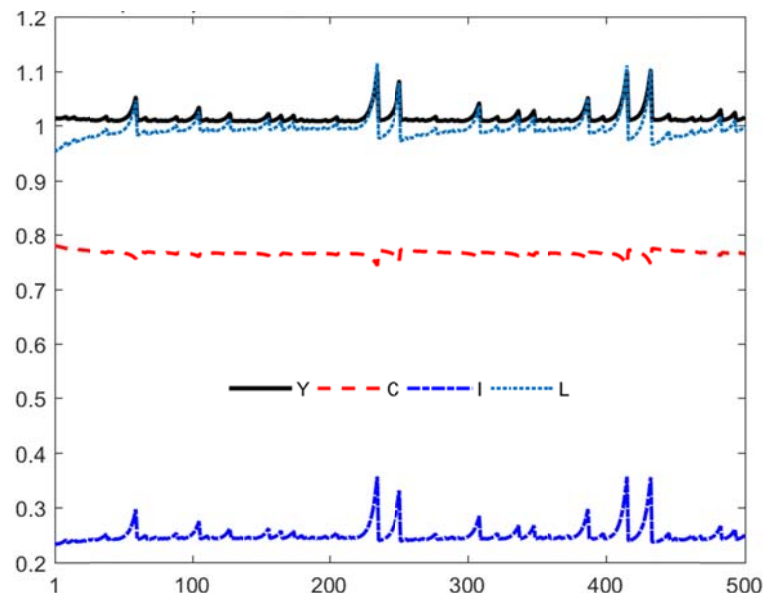
(1) Bubbles, no TFP shocks. Unit RA,  $\pi=0.5$



(2) Bubbles, no TFP shocks. Unit RA,  $\pi=0.2$



(3) Bubbles, no TFP shocks. High RA,  $\pi=0.5$



(4) Bubbles, no TFP shocks. High RA,  $\pi=0.2$

**Fig.3 Non-linear RBC model with incomplete capital depreciation (variable labor)**

Simulated paths of GDP (Y, continuous black line), consumption (C, red dashed line), investment (I, blue dash-dotted line) and hours worked (L, blue dotted line) are shown for 10 variants of the RBC model with incomplete capital depreciation and variable labor described in Sect. 3. Panel (i) of this Figure assumes the model variant considered in Col. (i) of Table 2. RA: risk aversion. GDP, C and I series are normalized by steady state GDP. The hours worked series is normalized steady state hours.

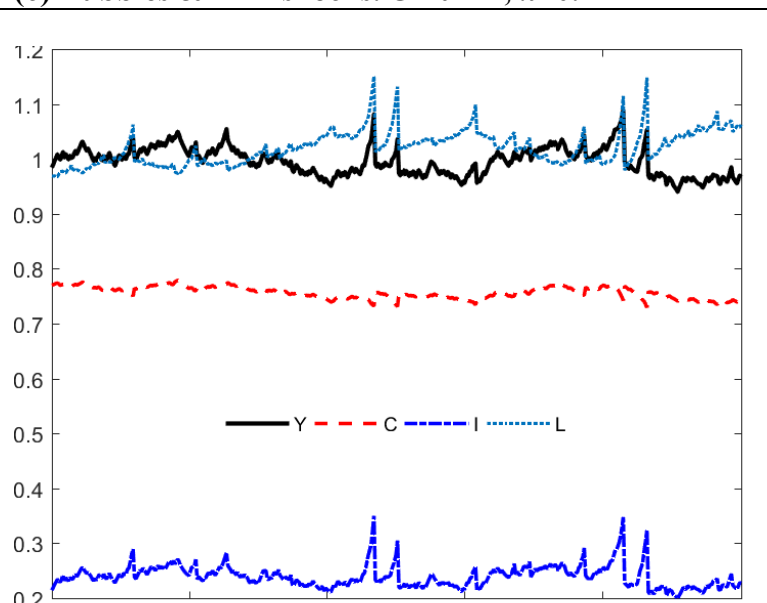
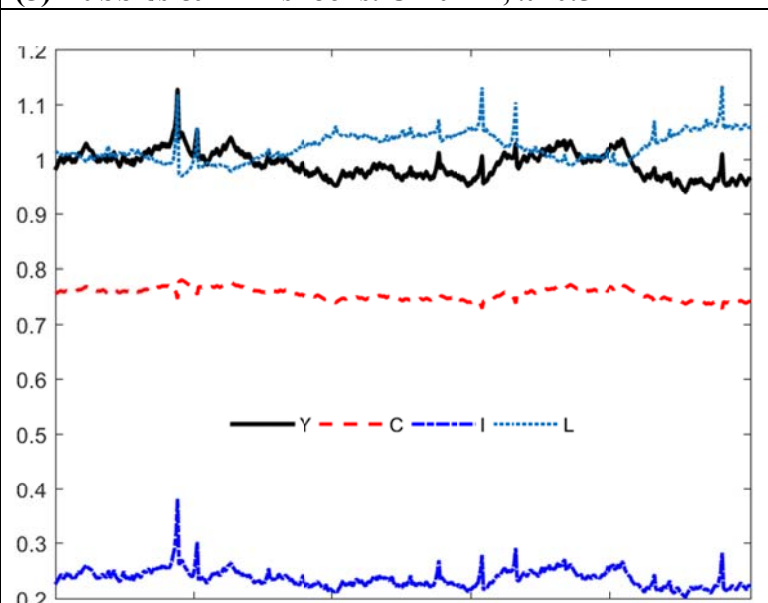
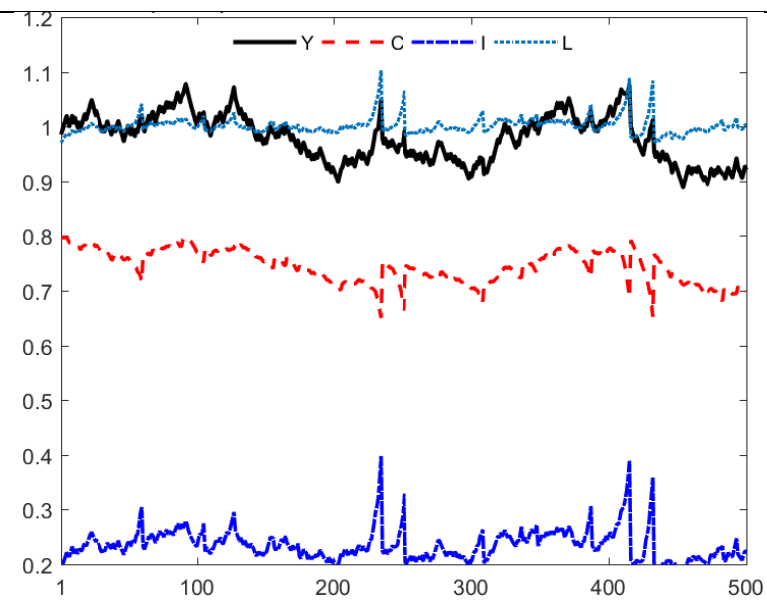
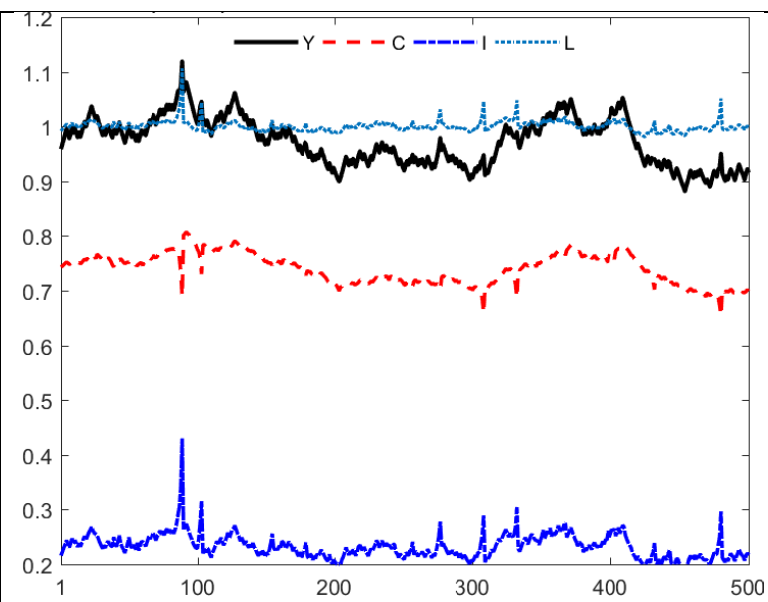
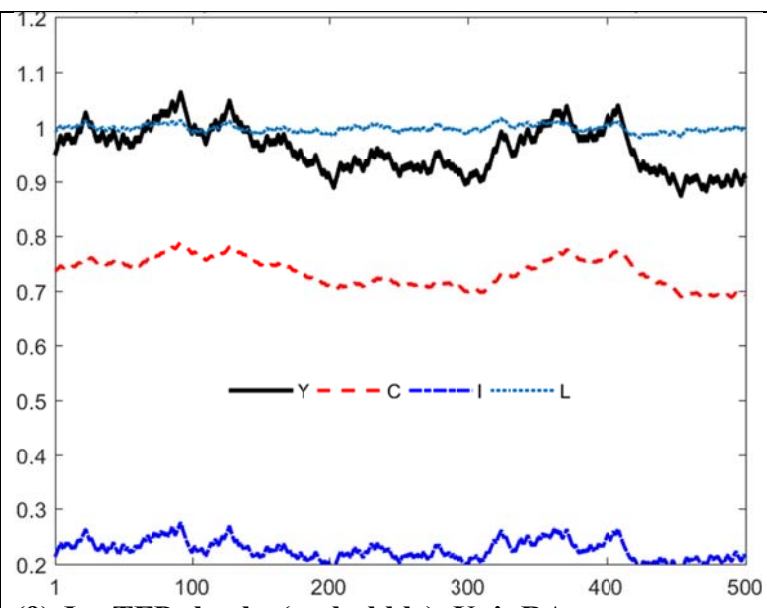
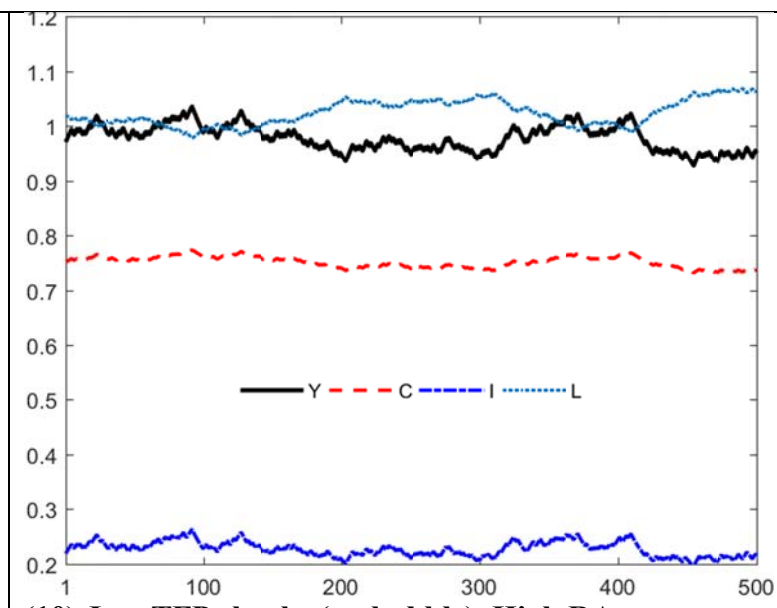


Figure 3. (continued)



(9) Jut TFP shocks (no bubble). Unit RA



(10) Just TFP shocks (no bubble). High RA

Figure 3. (continued)