The Macroeconomics of Sticky Prices with Generalized Hazard Functions*

Fernando Alvarez
University of Chicago and NBER

Francesco Lippi
LUISS University and EIEF

Aleksei Oskolkov
University of Chicago

October 26, 2020

Abstract

We give a thorough analytic characterization of a large class of sticky-price models where the firm's price setting behavior is described by a generalized hazard function. Such a function provides a tractable description of the firm's price setting behavior and allows for a vast variety of empirical hazards to be fitted. This setup is microfounded by random menu costs as in Caballero and Engel (1993) or, alternatively, by information frictions as in Woodford (2009). We establish two main results. First, we show how to identify all the primitives of the model, including the distribution of the fundamental adjustment costs and the implied generalized hazard function, using the distribution of price changes or the distribution of spell durations. Second, we derive a sufficient statistic for the aggregate effect of a monetary shock: given an *arbitrary* generalized hazard function, the cumulative impulse response of output to a once-and-for-all monetary shock is proportional to the ratio of the kurtosis of the steady-state distribution of price changes over the frequency of price adjustment. We prove that Calvo's model yields the upper bound and Golosov and Lucas's model the lower bound on this measure within the class of random menu cost models.

Key Words: Random Menu cost, Impulse response functions, Monetary Shocks, Generalized Hazard Function, Flexibility Index, sufficient statistic

JEL Classification Numbers: E3, E5

*We are grateful to Ricardo Caballero, Alberto Cavallo, Eduardo Engel, Andres Blanco, Isaac Bailey and Luca Dedola, for detailed suggestions and discussion of our paper. We also benefited from the comments of Klaus Adam, Erwan Gautier, Andrea Ferrara, Anton Nakov, Ricardo Reis, Raphael Schoenle, Rob Shimer, Nathan Zorzi, our discussant Andre Silva and participants to the 2019 ECB-PRISMA Conference in Lisbon, the 2019 workshop *Recent advances in macroeconomics* EIEF (Rome), the Macro-International Workshop at the University of Chicago, the Bank of Finland and CEPR Joint Conference on Monetary Policy Tools, Bank of Norway Workshop, the Hydra Conference in Athens, and the students in the 2019 Monetary Economics at The University of Chicago and LUISS for their comments. This paper originally circulated with the title "The Analytics of Monetary Shocks with Generalized Hazard Functions". This material is based, in part, on work supported by the National Science Foundation under grant number SES-1559459.

1 Introduction and Summary of Results

Most sticky price models relate the firm's price adjustment decision to its own price or markup, a natural proxy for the firm's desired adjustment. Caballero and Engel (1993a) model the firm's decision by employing a "generalized hazard function", a function relating the firm's price adjustment probability to its state. Such a function provides a tractable description of the firm's behavior and allows for a vast variety of empirical hazards to be fitted. Compared to the workhorse Calvo (1983) model, where the adjustment probability is constant, a generalized hazard function $\Lambda(x)$ allows it to depend on the state x, the firm's desired adjustment, such as the markup deviation from the desired level. Such state dependence is appealing theoretically, see e.g. Barro (1972); Sheshinski and Weiss (1977); Dixit (1991); Caplin and Spulber (1987); Golosov and Lucas (2007), and has been found to be relevant empirically, see e.g. Eichenbaum, Jaimovich, and Rebelo (2011); Gautier and Saout (2015). The notion of a generalized hazard function, and its derivation from first principles, were developed in seminal "menu-cost" papers by Caballero and Engel (1993a, 1999, 2007) and Dotsey, King, and Wolman (1999), and later revisited using information theoretical foundations by Woodford (2009) and Costain and Nakov (2011b). Several authors have since employed the generalized hazard function in applications and empirical work.[1]

This paper mostly follows the setup introduced by Caballero and Engel (2007) to frame a broad class of sticky price models where the firm's pricing decisions can be represented by a generalized hazard function $\Lambda(|x|)$. The symmetry of the function arises since we focus on economies where the idiosyncratic state is driftless, an accurate benchmark for low-inflation economies.[2] A large number of models are nested by this framework, including the two "extreme" versions: the canonical Calvo model with a constant hazard $\Lambda(x) = \lambda$ and unbounded x, and the Golosov and Lucas (2007) model with x bounded by the adjustment threshold $\pm X$ and a zero hazard on $|x| \in (0, X)$ with a spike that can be thought of as an "infinite hazard" at the adjustment thresholds.[3] Intermediate

[1] For recent applications see e.g. Costain and Nakov (2011a); Carvalho and Kryvtsov (2018); Sheremirov (2019); for empirical work see e.g. Berger and Vavra (2018); Petrella, Santoro, and de la Porte Simonsen (2018), and for related theoretical work Baley and Blanco (2019).

[2] See proposition 7 in Alvarez, Le Bihan, and Lippi (2016) for a result explaining why inflation has no first order effects on the propagation of monetary shocks in this class of models. See Alexandrov (2020) for an extension to the case of big inflation.

[3] The infinite hazard at the threshold should be thought of as an approximation of the behaviour at an sS barrier.

cases cover the so called Calvo-plus model by Nakamura and Steinsson (2010), the random menu cost problem of Dotsey and Wolman (2020), as well as the cases discussed above which explicitly use a generalized hazard function.

We employ this setup to prove two main analytical results that give a thorough understanding of the workings of sticky price models, their mapping to the data, and the propagation of monetary shocks. First we show how to identify *all primitives* of the model from readily available data. We establish an *invertible* mapping between the fundamental cost of price adjustment (menu cost or information cost) and the "reduced form" generalized hazard function. We consider two alternative foundations underlying this mapping. One, introduced in the seminal work by Caballero and Engel (1993b), assumes the firm can change its price upon paying a fixed (menu) cost ψ that is drawn every period from an *unrestricted* distribution of costs $G(\psi)$. We prove that the mapping between *any* given menu cost distribution $G(\psi)$ and the generalized hazard function $\Lambda(x)$ is invertible. This means that *any* non-decreasing generalized hazard function can be rationalized by a *unique* choice of the distribution of the random fixed costs $G(\psi)$. While the non-decreasing nature of the generalized hazard function was established by Caballero and Engel (1993a), we prove the invertibility of the mapping and give an explicit formula to recover $G(\psi)$ from any $\Lambda(x)$ non-decreasing in $|x|$. We also provide an identical result for an alternative foundation where the firm optimally selects the "probability" of adjustment λ in every period, subject to a cost $c(\lambda)$, a simplified version of Woodford (2009), where the cost is modeled in a rational inattention framework.[4] We show that every non-decreasing generalized hazard rate $\Lambda(x)$ can be rationalized by a convex cost function $c(\lambda)$.

We complete the mapping between theory and data by showing how to use the (observed) distribution of price changes $Q(\Delta p)$ and frequency of price changes N_a to fully identify the generalized hazard function $\Lambda(x)$.[5] A price change $\Delta p = -x$ is chosen by a firm with desired adjustment x that is given the option to adjust. A straightforward relation links the density of price changes $q(-x)$

In our results we make the different behaviour precise. In Proposition 3 we produce a rigorous approximation of the behaviour at barrier as a very high hazard rate, which justifies this analogy.

[4]See also Costain and Nakov (2011b) for a sticky price model where firms must pay a cost to increase the probability of a price change.

[5] Such data have been heavily used to discipline sticky price models over the past two decades, see e.g. Bils and Klenow (2004); Klenow and Malin (2010); Cavallo and Rigobon (2016).

to the hazard function and the cross-sectional distribution of desired adjustments, $f(x)$, namely $q(-x)N_a = \Lambda(x)f(x)$. Previous contributions such as Berger and Vavra (2018) retrieve $f(x)$ by postulating a parametric form for $\Lambda(x)$ and then using this relation. We show that, surprisingly to us, $\Lambda(x)$ and $f(x)$ are both fully encoded in $q(\Delta p)$ and N_a, and that it is possible to identify *both* functions using the distribution and frequency of price changes alone. We derive the expression to retrieve $f(x)$ and $\Lambda(x)$ from $q(\Delta p)$ and N_a in closed form. The recovery of the function $f(x)$ from observables relates to the work by Baley and Blanco (2019) who obtain all the moments of $f(x)$ even in the presence of drift and asymmetries. Using our first result, we can then recover the entire distribution of random menu cost $G(\psi)$. We propose an estimator for such distributions that is consistent with the theory and allows for unobserved heterogeneity among products. To illustrate our procedure, we use publicly available scraped-price data by Cavallo (2015) for the US, estimate the underlying distribution of price changes, measure its kurtosis, and recover $\Lambda(x)$, $f(x)$, and $G(\psi)$. Interestingly, accounting for measurement error and aggregation, and correcting for unobserved heterogeneity, we find values of kurtosis around 2, roughly consistent with a quadratic hazard function, and much smaller than those typically reported in the literature. Furthermore, we define a statistic $\mathcal{C}$ (for "Calvo-ness") that measures the fraction of price changes happening *independently* of the state of the firm. Using our characterization of the relationship between the observed distribution of price changes and the generalized hazard rate, we show that $\mathcal{C}$ is proportional to $q(0)$, the density of price changes near zero. We estimate $\mathcal{C}$ in Cavallo's (2015) data set and find it to be about 6%, i.e. about 94% of price changes show some state dependence.

The second main result of the paper gives a sufficient statistic for the aggregate effect of a monetary shock. We establish that the cumulative impulse response (CIR) of output to a once-and-for-all monetary shock in *any* model characterized by a generalized hazard function $\Lambda(x)$ is a simple function of two steady state statistics: the Kurtosis of the distribution of price changes divided by six times the frequency of price changes. The CIR, namely the area under the output impulse response function, is a convenient summary measure of the non-neutrality of monetary shocks. The notion of CIR was introduced in Alvarez, Le Bihan, and Lippi (2016), which showed that the "kurtosis result" holds for a Calvo-plus model (and multi-product firms), a class that implies a constant hazard function $\Lambda(x) = \lambda$ in the inaction region. Alvarez, Lippi, and Paciello (2016) showed that the kurtosis result holds in a large class of rational inattention models, as proposed by Reis (2006), which are purely time dependent. This paper provides a substantive generalization of the previous cases: we establish that the kurtosis result holds for *any* symmetric $\Lambda(x)$ function, allowing for both finite and infinite boundaries of the inaction region. This includes decreasing or non-monotone hazard functions (which are not rationalized by random menu cost models), and hazards with discontinuities corresponding to mass points in the distribution of menu cost. For instance, this provides a rigorous (negative) answer to Dotsey and Wolman (2020) who conjecture, based on numerical simulations of a different model, that the kurtosis result may fail to apply in a model with random menu costs.[6]

Our analysis shows that within the class of non-decreasing generalized hazard functions the largest Kurtosis is six, attained by the constant hazard rate model, like the pure Calvo (1983) case. The smallest, equal to one, corresponds to the pure menu cost model of Golosov and Lucas (2007). This result is interesting because non-decreasing hazard rates describe the entire set of models with random menu costs and information gathering, and thus it establishes Calvo as an upper bound within this broad class. For a Kurtosis higher than six, like in the pure Calvo model, one would need to come up with an economic foundation for a (locally) *decreasing* hazard function.

We develop several extensions in the appendices. Appendix E generalizes our mapping between the theory and data. We show that, under regularity conditions, the survival function $S(t)$, measuring the distribution of *durations* of unchanged prices, uniquely identifies $\Lambda(x)$ as well. Appendix F studies the scope of the Flexibility Index, $\mathcal{F}$, a notion introduced by Caballero and Engel (2007) to analyze the link between the microeconomic behavior and aggregate stickiness and used by some authors as a summary measure of monetary non-neutrality, see e.g. Berger and Vavra (2018); Petrella, Santoro, and de la Porte Simonsen (2018). We show that the index can sometimes be a poor summary of monetary non-neutrality, both over the long-term as well as over the short-term.

[6]One reason the "kurtosis result" fails in their numerical analysis is likely a fraction of firms wit flexible prices. Heterogeneity across firm types makes it essential to properly aggregate. Failing to do so will obfuscate the result, which holds for each firm's type. We analyze the case with heterogenous firms in equation (44) and discussion around it.

Structure of the paper. The next section provides two foundations for the generalized hazard function. In the first one (Section 2.1), firms choose when to change prices subject to random menu costs, distributed according to CDF G. In the second one (Section 2.2), firms choose the intensity with which they can change prices, subject to a cost function c. In both models, the optimal decision rule is summarized by a generalized hazard function Λ. We show that in both models, given Λ, one can recover the primitive cost, either G or c. Section 3 characterizes the steady-state statistics of a model where the firms' decisions follow a generalized hazard function. Section 4 shows how to recover Λ starting from an observed distribution of the size of price changes and presents our estimation using Cavallo's (2015) data set. Section 5 discusses the propagation of a once-and-for-all small aggregate shock in an economy characterized by a generalized hazard function, and proves that its effect can be summarized by a simple sufficient statistic. Section 6 concludes discussing extensions and future work. The appendix contains selected proofs, the rest of which are relegated to an online appendix.

2 Foundations of the Generalized hazard function $\Lambda(x)$

The generalized hazard function is a building block of several macro models featuring sticky prices. It is a function that maps the state of the firm, x, e.g. the deviation of the current markup from the profit maximizing one, into the *likelihood* of a price adjustment $\Lambda(x)$. Such a function is appealing to scholars because it allows for substantive flexibility in fitting cross-sectional data on price setting behavior while, at the same time, having explicit microeconomic foundations. This section presents two alternative settings for such foundations, and provides an invertible mapping that allows one to recover the foundations from a given hazard function.

Our first setup uses a random menu cost model, first proposed by Caballero and Engel (1993a) and elaborated in Caballero and Engel (1999, 2007). A particular case, the Calvo-plus model, was analyzed by Nakamura and Steinsson (2010). The second setup relates to models of inattention as in Woodford (2009), where firms choose the arrival rate of opportunities to change prices.[7]

[7] In Woodford (2009) the form of the firm's problem and the specification of $c(\cdot)$ are derived assuming constraints on information flows.

Both setups feature a firm that maximizes the present discounted value of a per-period profit function given by $-Bx^2$, a second order approximation of the profit function, where x is the price gap, and the parameter $B > 0$ measures the curvature of the profit function. If prices are not changed, the price gap x evolves as a standard Brownian Motion with zero drift and variance σ^2. The lack of drift indicates that the economy under consideration has no inflation.[8] The two setups differ in the friction that prevents the firm from setting $x = 0$ at all times. In the first, the friction is due to the presence of random fixed costs of price adjustment; in the second the friction is due to an information cost.

2.1 The Random Menu Cost Model

The Calvo-plus model supplements the traditional Calvo model with the possibility that the firm can change its price by paying a fixed menu cost at any time. The advantage of this model is to eliminate a long tail of delayed adjustments that seems counterfactual. The generalized model allows the firm to draw a fixed menu cost ψ from a distribution G at random times – arriving at a Poisson rate $\kappa > 0$.

As in Caballero and Engel (1993a), we call the difference between the current price of the firm and its ideal price a “price gap”. We will specify the process for the demand and cost of the firm, so that the price gap is the state of the firm's problem. The menu costs drawn by the firm can be zero or strictly positive. If the cost is zero the firm changes its price to the ideal one (i.e. it “closes its price gap”). If the firm draws a strictly positive cost, it will either ignore it or change its price depending on the value of the “price gap” relative to the realization of the fixed cost. In particular, the optimal decision rule will be characterized by a threshold rule that gives the maximum adjustment cost that the firm is willing to pay for adjustment. For all fixed costs smaller than the threshold the firm changes its price, while for larger costs it keeps the price unchanged.

We also allow the firm to have a price change at any time by paying a (relatively large) fixed

[8] See The Online appendix B in Alvarez and Lippi (2014) for a microfoundation of this model. The focus on a model with zero inflation provides an accurate approximation for economies where inflation is low, as the effects on decision rules are of second order when inflation is close to zero, as shown theoretically and validated empirically in Alvarez et al. (2019); Alvarez and Lippi (2020).

cost, which we denote by $\Psi > 0$ and refer to as the "deterministic fixed cost". If $\Psi = \infty$, then the firm has no such alternative. We can write the value function of the firm, $v(x)$, as:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi}\min\left\{\psi + \min_{\tilde{x}} v(\tilde{x}) - v(x)\,,\,0\right\}dG(\psi)\,,\,r\left(\Psi + \min_{\tilde{x}} v(\tilde{x})\right)\right\}$$

Two points are worth making. First, given the symmetry of Bx^2, the value function is *symmetric around* $x = 0$. A proof can be constructed by a simple guess and verify argument. Second, if $\Psi = \infty$ then $X = \infty$, and thus there is no second branch in the Bellman equation. Note that as long that either $r > 0$ and/or that $\kappa > 0$, the value function v is finite and well defined in the case of $\Psi = \infty$.

The term $\min_{\tilde{x}} v(\tilde{x})$ is the value right after adjustment, and given the symmetry of the return function, we have $v(0) = \min_{\tilde{x}} v(\tilde{x})$. Thus we can simply write that for all x

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\int_0^{\Psi}\min\left\{\psi + v(0) - v(x)\,,\,0\right\}dG(\psi)\,,\,r\left(\Psi + v(0)\right)\right\}$$

For the case where $\Psi < \infty$ we can use that the optimal decision rule has a threshold $X < \infty$ such that if $|x| \geq X$ the firm pays the fixed cost Ψ. Thus we can write

$$rv(x) = \begin{cases} Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\displaystyle\int_0^{v(x)-v(0)}\psi dG(\psi) + \kappa\left[v(0) - v(x)\right]G\left(v(x) - v(0)\right), & |x| \leq X \\ r\left(v(0) + \Psi\right), & |x| > X \end{cases} \tag{1}$$

Note that we can define the threshold function $\bar{x} : [0, \Psi] \to [0, X]$ as solving

$$v(\bar{x}(\psi)) = v(0) + \psi \text{ for all } \psi \in [0, \Psi] \tag{2}$$

It is easy to see that v is increasing in $|x|$, since the period cost Bx^2 is strictly increasing in $|x|$, the uncontrolled process is a brownian motion, and the adjustment cost is independent of x. Since v is strictly increasing in $[0, X]$, then $\bar{x}'(\psi) = 1/v'(\bar{x}(\psi)) > 0$. We can let the function $\bar{\psi}(x)$ be the inverse of $\bar{x}(\psi)$.

For simplicity, in the characterization of the problem that follows we will assume a distribution function G with a continuous density. We require G to be continuously differentiable at all points, with the possible exception of $\psi = 0$. For completeness, Appendix F considers the case of a discrete distribution G, where ψ takes finitely many values.[9] In either case we have the following smooth pasting and optimal return point conditions:

$$v'(-X) = v'(X) = v'(0) = 0 \tag{3}$$

We are now ready to define the *generalized hazard rate*, $\Lambda : (-X, X) \to \mathbb{R}_+$, which gives the probability (per unit of time) that a firm with $x \in (-X, X)$ will change its price. It is defined by the optimal decision rule, or the value function, as well the Poisson arrival rate $\kappa > 0$ and the distribution of fixed cost G:

$$\Lambda(x) = \kappa \, G\left(v\left(x\right) - v\left(0\right)\right) \text{ for all } x \in (-X, X) \quad . \tag{4}$$

The function Λ is symmetric around $x = 0$ and weakly increasing in $|x|$. It is continuous at x if G is continuous at $\psi = v\left(x\right) - v\left(0\right)$, and bounded above by κ. While the function Λ is not defined at $x = \pm X$, we abuse notation and let $\Lambda(X) = \lim_{x \to X} \Lambda(x) = \kappa \, G\left(\Psi\right) = \kappa$.

2.1.1 Rationalizing a given generalized hazard Λ

We next show that any increasing, differentiable, symmetric and bounded hazard rate Λ can be rationalized as the solution to the firm problem in equation (1) by a unique menu cost distribution G and two parameters $\{\kappa, \Psi\}$. Our proof is constructive: we provide an algorithm to compute $\{G, \kappa, \Psi\}$ from Λ, proving existence and uniqueness. Indeed G is obtained by solving a linear ordinary differential equation of the second order. Section 2.2 describes an alternative problem of the firm that also generates a non-decreasing generalized hazard function. We find this interesting because it allows us to relate to setups costly information collection, as in Woodford (2009).

The main result in this section shows how to recover the distribution G, with a density $G' = g$,

[9] The two cases differ on whether $\bar{x}(\cdot)$ is a continuous function, and on whether the value function $v(\cdot)$ is twice differentiable everywhere or it has jump discontinuities on finitely many values. Indeed in the latter case we need to rewrite the value function since $v''(x)$ is not defined at all points.

given Λ and the values of three parameters: r, B, and σ^2. Three remarks are in order. First, the values of the fixed costs ψ are measured relative to B, and thus the optimal decision rules depend only on the distribution of ψ/B. Second, we show that σ^2, while in principle unobservable, is encoded in the frequency and variance of price changes. Thus, once Λ is given, we can recover all the parameters of the firm's problem, except the discount rate r. Third, while in this section we consider the case where G is differentiable for $\psi > 0$ to simplify the exposition, Appendix F considers discrete distributions of costs which imply an hazard Λ that is a step function. In this case we can recover G starting from Λ by solving a system of linear equations.

Assume the firm faces a distribution G of the menu costs with a density g for all $\psi > 0$, and possibly a mass point at $\psi = 0$. In this case its Bellman equation solves

$$r v(x) = B x^2 + \frac{\sigma^2}{2} v''(x) + \int_0^{v(x)-v(0)} \kappa[\psi + v(0) - v(x)] g(\psi) d\psi + \kappa[v(0) - v(x)] G(0) \tag{5}$$

for all $x \in [-X, 0]$, and we can use the symmetry of v to define it as $v(x) = v(-x)$. The boundary conditions are $v'(X) = 0$ and $v(X) = v(0) + \Psi$, the smooth pasting and value matching. Note that in the interior $(0, X)$ the function v solves a *non-linear* ordinary differential equation.

Before the main result on the existence of a unique invertible mapping between Λ and G, we state an intermediate result that provides a solution for the value function v and a new auxiliary function that will be used to solve the general problem. Consider the function Λ describing the probability per unit of time of a price adjustment if the price gap is $|x| < X$. We have the following:

Lemma 1. Let the function u solve the linear ordinary differential equation

$$[r + \Lambda(x)]\, u(x) = 2Bx + \frac{\sigma^2}{2} u''(x) \text{ for } x \in [0, X] \tag{6}$$

with boundary conditions $u(0) = u(X) = 0$. The solution for u is unique. Moreover, v is given by

$$v(x) = u'(0)\frac{\sigma^2}{2r} + \int_0^x u(z) dz \quad \text{for } x \in [0, X]\ . \tag{7}$$

The auxiliary function u can readily be used to compute the value function and, as shown below, to characterize the distribution of costs that rationalizes the postulated hazard function. We now state the main result of this section:

THEOREM 1. Fix a discount rate $r > 0$, the curvature of the profit function $B > 0$, the volatility of shocks $\sigma > 0$, and the threshold X, with $X \in \mathbb{R}_+ \cup \{+\infty\}$. Consider a generalized hazard function $\Lambda(\cdot) : (-X, X) \to \mathbb{R}_+$ that is symmetric around zero, increasing in $|x|$, differentiable on $(0, X)$, and bounded. There exist real numbers $\{\kappa > 0, \Psi > 0\}$, both positive, and a cost distribution $G(\cdot) : [0, \Psi] \to [0, 1]$ with a density $g(\cdot)$, continuous on $(0, \Psi)$, and possibly a mass point $G(0) > 0$, that uniquely rationalizes Λ with a value function that solves equation (5). Using the auxiliary function u in Lemma 1 and $U(x) = \int_0^x u(z)dz$ for $x \in (0, X)$,

$$\kappa = \lim_{x \uparrow X} \Lambda(x) \quad , \quad \Psi = U(X) \quad , \quad G(0) = \frac{\Lambda(0)}{\kappa} \tag{8}$$

$$g\left(U(x)\right) = \frac{\Lambda'(x)}{u(x)\kappa} \text{ for all } x \in (0, X) \;\; \text{with} \;\; \psi = U(x) \tag{9}$$

The theorem allows us to retrieve the primitives of a fully specified price setting problem starting from any given non-decreasing hazard function Λ. Note that whenever $\Lambda(0) > 0$ the model implies a mass point at $\psi = 0$. Intuitively, rationalizing a non-zero probability of adjustment when the gap is small requires a mass point of zero menu costs. Also note that $g(\cdot) > 0$ requires $\Lambda'(\cdot) > 0$.

Application: a quadratic hazard function. We conclude with an application to a quadratic generalized hazard function $\Lambda(x) = \Lambda_0 + \Lambda_2 x^2$ where $\Lambda_0 \geq 0, \Lambda_2 \geq 0$ and $|x| \in [0, X]$.[10] We can solve for the auxiliary function $u(x)$ using Lemma 1. This yields a polynomial:

$$u(x) = \sum_{i=0}^{\infty} a_{2i+1}\, x^{2i+1} \tag{10}$$

[10] If Λ is symmetric and smooth, it often admits a quadratic approximation close to zero. This feature, mentioned by Caballero and Engel (2007) and Berger and Vavra (2018), makes quadratic generalized hazard functions especially appealing. In Appendix B we show that if Λ does not admit a quadratic approximation around $x = 0$, the underlying density g exhibits non-generic behavior.

satisfying the ODE in equation (6) and the boundary conditions $u(0) = u(X) = 0$. Straightforward application of the method of undetermined coefficients gives the recursive relation

$$a_3 = \frac{(r + \Lambda_0)a_1 - 2B}{3\sigma^2} \tag{11}$$

$$a_{2i+1} = \frac{(r + \Lambda_0)a_{2i-1} + \Lambda_2 a_{2i-3}}{\sigma^2\, i(2i+1)} \quad \text{for} \quad i = 2, 3, ... \tag{12}$$

All coefficients are determined as a function of a_1, which is pinned down by the boundary condition $u(X) = 0$. Application of Theorem 1 gives $U(x) = \sum_{i=1}^{\infty} \frac{a_{2i-1}}{2i} x^{2i}$, the value function $v(x) = a_1 \frac{\sigma^2}{2r} + U(x)$, the arrival rate $\kappa = \Lambda_0 + \Lambda_2 X^2$, the distribution function $G(\psi)$ with

$$G(0) = \frac{\Lambda_0}{\Lambda_0 + \Lambda_2 X^2} \tag{13}$$

Note that if $\Lambda_0 > 0$ the proposition implies a mass point at $\psi = 0$. For $|x| \in [0, X]$, the proposition gives the menu cost density function

$$g(U(x)) = \frac{2\Lambda_2 x}{\kappa u(x)} \tag{14}$$

The limit of the density is finite and positive, $\lim_{\psi \downarrow 0} g(\psi) = \frac{2\Lambda_2}{\kappa a_1}$. This happens because $v(x)$ is smooth and symmetric, so $u(x) = v'(x)$ admits a linear approximation close to zero.

2.2 An Optimal Adjustment Intensity Model

In this section we describe an alternative setup that yields a similar mapping between an underlying cost function and the generalized hazard. Now the firm does not face random menu cost. Instead, it directly controls the arrival rate of a free opportunity to change prices. At each moment the firm must pay a flow cost $c(\ell)$ to obtain an arrival rate ℓ. We assume that the flow cost is increasing and convex. This will give rise to the choice of the optimal rate of price changes as a function of the price gap, leading to a generalized hazard function Λ. As in the previous case, we also allow the firm to pay a deterministic menu cost Ψ to change its price with certainty. This Ψ will give rise to a barrier X, and we allow $\Psi = \infty$, in which case this will never be used, so $X = \infty$.

The main result of this section is that, analogously to the previous setup, any increasing symmetric function Λ can be rationalized by some increasing and convex cost function c. One difference between the two setups is that the resulting Λ in this setup does not need to be bounded above. This justifies the use of some of our examples later on. Additionally, this setup imposes fewer constraints on the tails of the implied distribution of price changes.

The firm's problem is:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \min_{\ell\geq 0}\{\ell\,(v(0) - v(x)) + c(\ell)\}\,,\, r\,(\Psi + v(0))\right\} \tag{15}$$

We assume that the cost function $c : \mathbb{R}_+ \to \mathbb{R}_+$ is increasing and convex in ℓ, and that $c(\ell) \to \infty$ as $\ell \to \infty$. We can also allow c to have finitely many flat segments, and do not assume that c is continuously differentiable. The possibility of kinks in c may be needed to rationalize constant segments on Λ. Allowing for flat segments in c implies that the minimizer ℓ^* may be an interval for some x, which we can represent with a discontinuity in Λ at that value of x. We can now state a result that echoes the one in Theorem 1:

Proposition 1. Fix a discount rate, curvature, variance, and a value of the threshold (r, B, σ^2, X), all positive. Let $\Lambda(\cdot) : (-X, X) \to \mathbb{R}_+$. Assume that $\Lambda(\cdot)$ is symmetric around zero, $\Lambda(x) = \Lambda(-x)$, increasing in $|x|$, and differentiable on $(0, X)$. Then, there exists an increasing convex cost function $c(\cdot) : \mathbb{R}_+ \to \mathbb{R}_+$ that uniquely rationalizes the postulated decision rule as in equation (15). Moreover, the marginal cost $c'(\cdot)$ can be constructed by solving a second order linear ordinary differential equation.

The proof of the statement follows the same logic used in the proof of Theorem 1. Appendix B provides more details on the solution of this model. Note that observation on the frequency and size of price changes cannot in general distinguish between the random menu cost model of Section 2.1 and the optimal intensity of price adjustment of this section. In this sense, the generalized hazard function Λ is a more fundamental object. Furthermore, as explained above, the model of this section allows a slightly larger set of generalized hazard functions Λ.

3 Steady State observable statistics

In this section we show how to use the hazard function Λ to derive several observable statistics produced by our model in the steady state. In particular, we solve for the implied invariant distribution of price gaps, with density $f(x)$, the number of price changes per unit of time, N_a, and the distribution of price changes, with density $q(\Delta p)$. We focus on two moments of this distribution, the variance and the Kurtosis, denoted $Var(\Delta p)$ and $Kurt(\Delta p)$. The setup allows for $\Psi \in \bar{\mathbb{R}}_+ \equiv \mathbb{R}_+ \cup \{\infty\}$. If Ψ is finite then the inaction range is bounded, $X < \infty$. Otherwise, the support is unbounded, $X = \infty$. Both cases are encompassed by the analysis of this section. The starting point of this section is the function Λ that summarizes the firm's optimal decisions:

Assumption 1. Let $\Lambda : (-X, X) \to \mathbb{R}_+$, be non-negative, piece-wise continuous, symmetric, i.e. $\Lambda(x) = \Lambda(-x)$ for all x, with at most finitely many discontinuities $x_k \geq 0$, and let $\mathbb{J} \equiv \{x_k\}$. If $X = \infty$, we assume that there is a $\lambda > 0$ and $0 < x_H < \infty$ such that $\Lambda(x) \geq \lambda$ for all $|x| > x_H$.

Note that if Λ is the solution to the firm problem studied in Section 2, then $\Lambda(x)$ must be weakly increasing for $x > 0$, although Assumption 1 does not impose that.

Next we define the invariant distribution of price gaps, with density $f(\cdot) : (-X, X) \to \mathbb{R}_+$. Importantly, f must be continuous everywhere, continuously differentiable at $|x| \in (0, X)$, twice continuously differentiable at all $|x| \in (0, X)/\mathbb{J}$, and symmetric around $x = 0$. Given the symmetry, we only define f on positive real values. It solves the following equations:

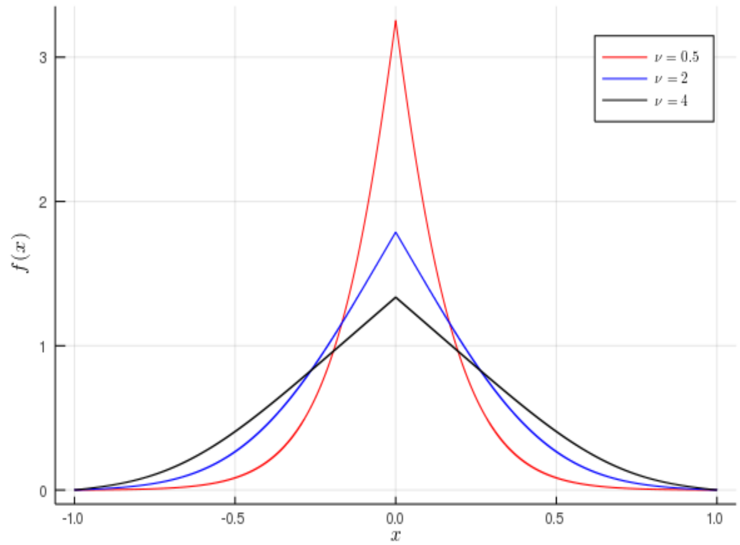
$$f(x)\Lambda(x) = \frac{\sigma^2}{2} f''(x) \text{ for all } x \in [0, X)\,,\ x \neq 0 \text{ and } x \notin \mathbb{J} \tag{16}$$

$$\text{with boundary conditions: } \frac{1}{2} = \int_0^X f(x)dx \quad \text{and} \quad \lim_{x \to X} f(x) = 0\ . \tag{17}$$

Note that if $\Psi < \infty$, then $f(X) = 0$ is an implication of X being an exit point, i.e. a barrier. Otherwise it is a requirement for integrability. Figure 1 plots three examples of the invariant distribution of price gaps which solves equation (16)-(17) for a hazard function with power form $\Lambda(x) = \kappa \left(\frac{x}{X}\right)^\nu$. The quadratic case, $\nu = 2$, has been considered for instance by Caballero and Engel (1993a); Berger and Vavra (2018).

Figure 1: Density function $f(x)$ for the invariant distribution of gaps

Power hazard function: $\Lambda(x) = \kappa\left(\frac{x}{X}\right)^{\nu}$, $X < \infty$



Frequency of price changes N_a. There are two types of price changes: those that occur when x reaches X, if it is finite, and those that occur when the firm draws a low enough fixed cost. Since X is an exit point, the number of price changes of the first type is given by $-2\frac{\sigma^2}{2}f'(X)$. The sign is negative because $f'(X)$ is negative. The 2 in front is becaus the same number of price changes happens when x reaches X as when x reaches $-X$. Note that if $X = \infty$ then $f'(X) = 0$. The second type of price changes occurs when $|x| < X$, which happens with density $f(x)$, and draws a sufficiently low fixed cost, which happens with probability $\Lambda(x)$ per unit of time. This gives

$$N_a = 2\left[\int_0^X f(x)\Lambda(x)dx - \frac{\sigma^2}{2}f'(X)\right] \quad . \tag{18}$$

We remark for future reference that, as shown in Alvarez, Le Bihan, and Lippi (2016) for a very wide class of models that includes the ones in this paper, the following relation holds for any feasible policy in this class of menu cost problems:[11]

$$N_a Var(\Delta p) = \sigma^2 \tag{19}$$

[11] The key assumption for this result to hold is that the price gap is closed upon adjustment. This assumption is not true in e.g. models with high inflation or models with price plans, see Alvarez and Lippi (2020).

This equation will be useful later in applications. We will use s for the fraction of price changes that occur before hitting the boundary $\pm X$. We can use equation (18) to replace the Kolmogorov forward equation for f, and integrate by parts to obtain that:

$$s \equiv \frac{\int_{-X}^{X} \Lambda(x) f(x) dx}{N_a} = 1 - \frac{\sigma^2 |f'(X)|}{N_a} = 1 - \frac{|f'(X)|}{|f'(0)|} \text{ since } N_a = \sigma^2 |f'(0)| \tag{20}$$

where $|f'(0)|$, with a slight abuse of notation, is the absolute value of either the right or left derivative of $f(x)$ evaluated at $x = 0$.

Distribution of price changes. Recall that upon any price change the firm "closes" its gap x, i.e. the size of the adjustment is $\Delta p = -x$. If $X < \infty$ then the distribution of price changes has a mass point at $\Delta p = -X$. The mass of such price changes is equal to $\frac{\sigma^2}{2}|f'(X)|$. There are also price changes of size $|\Delta p| < X$ that occur when a firm has $x < X$ and draws a sufficiently low fixed cost. This occurs with probability $\Lambda(x)$ per unit of time for a firm with price gap x. Recall also that at steady state, there is a density $f(x)$ of firms with price gap x. This density is symmetric around zero. The distribution of price changes is thus symmetric around zero as well. It has the following form:

$$\Delta p = \begin{cases} -x & \text{w/ density } q(-x) \equiv \frac{\Lambda(x) f(x)}{N_a} \text{ for } x \in (0, X) \\ -X & \text{w/ probability } \frac{\frac{\sigma^2}{2}|f'(X)|}{N_a} \end{cases} \tag{21}$$

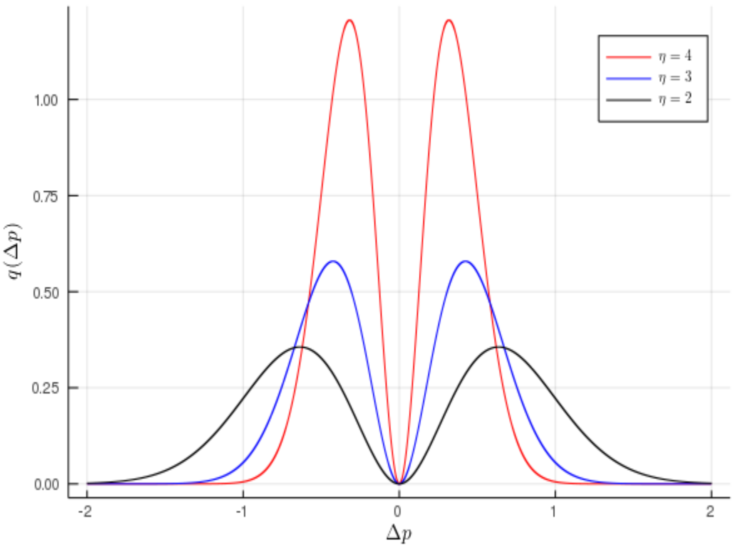
Note that $1 - s$, as defined in equation (20), is also twice the size of the mass point at the boundary of the support of this distribution.

Figure 2 plots a few examples of the density of price changes implied by a quadratic hazard function $\Lambda(x) = \kappa x^2$ with an unbounded support $X = \infty$. This uses the definition $q(-x) \equiv \frac{\Lambda(x) f(x)}{N_a}$ from equation (21), where the density of price gaps f solves the Kolmogorov forward equation equation (16). The generalized hazard function and frequency of price changes alone are sufficient to construct both f and q.

We note that in the quadratic case the distribution of price changes is indexed by a single parameter $\eta \equiv \left(\frac{2\kappa}{\sigma^2}\right)^{\frac{1}{4}}$ determining its shape, and features no mass points at the boundary of the inaction region since $X = \infty$. This means $s = 1$ in terms of equation (20). The parameter η is

Figure 2: Density function $q(\Delta p)$ of the distribution of price changes

Quadratic Hazard function: $\Lambda(x) = \kappa x^2$, $X = \infty$, shape parameter: $\eta \equiv \left(\frac{2\kappa}{\sigma^2}\right)^{\frac{1}{4}}$



recurring in the class of generalized hazard functions of the power form, and generally determines the shape of the distributions of price changes.

For future reference we define two useful moments. The variance and the Kurtosis of the price changes $Kurt(\Delta p)$ can be defined using the distribution in equation (21):

$$Var(\Delta p) = \frac{2\left[\int_0^X x^2 \Lambda(x) f(x) dx - X^2 \frac{\sigma^2}{2} f'(X)\right]}{N_a} \tag{22}$$

$$Kurt(\Delta p) = \frac{2\left[\int_0^X x^4 \Lambda(x) f(x) dx - X^4 \frac{\sigma^2}{2} f'(X)\right]}{N_a} \frac{1}{\left[Var(\Delta p)\right]^2} \tag{23}$$

Standardization. It is useful to rescale the firm's decision rule to isolate the role of the shape of Λ and of other parameters. Standardization clarifies which objects matter conceptually, and also helps to bring the model to the data, as shown in Section 4. Let's start with a price-setting problem represented by the triplet $\{X, \Lambda, \sigma^2\}$ with $\sigma^2 > 0$ and $\Lambda : (-X, X) \to \mathbb{R}_+$ satisfying Assumption 1. Given the triplet $\{X, \Lambda, \sigma^2\}$, we can compute the corresponding density of price changes $q(\cdot) : (-X, X) \to \mathbb{R}$, the variance of price changes $Var(\Delta p)$, the frequency of price changes N_a, and the share of price changes away from the boundaries s. We have the following result:

Proposition 2. Consider an economy characterized by $\{X, \Lambda, \sigma^2\}$, and associated $q, Var(\Delta p), N_a$ and s. For any $b > 0$ define another economy $\{\tilde{X}, \tilde{\Lambda}, \tilde{\sigma}^2, \}$ where $\tilde{X} = b\,X$, $\tilde{\Lambda}(z) = \Lambda(z/b)$ for all $z \in (-\tilde{X}, \tilde{X})$, and $\tilde{\sigma} = b\,\sigma$. These economies feature: (i) the same frequency of price changes, $\tilde{N}_a = N_a$, (ii) the same fraction of price changes away from the boundaries, $\tilde{s} = s$, and (iii) the same shape of the density of price changes, namely: $\tilde{q}(z) = q(z/b)/b$ for all $z \in (-X\,b, X\,b)$.

Note that we can choose $b^2 = 1/Var(\Delta p)$, for instance, so that the variance of price changes in the rescaled economy is one, $\tilde{V}ar(\Delta p) = 1$. This new economy can then be referred to as "standardized". The proposition shows that $Kurt(\Delta p)$ and the share s only depend on the shape of Λ, described by $\hat{\Lambda}$. In general, the shape cannot be summarized by a finite number of parameters, but in some situations a single parameter will suffice. For instance, below we consider a case where Λ is a power function and $\hat{\Lambda}$ is described by a single parameter.

In addition to the standardization described above, we can also consider transformations akin to changing the time units, thus only affecting the frequency of adjustment, but not the distribution of price changes. In particular, consider a scalar $k > 0$, and define $\hat{\Lambda}(x) = \Lambda(x)/k$ for all x, $\hat{\sigma}^2 = \sigma^2/k$ and $\hat{X} = X$. It easy to see that $\{\hat{X}, \hat{\Lambda}, \hat{\sigma}^2\}$ has $\hat{N}_a = kN_a$ and $\hat{Q}(x) = Q(x)$ for all x.[12]

A useful approximation. We conclude with a proposition showing that for the case in which $\Psi < \infty$, so that $X < \infty$, the invariant distribution can be accurately approximated by one corresponding to a generalized hazard function Λ with unbounded support and arbitrarily large values for $x > X$. This approximation is useful because the case with unbounded support is somewhat simpler to analyze, since it does not involve discussing the mass points at the boundary of the inaction region.

Proposition 3. Let $X < \infty$ and let $\Lambda : [0, X) \to \mathbb{R}_+$ be a continuous generalized hazard function, where $f : [0, X] \to \mathbb{R}_+$ is its corresponding invariant density, assumed to be symmetric. Let $\Lambda_k : [0, \infty) \to \mathbb{R}_+$ be defined as $\Lambda_k(x) = \Lambda(x)$ if $x < X$ and $\Lambda_k(x) = k$ otherwise. Let also $f_k : [0, \infty) \to \mathbb{R}_+$ be the invariant density associated with Λ_k, also assumed symmetric for negative $x's$. Then f_k converges uniformly to f in $[0, X]$ as $k \to \infty$.

[12] In Appendix C we consider an alternative normalization, suitable for the case where $X < \infty$.

4 From Price Changes to Price Gaps and Hazards

In this section we show how to recover the invariant density of price gaps f and the adjustment hazard Λ from the observable distribution of price changes. These two objects then allow us to recover the underlying distribution G of adjustment costs ψ in a random menu cost model of Section 2.1. We apply the algorithm to data taken from Cavallo (2015), fitting the distribution of price changes Q and recovering f, Λ, and G. For future reference, we pay particular attention to estimate the kurtosis of the distribution of price changes.

To do this, we first characterize the restrictions that an increasing hazard function Λ imposes on Q and establish a mapping from the observables (price changes) to the distribution of price gaps f and adjustment hazard Λ. We then propose a non-parametric identification strategy to identify the distribution of price changes which takes into account unobserved heterogeneity across different products consistent with the theory as described in Proposition 2 . We illustrate these results fitting a flexible functional form to the distributions Q for several product categories in the dataset from Cavallo (2015). Interestingly, using this data set which have no time aggregation, arguably minimum measurement error, and accounting for unobserved heterogeneity, we find distributions with much smaller Kurtosis than in the literature. From this estimated distribution, we then recover f and Λ, and from them obtain the distribution of random menu cost G using the characterization in Theorem 1.

Identification of f and Λ. We start with a lemma that describes the properties of the distribution of price changes generated by a generalized hazard function. It only requires Assumption 1:

PROPOSITION 4. Let Q be the CDF of price changes corresponding to a generalized hazard function Λ satisfying Assumption 1. Then, Q is absolutely continuous on $(-X, X)$, so that $Q(x) = Q(-X) + \int_{-X}^{x} q(s)ds$ for $x < X$. The density $q(\cdot) : (-X, X) \mapsto \mathbb{R}_+$ is symmetric around zero, $q(x) = q(-x)$, and continuous at $x \notin \mathbb{J}$. Q has mass points if and only if $X < \infty$, in which case they are at $-X$ and X, and is fully identified by the collection of all its moments.

The next proposition, which is one of the main results of the paper, obtains the density f of price gaps from the distribution of price changes. The idea is simple: we integrate the Kolmogorov forward equation twice and replace σ^2 as in equation (19). Once we have f, it is straightforward to get Λ using $f(x)\Lambda(x) = q(x)N_a$.

Theorem 2. Let Δp be price changes, and let Q and q be the CDF and corresponding density of price changes corresponding to a generalized hazard function Λ satisfying Assumption 1. Let N_a be the frequency of price changes. The density for the invariant distribution $f(x)$ is given by

$$f(x) = \frac{2}{Var(\Delta p)} \left[\int_x^X (1 - Q(z))\, dz \right] \text{ for all } x \in (0, X) \tag{24}$$

and $f(-x) = f(x)$, where $Var(\Delta p)$ is the variance of the price changes computed using Q. The generalized adjustment hazard $\Lambda(x)$ is given by

$$\Lambda(x) = \frac{N_a Var(\Delta p)}{2} \frac{q(x)}{\int_x^X (1 - Q(z))\, dz} \text{ for all } x \in [0, X) \tag{25}$$

and $\Lambda(-x) = \Lambda(x)$.

Recall that the function Λ implied by the models of Section 2.1 and Section 2.2 is increasing in $x \in (0, X)$. If Λ is increasing in $(0, X)$, the right hand side of equation (25) must be increasing. At any x where Λ is differentiable,

$$\frac{\Lambda'(x)}{\Lambda(x)} = \frac{q'(x)}{q(x)} + \frac{1 - Q(x)}{\int_x^X (1 - Q(z))\, dz} \geq 0 \text{ for all } x \in (0, X), x \notin \mathbb{J} \tag{26}$$

The model of Section 2.1 also implies that $\Lambda(\cdot)$ is bounded above on $(0, X)$. If this is the case, the right hand side of equation (25) must be bounded. If Λ is increasing, this is equivalent to

$$\lim_{x \to X} \frac{q(x)}{\int_x^X (1 - Q(z))\, dz} = \lim_{x \to X} \frac{q'(x)}{-(1 - Q(x))} \leq C \tag{27}$$

for some constant C. Moreover, if $X < \infty$, then $\lim_{x \to X}(1 - Q(x)) > 0$ and hence $\lim_{x \to X} q'(x)$ must be finite. If $X = \infty$, then $\lim_{x \to X} \frac{q''(x)}{q(x)} \leq C$ by L'Hopital rule. Note that if q has exponential tails, equation (27) is satisfied even if $X = \infty$. Moreover, since the model of Section 2.2 does *not* imply a bounded Λ it does not require equation (27).

A simple measure of state dependence. The expression for Λ in Theorem 2 evaluated at $x = 0$ can be used to measure a simple index of the lack of state dependence in pricing. We label it as $\mathcal{C}$, for "Calvo-ness":

$$\mathcal{C} \equiv \frac{\Lambda(0)}{N_a} \tag{28}$$

The index $\mathcal{C}$ measures the fraction of price changes that happen independently of the price gap x. In terms of the random menu cost model, it measures the fraction of price changes with no adjustment cost paid. Alvarez, Le Bihan, and Lippi (2016) use the same statistic to index multi-product version of the Calvo$^+$ model. In that special case, the function Λ is constant, equal to $\Lambda(0)$ for all $|x| < X$, and then it jumps to infinity, i.e. there is a barrier in this case $\mathcal{X}$ is simply equal to the fraction of price changes that do not occur at the barriers $\pm X$. Clearly, the setup here is much more general, and the definition captures all the price changes that are unrelated to the value of the price gap. Hence, $\mathcal{C}$ is a broad measure of lack of state dependence. The next corollary of Theorem 2 shows that $\mathcal{C}$ can be measured using data on the distribution of price changes.

Corollary 1. The fraction of price changes independent of the price gap $\mathcal{C}$ defined in equation (28), is given by

$$\mathcal{C} = \frac{Var(\Delta p)}{2\,\mathbb{E}[|\Delta p|]}\, q(0)\,. \tag{29}$$

Moreover, using equation (18) for N_a, $\mathcal{C} \le 1$ if Λ is increasing.

The expression in Corollary 1 is intuitive: the fraction of price changes independent of the state is proportional to the density of price changes at zero, a magnitude that can be estimated. The constant of proportionality is a ratio of two easily measurable statistics. The importance of Corollary 1 is that the right hand side of equation (29) involves three observable quantities which depend exclusively on the distribution of price changes: the density at zero, $q(0)$, and two of its moments: the variance $Var(\Delta p)$ and the expected absolute value $\mathbb{E}[|\Delta p|]$.

Unobserved Heterogeneity. Armed with Theorem 2 we can recover f, Λ, and the model primitives, like the distribution of menu cost, using Theorem 1. As an intermediate step, we discuss how to account for a simple, yet pervasive, form of unobserved heterogeneity in the estimation of q. We assume that products in a narrowly defined category have the same distribution of price changes up to an (unobserved) shift in the size, i.e. the distributions have the same shape but different scale. Proposition 2 discusses exactly this type of transformation that changes the scale without affecting shape. The reason we want to account for this form of unobserved heterogeneity is that, as is well known, a mixture of distributions with identical kurtosis but different variances has itself a higher kurtosis.[13] The setup is similar to a random effect model, yet without assuming any functional form for the distributions. In particular we use a variation of Kotlarski (1967)'s lemma. The products (within a category) are indexed by i, and t is the chronological number of adjustment. Let I be the set of all products and $T(i)$ be the set of adjustment instances for a product $i \in I$. We use the following specification:

$$\Delta p_{it} = b_i \Delta \tilde{p}_t \text{ for } i \in I \text{ and } t \in T(i) \tag{30}$$

Here b_i corresponds to the scaling factor b in Proposition 2. The six identification assumtpions are

1. $\#T(i) > 1$, so there are at least two price changes for each i
2. $\Delta \tilde{p}_t$ are drawn from a distribution Q, described by Proposition 4, for all $t \in \bigcup_{i \in I}$
3. $\Delta \tilde{p}_t$ and $\Delta \tilde{p}_s$ are statistically independent for all $t, s \in \bigcup_{i \in I} T(i)$
4. $b_i \geq 0$ are drawn from a distribution H for all $i \in I$
5. $\Delta \tilde{p}_t$ and b_i are statistically independent for all $i \in I$ and $t \in \bigcup_{i \in I} T(i)$
6. $\mathbb{E}[(\Delta \tilde{p}_t)^2] = 1$ for all $t \in \bigcup_{i \in I} T(i)$

That the distribution Q is described by Proposition 4 means, in particular, that it is symmetric around zero. The last assumption is a normalization, since the variances of H and Q are not identified together. We can show the following result:

Proposition 5. Consider two pairs of integer numbers (j, k) and (j', k') such that $j + k = j' + k'$.

[13] See Appendix C for the formal treatment of this result.

Under the assumptions stated above we have:

$$\frac{\mathbb{E}[(\Delta\tilde{p}_t)^j]\mathbb{E}[(\Delta\tilde{p}_t)^k]}{\mathbb{E}[(\Delta\tilde{p}_t)^{j'}]\mathbb{E}[(\Delta\tilde{p}_t)^{k'}]} = \frac{\mathbb{E}[(\Delta p_{it})^j(\Delta p_{is})^k]}{\mathbb{E}[(\Delta p_{it})^{j'}(\Delta p_{is})^{k'}]} \tag{31}$$

for any (t, s) with $t \neq s$.

This proposition has two important implications. First, we can establish a recursive expression for the even moments of the distribution of $\Delta\tilde{p}_t$:

$$\mathbb{E}[(\Delta\tilde{p}_t)^{2k+2}] = \mathbb{E}[(\Delta\tilde{p}_t)^{2k}] \cdot \frac{\mathbb{E}[(\Delta p_{it})^{2k+2}]}{\mathbb{E}[(\Delta p_{it})^{2k}(\Delta p_{is})^2]} \text{ for all } k \geq 0 \tag{32}$$

which only uses equation (31) and the normalization assumed above that $\mathbb{E}[(\Delta\tilde{p}_t)^2] = 1$. Starting from the normalized second moment, we can construct all even moments recursively using equation (32), thus obtaining a *non-parametric identification* of the density q.[14] Second, for future reference we display an expression for the Kurtosis of $\Delta\tilde{p}$:

$$Kurt(\Delta\tilde{p}_t) = \frac{\mathbb{E}[(\Delta p_{it})^4]}{\mathbb{E}[(\Delta p_{it})^2(\Delta p_{is})^2]} = \frac{Kurt(\Delta p_{it})}{1 + \text{corr}(\Delta p_{it}^2, \Delta p_{is}^2)CV(\Delta p_{it}^2)CV(\Delta p_{is}^2)} \text{ for } t \neq s \tag{33}$$

The first equality is how we estimate kurtosis, correcting for this unobserved heterogeneity. The second equality shows how our method to measure kurtosis amounts to a correction of the kurtosis computed by pooling different goods without accounting for heterogeneity. Whenever the squares of price changes of individual products are positively correlated, as we have systematically found in the data, the correction leads to a substantial downward adjustment of the estimated kurtosis.

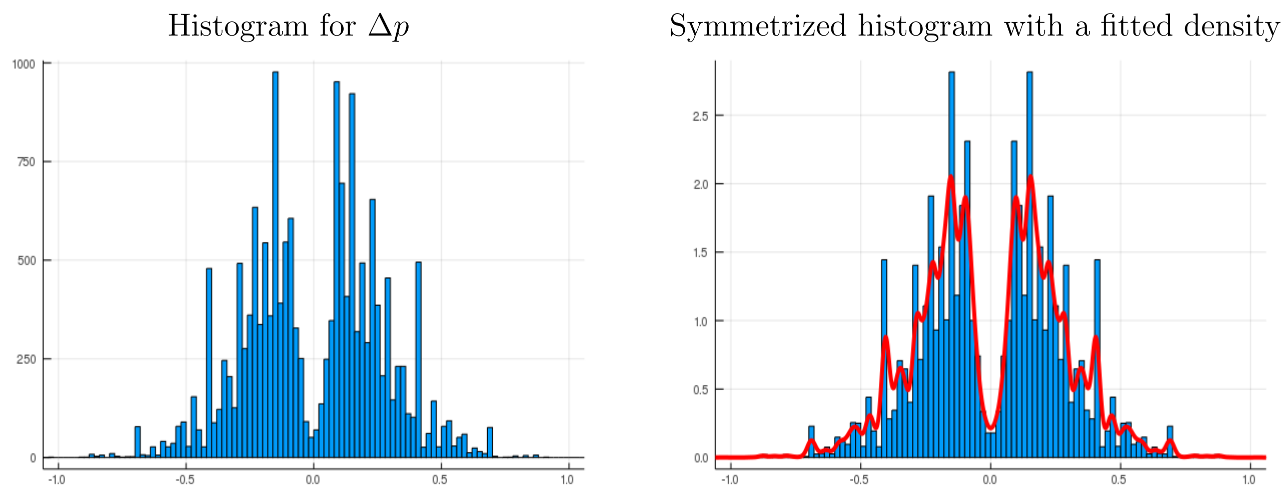
Data and estimation. We use the open access data from Billion Prices Project presented by Cavallo (2015).[15] We have chosen scraped price data to reduce the measurement error present in other data sets for example due to time aggregation using average revenue. It is important to avoid this form of measurement error to accurately estimate the kurtosis of the distribution of price changes, one of the goals of this section. The time span of our sample is between May 2008 and June 2010. From daily data on prices we construct the series of spells together with the size

[14] This fully characterizes the distribution of $\Delta\tilde{p}_t$, since its odd moments are equal to zero due to symmetry.

[15] Link: `http://www.thebillionpricesproject.com/datasets/`. We use the US store number 1.

of the price change at the end of each spell. We trim the sample at price changes larger than 150 log points size in absolute value.[16] To fit a symmetric density, for each value Δp in the sample we use points in the band around it and around $-\Delta p$ as well. The left panel of Figure 7 plots the histogram of price changes for a narrowly defined product category. The right panel presents the symmetrized histogram with a fitted density. This fitted density is *not* the underlying density q, since it is confounded by the unobserved values of b_i.

Figure 3: Distribution of price changes in a narrow category



Pooling all products for category 561, COICOP label "Non-durable household goods"

Table 1 presents summary statistics on the seven categories we use, as well as the estimated kurtosis. The kurtosis is estimated in two ways: first by (incorrectly, according to our assumptions) pooling different products in the same category (p), and second by accounting for product heterogeneity (u). To implement the latter procedure, we use equation (33). This equation is a particular case of equation (31) with $j = 4$, $k = 0$, and $j' = k' = 2$. Importantly, the expectation is taken over $t \neq s$, and any such pair (t, s) can be taken to estimate it. Our estimator is constructed as follows: for any pair (j, k), we estimate $\mathbb{E}[|\Delta p_{it}|^j |\Delta p_{is}|^k]$ by

$$\frac{1}{\#I} \sum_{i \in I} \frac{1}{\#T(i)(\#T(i)-1)} \sum_{t,s \in T(i), t \neq s} |\Delta p_{it}|^j |\Delta p_{is}|^k \tag{34}$$

[16] We remove 87 (larger than 150 log points) out of 326,570 price changes for products with at least three spells.

Table 1: Summary statistics and kurtosis estimates

Category	Number Products	Number P. changes	$\hat{\mathbb{E}}(\Delta p_{it})$	$\hat{\sigma}(\Delta p_{it})$	Kurtosis Pooled	Kurtosis w/Unobs. Heterog.	$\mathcal{C}_{pooled}$	$\mathcal{C}$ w/Unobs. Heterog.
111	3437	74464	0.002	0.341	3.418 (0.162)	1.656 (0.071)	0.077	0.071
119	3225	56527	0.002	0.328	3.831 (0.092)	1.955 (0.050)	0.085	0.064
1212	2551	30343	-0.001	0.245	3.524 (0.272)	2.052 (0.162)	0.040	0.039
122	1401	27321	0.002	0.342	2.956 (0.089)	1.677 (0.051)	0.118	0.091
118	1388	30111	0.003	0.308	3.624 (0.240)	2.044 (0.118)	0.080	0.078
117	1154	20995	0.007	0.309	3.487 (0.135)	1.989 (0.047)	0.071	0.058
561	1032	17724	0.002	0.260	3.324 (0.221)	1.778 (0.133)	0.034	0.030

Categories legend: 111 "bread and cereals", 119 "other food products", 1212 "electric appliances for personal care", 122 "soft drinks", 118 "sugar, honey, and confectionary", 117 "vegetables", 561 "non-durable household goods"

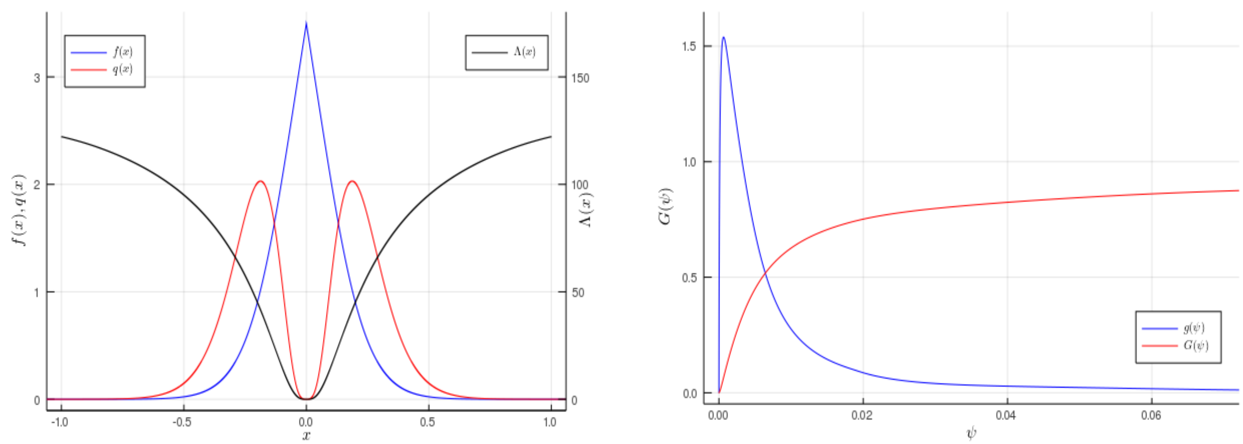
where # denotes the number of elements in the set. Note that this estimator includes all available price changes $t, s \in T(i)$ for every product, maximizing the use of the data. Individual products in the sample have around 20 price changes each. Distributions are centered around zero, with the mean being around one hundredth of the standard deviation. It is evident from the table that properly accounting for heterogeneity reduces the estimated kurtosis to about half. This points to a substantial correlation in absolute values of the consecutive squared price changes. In Appendix D we tabulate implied correlations recovered from equation (33). These turn out to be in the range between 0.29 and 0.45.

To estimate Q we use a Gamma distribution. In principle, the distribution could be estimated non-parametrically, since every moment is identified. In practice, this would require estimating a large number of moments, substantially decreasing precision. Instead, we estimate the Kurtosis and use the unit variance restriction to fit the scale and size. The fitted density q is presented on the left panel of Figure 4 together with the underlying generalized hazard function Λ and the density of price gaps f. In the Appendix D, we detail our algorithm and show extended results from fitting a mixture of two Gamma distributions, for which we estimate moments using Proposition 5 to fit five parameters: scale and size of the distributions and the weight.

Finally, the right panel contains the distribution of menu cost recovered from the resulting hazard Λ. The units on the horizontal axis correspond to the annual profit of the firm. There is no mass point at zero, since the recovered generalized hazard function has $\Lambda(0) = 0$. Note that the model with random menu cost can only rationalize a bounded generalized hazard function. Gamma distribution is convenient, since by Theorem 2 it implies a bounded Λ. In the Appendix D, we show the procedure to recover the cost function c corresponding to the model in Section 2.2. This model allows for an unbounded Λ, so we use a power specification $\Lambda(x) = \kappa x^{\nu}$, deriving the moments of Q and using the analytical expressions to fit the parameters.

Figure 4: Estimated distribution of price changes and implied cost functions

Estimated $q(\cdot)$, recovered $f(\cdot)$ and $\Lambda(\cdot)$ | Recovered CDF and density of menu costs



Estimating the degree of state dependence. We now turn to measuring $\mathcal{C}$. We do this in two ways. First, we ignore the unobserved heterogeneity and assume that the price data for the narrowest category of goods all come from the same primitives of the model. These primitives are the generalized hazard function Λ (including the value of X, the barrier) and σ^2. Recall from Theorem 2 that these objects fully describe the data-generating process, and this mapping is injective. In this exercise we just estimate all the objects on the right hand side of equation (29).

The results are shown in the column labelled "$\mathcal{C}_{pooled}$" in Table 1. Their average across categories is 0.072, i.e. just above 7% of price changes are independent of the state. This small number is due to the small value of the density q at $\Delta p = 0$, which is apparent from the right hand side panel of Figure 7.

Second, we account for unobserved heterogeneity of the type described above. If there is heterogeneity of this type across products in the narrowest category, using the simple expression for $\mathcal{C}$ from equation (29) produces an upward bias in the estimate. We derive an unbiased estimator, the result being analogous to the one in Proposition 5. We express this estimator as a function of the pooled estimator $\mathcal{C}_{pooled}$ and a correction due to the unobserved heterogeneity:

Proposition 6. Under the assumptions 1-5 stated above,

$$\mathcal{C} = \mathcal{C}_{pooled} \left(1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} \right) < \mathcal{C}_{pooled} \tag{35}$$

where the two components are given by

$$\mathcal{C}_{pooled} = \frac{q(0)Var(\Delta p_{it})}{2\,\mathbb{E}[|\Delta p_{it}|]} \text{ and } 1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \frac{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|^2]} \text{ for } t \neq s$$

The estimate for $\mathcal{C}_{pooled}$ is obtained from the pooled data, and the correction for unobserved heterogeneity is measured using the (short) time dimension of the panel. The last column, labelled "$\mathcal{C}$ w/Unobs. Heterogeneity", in Table 1 contains the estimation results. Averaging across categories, the fraction of price changes independent of the state is 0.062 or just above 6%.

The correction multiplier is smaller than one because $1/b_i$ and b_i^2 are negatively correlated. Of course, if data for all products i in the narrowest category come from the same model primitives (Λ, X, σ^2), then there is no variation in b_i, and $\mathcal{C} = \mathcal{C}_{pooled}$.

5 A Sufficient Statistic for Monetary Shocks

This section characterizes the real output effect of monetary shocks using a simple summary statistic, the cumulative output generated by a once and for all monetary shock. This is the area under the output's impulse response function. It combines in a single value the persistence and the size of the output response. The key result we present is that for small monetary shocks, like the ones typically considered in the literature, the area is completely encoded by the kurtosis and the frequency of price changes. These two moments are thus sufficient to compare different models.

We also find that, among the models with non-decreasing adjustment hazards, the kurtosis of price changes is maximized in the Calvo model. As was established above, only a non-decreasing generalized hazard function can be rationalized by random menu costs. Calvo model is the limiting case with no randomness and no option to adjust, so it minimizes the amount of selection and hence maximizes the output response. To establish this, we develop a general result that compares kurtoses generated by two different hazard functions.

The contribution to the cumulative impulse response of a firm with price gap x is

$$m(x) = -\mathbb{E}\left[\int_0^\tau x(t)dt \,|\, x(0) = x\right] \tag{36}$$

where τ is the stopping time defined as the first time when $x(t)$ hits $\pm X$ or a reduction in adjustment costs causes the firm to change price. This stopping time is stochastic, so the expectation accounts for both the diffusion of the firm's price gap and the possible event of adjustment that happens with a Poisson intensity $\Lambda(x(t))$. In words, $m(x)$ is the expected (cumulative) price gap of a firm that starts with a gap x. Notice that in the Calvo case, where $\Lambda(x) = \lambda$ is independent of x, we immediately obtain $m(x) = -x/\lambda$, where $1/\lambda$ is the expected duration of a price spell.

The definition above uses the steady state decision rule $\Lambda(x)$, thus ignoring the general equilibrium feedback effect of the shock on the firm's decision. In Proposition 7 of Alvarez and Lippi (2014) it is shown that, given a combination of the general equilibrium setup in Golosov and Lucas (2007) and the lack of the strategic complementarities, these general equilibrium effects are of second order. In addition, we use the fact that after the first price change the expected contribution to output of each firm is zero, since positive and negative output contributions are equally likely, so $m(0) = 0$. This allows us to characterize the propagation of the monetary shocks without tracking the time evolution of the whole price gap distribution.

The expectation in the right hand side of equation (36) is with respect to the process for x, a jump-diffusion with jump intensity $\Lambda(x)$, diffusion variance σ^2, and zero drift. The function $m : [-X, X] \to \mathbb{R}$ is once continuously differentiable, antisymmetric around $x = 0$, and satisfies:

$$m(x)\Lambda(x) = -x + \frac{\sigma^2}{2} m''(x) \text{ for all } x \text{ at which } \Lambda \text{ is continuous} \tag{37}$$

$$0 = m(X) \text{ if } X < \infty \text{ and } \lim_{x\to\infty} \frac{|m(x)|}{x} \leq \frac{1}{\inf_y \Lambda(y)} \text{ if } X = \infty \quad . \tag{38}$$

Now we can define the cumulative impulse response to a monetary shock of size δ as

$$\mathcal{M}(\delta) = \int_{-X}^{X} m(x - \delta) f(x) dx \quad . \tag{39}$$

This is simply the aggregate contribution of the firms to the cumulative impulse response. The response of a firm with the price gap x before the shock is $m(x - \delta)$.

Let $\{X, \Lambda, \sigma^2\}$ characterize an economy, with its corresponding invariant density f and firm's contribution to CIR, m. Let $\{\tilde{X}, \tilde{\Lambda}, \tilde{\sigma}^2\}$ be the standardized economy, defined as in Proposition 2, that has its associated $\{\tilde{f}, \tilde{m}\}$ with $\tilde{m}$ defined as $\tilde{m}(z) = m(z/b)/b$ for $b^2 = 1/Var(\Delta p)$ and satisfying the corresponding ODE with the boundary conditions for $\tilde{\sigma}^2$ and $\tilde{X}$. Define $\tilde{\mathcal{M}}(\delta)$, the cumulative impulse response of output to a monetary shock for the standardized economy, as

$$\tilde{\mathcal{M}}(\delta) = \int_{-\tilde{X}}^{\tilde{X}-\delta} \tilde{m}(x)\, \tilde{f}(x + \delta)\, dx \tag{40}$$

The next proposition relates the CIR of output in an economy to the one of its standardized version by scaling the monetary shock with the steady-state standard deviation of price changes. In words, for small monetary shocks the dispersion of price changes is immaterial, although in general the size of monetary shocks should be measured relative to the steady-state dispersion of price changes.

Proposition 7. Let $\mathcal{M}$ and $\tilde{\mathcal{M}}$ be the cumulative impulse responses of an economy $\{X, \Lambda, \sigma^2\}$ with $Std(\Delta p) = Var(\Delta p)^{1/2}$ and the corresponding standardized economy $\{\tilde{X}, \tilde{\Lambda}, \tilde{\sigma}^2\}$. Then

$$\mathcal{M}(\delta) = \tilde{\mathcal{M}}\left(\frac{\delta}{Std(\Delta p)}\right) Std(\Delta p) \tag{41}$$

and thus $\mathcal{M}'(0) = \tilde{\mathcal{M}}'(0)$.

The proof is immediate, using the properties of $\tilde{m}$ and $\tilde{f}$ established above, differentiating equation (40), and evaluating at $\delta = 0$. Summarizing, Proposition 7 says that for small monetary shocks, the steady state standard deviation of the price changes is not important. For large shocks it clearly is. For example, take the case $X < \infty$. For $\delta \geq 2\bar{X}$, we have $\mathcal{M}(\delta) = 0$, because the shock displaces all the firms far enough, and they adjust immediately. Since the standardized version has $\tilde{X} = X/Std(\Delta p)$, this shows the importance of the size of the shock for large values.[17] The marginal version of this cumulative impulse response is

$$\mathcal{M}'(0) = -\int_{-X}^{X} m'(x) f(x) dx \tag{42}$$

This term can be used for a linear approximation of $\mathcal{M}$ around zero. Our main result is that it can be expressed as a function of two sufficient statistics: $Kurt(\Delta p)$, the kurtosis of the steady state distribution of price changes, and N_a, frequency of price changes.

Theorem 3. Let $\Lambda(x)$ be any function satisfying Assumption 1. Then the cumulative impulse response for a small monetary shock is given by the ratio of two steady state statistics:

$$\mathcal{M}(\delta) = \frac{Kurt(\Delta p)}{6N_a}\delta + o(\delta^2) \tag{43}$$

The approximation is accurate up to second order terms, so the remainder is of order δ^3. This happens since $\mathcal{M}''(0)$ is zero, which follows from $\mathcal{M}$ being an antisymmetric function, because m

[17] A similar result was shown in Alvarez and Lippi (2014) for the case of multiproduct firms, which only overlap with the current set up for the Golosov and Lucas case — with one product per firm.

is antisymmetric and f is symmetric.

Our results from Theorem 1 and Proposition 15 show that only weakly increasing Λ can be rationalized by the solution of a firm problem subject to random menu costs. But Assumption 1 allows for a very large class of generalized hazard functions, including decreasing and non-monotone ones. Theorem 3 holds for such functions too. It makes no reference to the micro-foundations behind Λ and hence also applies to setups where firms's behaviour is not described by an increasing hazard. An example is the model in Woodford (2009), where firms conduct costly reviews and have imperfect recall and access to their state. Also Costain and Nakov (2011b) use generalized hazard functions, without linking them to random menu costs.

Aggregation across heterogenous firms. We briefly discuss how the above results can be applied to economies composed of heterogenous firms. Assume that there are S groups of firms with different parameters, each with an expenditure weight $e(s) > 0$, $N(s)$ price changes per unit of time, and a distribution of price changes with kurtosis $Kurt(s)$. In this case, after repeating the arguments above for each group and aggregating, we obtain that the area under the IRF of *aggregate* output for a small monetary shock δ is

$$\mathcal{M}(\delta) = \delta \mathcal{M}'(0) + o(\delta^2) = \frac{\delta}{6} \sum_{s \in S} \frac{e(s)}{N_a(s)} Kurt(s) + o(\delta^2) = \frac{\delta}{6} D \sum_{s \in S} d(s)\, Kurt(s) + o(\delta^2) \tag{44}$$

where D is the expenditure-weighted average duration of prices $D \equiv \sum_{s \in S} \frac{e(s)}{N_a(s)}$, and $d(s) \equiv \frac{e(s)}{N_a(s) D}$ are weights that take into account both relative expenditures and durations. When all groups have the same durations, then $d(s) = e(s)$ and $\mathcal{M}$ is proportional to the average of the kurtosis of the sectors. As explained in Section 4, and shown in Proposition 10 in Appendix C, this average is also different from the kurtosis of the pooled data. This applies even if all the groups have the same kurtosis.[18] However, if groups are heterogenous in duration (or expenditures), then the kurtoses of the groups with longer duration (or higher expenditures) receive a higher weight in the computation of $\mathcal{M}$. Suppose for instance that a fraction of firms have flexible prices (zero duration in our model, or infinitely many price changes per unit of time), as in Dotsey and Wolman (2020).

[18] The effect of heterogeneity in $N_a(\Delta p_i)$ on aggregation is well known for the Calvo model: D is different from the average of $N_a(\Delta p_i)$'s, see for example Carvalho (2006) and Nakamura and Steinsson (2010).

The above formula implies that the group of the flexible price firms are excluded (zero duration yields a zero weight), and that the cumulative impulse response (CIR) is computed on the mass of firms with sticky prices. Notice that this is different from computing the CIR as the ratio of the cross-sectional average kurtosis and the average frequency. Since the latter is diverging because of the firms with flexible prices, the CIR computed this way would be zero, while obviously it is not.

Kurtosis. The next proposition shows the properties of generalized hazard functions that determine the Kurtosis of price changes. We will concentrate on the case where we will hold the adjustment frequency constant. Recall that that fixing the frequency of price changes can be accomplished as fixing the units of time. This procedure allows us to isolate the effect of a change in Λ on selection from its effect on the frequency. Moreover, with the frequency fixed, the kurtosis of price changes directly maps into the approximate cumulative impulse response.

Proposition 8. Fix N_a and consider two hazard functions $\Lambda_1(x)$ and $\Lambda_2(x)$ with the corresponding boundaries X_1 and X_2, where $0 < X_2 \leq X_1 \leq \infty$. Let $\Lambda_1(0) > \Lambda_2(0)$ and let the function $\Lambda_1(x) - \Lambda_2(x)$ change sign at most once. Then, $\Lambda_1(x)$ generates a higher kurtosis of price changes.

The condition that $\Lambda_1 - \Lambda_2$ changes sign only once means that it is positive at first and maybe negative for x far from zero. This is to say that Λ_1 generates more adjustment for smaller x, and Λ_2 generates more for larger ones. Selection is therefore more pronounced with Λ_2, and the kurtosis of price changes is lower. There are two interesting corollaries of this result. The first is that for a fixed X the highest kurtosis is attained by the constant generalized hazard function. This corresponds to the Calvo$^+$ case:

Corollary 2. Fix N_a, the number of adjustments per unit of time, and $X < \infty$. The function $\Lambda(x)$ that is constant on $(-X, X)$ maximizes the kurtosis of the price changes over all functions $\Lambda(x)$ that are weakly increasing on $(0, X)$ and satisfy Assumption 1.

Second, a constant hazard function in combination with the infinite boundary X maximizes the kurtosis of price changes over *all* weakly increasing hazards. This is the pure Calvo case:

Corollary 3. Fix N_a. The constant function $\Lambda(x) \equiv \lambda$ maximizes the kurtosis of the price changes over all weakly increasing functions $\Lambda(x)$ satisfying Assumption 1.

By Theorem 3, it also means a constant Λ maximizes $\mathcal{M}'(0)$ for a fixed N_a. This highlights the role of selection. A strictly increasing rate of adjustment Λ implies positive selection, so the firms with larger deviations are more likely to adjust. When Λ is flat, there is no selection, so the price changers are drawn randomly from the population. Shocks are accommodated more slowly in this case, because the adjustment frequency does not depend on how much a firm needs to adjust, so the response of price takes longer, and hence the response of output is larger.

Finally, Proposition 8 sheds some light on the relationship between the strength of state dependence and the magnitude of output response. As we noted before, one measure of the strength of state dependence is the index $\mathcal{C} \equiv \Lambda(0)/N_a$, the share of adjustment happening independently of the price gap. We can show that, holding constant the shape of Λ (captured by its curvature) and adjustment frequency, this index co-moves with the Kurtosis. Hence, a higher share of adjustment independent of x means a stronger output response for the same shape of the hazard.

Define the curvature of the function Λ as

$$k(x) = \frac{\Lambda''(x)x}{\Lambda'(x)} \tag{45}$$

To understand what it means for two functions to have the same curvature, take some arbitrary Λ and decompose it into two parts, the intercept and the rest: $\Lambda(x) = \Lambda(0) + (\Lambda(x) - \Lambda(0))$. Now consider two simple linear transformations of the two parts of the hazard:

$$\Lambda_1(x) = a_1\Lambda(0) + b_1(\Lambda(x) - \Lambda(0)), \quad \Lambda_2(x) = a_2\Lambda(0) + b_2(\Lambda(x) - \Lambda(0)) \tag{46}$$

The transformation scales the intercept and the rest with different numbers, changing the strength of state dependence but broadly preserving the shape (it is easy to see that both Λ_1 and Λ_2 have the same curvature as Λ). When Λ_1 and Λ_2 generate the same adjustment frequency, the one with a weaker state dependence (higher $\mathcal{C}$) corresponds to a higher Kurtosis.

Corollary 4. Consider two generalized hazard functions $\Lambda_1(x)$ and $\Lambda_2(x)$ with the same boundary $X \leq \infty$. Furthermore, assume that they have the same curvature k everywhere and the frequency of adjustment N_a. Then $Kurt_1(\Delta p) > Kurt_2(\Delta p)$ if and only if $\mathcal{C}_1 > \mathcal{C}_2$.

An immediate implication of the Corollary 4 is that for two economies with the same frequency of price changes and the same curvature of the generalized hazard function, the one with higher value of $\mathcal{C}$ has a higher cumulative impulse response after a monetary shock.

5.1 Illustration with a power hazard function

In this section we describe the case where the generalized hazard function is a power function with the power parameter ν. In particular, we let $\Lambda(x) = \kappa |x/X|^{\nu}$ on $(-X, X)$ for some $\nu \geq 0$. This functional form nests Calvo-plus models with $\nu = 0$ and quadratic generalized hazard functions with $\nu = 2$.

We use this example to illustrate how the parameters affecting the shape of Λ determine the Kurtosis of price changes. To do this, we first show that, with ν fixed, the Kurtosis of price changes varies one-to-one with the share of adjustments from strictly between the barriers, s. This highlights the role of selection: the output response is weaker when fewer firms reach the boundaries, because fewer firms are close to adjustment right before a monetary shock happens. Second, we show that for any s the Kurtosis of price changes decreases monotonically with the power ν, which governs the shape of Λ.

We now describe the invariant density f. Upon a renormalization, we can solve for a symmetric density $\hat{f}(z)$ defined by $\hat{f}(z) = X f(zX)$. The function $\hat{f}$ satisfies

$$\rho z^{\nu} \hat{f}(z) = \hat{f}''(z) \text{ with } \hat{f}(1) = 0 \text{ and } \int_0^1 \hat{f}(z) dz = \frac{1}{2} \tag{47}$$

where $\rho \equiv 2\kappa X^2/\sigma^2$. The solution to equation (47) is given by

$$\hat{f}(z) = c_1 \sqrt{z}\, \mathcal{I}_{\frac{1}{\nu+2}} \left(\frac{2\sqrt{\rho}}{\nu+2} z^{\frac{\nu+2}{2}} \right) + c_2 \sqrt{z}\, \mathcal{K}_{\frac{1}{\nu+2}} \left(\frac{2\sqrt{\rho}}{\nu+2} z^{\frac{\nu+2}{2}} \right) \text{ for all } z \in [-1, 1] \tag{48}$$

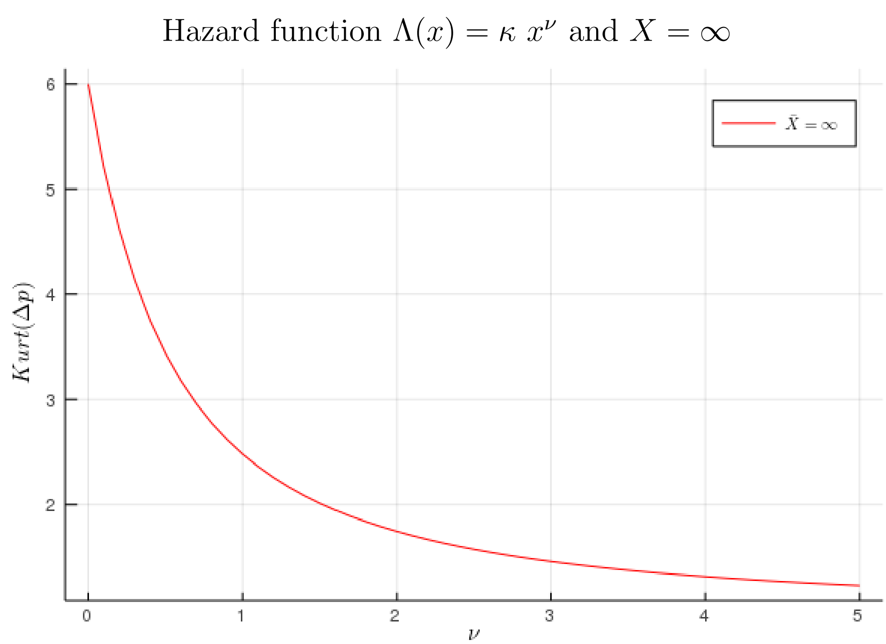
This is a combination of modified Bessel functions of the first and second kind $\mathcal{I}_{\frac{1}{\nu+2}}$ and $\mathcal{K}_{\frac{1}{\nu+2}}$ of order $\frac{1}{\nu+2}$, where the constants c_1, c_2 are chosen to satisfy the two boundary conditions described in equation (47). The form $\hat{f}$ depends on the parameters (ρ, ν) that capture adjustment coming from random menu costs. Note that if $X = \infty$ then $c_1 = 0$.

From the previous result, we can see that if two models have the same (ν, ρ), then the distribution of price changes in one is a rescaling of that in the other. The dimensionless statistics such as the kurtosis, the fraction of adjustment strictly within the boundaries s, or equivalent the mass of Q on $\pm X$, are the same. We can summarize this result as follows:

Proposition 9. Let $\Lambda(x) = \kappa |x/X|^{\nu}$. The Kurtosis of price changes, the share of adjustments strictly between the boundaries, and the frequency of price changes satisfy: $Kurt(\Delta p) = \hat{K}(\rho, \nu)$, $s = \hat{S}(\rho, \nu)$, and $N_a = \frac{\sigma^2}{X^2}\hat{N}(\rho, \nu)$ respectively, where these functions have no other parameters. For fixed ν, the function $\hat{S}(\cdot, \nu)$ is increasing in ρ, and $\hat{K}(\cdot, \nu)$ is decreasing in ρ.

Using this proposition we can fix s, say to $s = 1$, or $X = \infty$, then $Kurt(\Delta p)$ is just a function of ν only, displayed in Figure 5. Alternatively, fixing ν we have that $Kurt(\Delta p)$ is only a function of the fraction of price changes strictly between barriers, s as displayed in Figure 6 below.

Figure 5: Kurtosis of power hazard function as ν varies



[Figure 5](#) displays the value of $Kurt(\Delta p)$ for the case where $X = \infty$ as a function of ν. Note that Kurtosis goes from 6, corresponding to $\nu = 0$, or pure Calvo, to a value of 1, corresponding to $\nu \to \infty$ which approximates Golosov and Lucas. Increases in ν clearly change the shape of Λ, making it more convex, which is reflected in lower kurtosis of price changes. This illustrates how the shape of Λ determines the selection effect on price changes. Note that for $\nu = 2$, the quadratic case, $Kurt(\Delta p) \approx 1.75$.

Figure 6: Kurtosis behavior with a power hazard function as s varies

Hazard function $\Lambda(x) = \kappa \ \left(\frac{x}{X}\right)^{\nu}$ and $X < \infty$

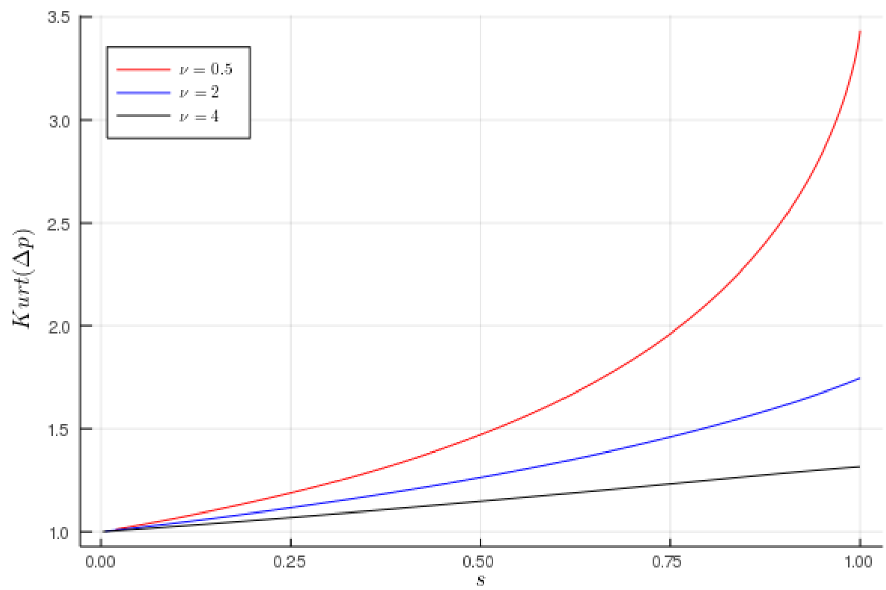


Figure 6 displays the value of $Kurt(\Delta p)$ for the case of $X < \infty$ as a function of s, the fraction of price changes strictly between the boundaries. We display such relationship for three values of ν. Note that fixing ν, as we change ρ, and obtain a larger share of price changes strictly inside the barriers s, which corresponds to a lower $Kurt(\Delta p)$. This illustrate the smaller selection effect of price changes when barriers are hit less often. Recall that $s = 0$ is equivalent to $X = \infty$. For each value of s, the difference lines shows that curvature ν corresponds to lower $Kurt(\Delta p)$.

6 Conclusion

We discuss the economic foundations of the generalized hazard function, a flexible modeling block used in several sticky-price setups, and map it to the primitives underlying the firm's optimization problem. Our main contribution is the identification of the generalized hazard function using observable objects, such as the distribution of price changes (or the distribution of durations), by means of simple closed form equations. In Appendix E we complement these results by establishing that, under regularity conditions, the survival function $S(t)$, measuring the distribution of *durations* of unchanged prices, uniquely identify $\Lambda(x)$ as well.[19]

Our empirical strategy accounts for unobserved heterogeneity across products, which we show is important in estimating the Kurtosis of price changes, and hence in quantifying the real effects of monetary shocks. On the analytical side, our main output is the generalization of the "Kurtosis result" from Alvarez, Le Bihan, and Lippi (2016) to a considerably larger set of models. We prove that the Calvo model yields the maximum amount of monetary non-neutrality within this broad class. In a narrower class where the inaction region is bounded, the upper bound is Calvo$^+$.

Our analytical setup can be extended to study multiproduct firms, as in Midrigan (2011); Alvarez and Lippi (2014), and to study the entire profile of the IRF (as opposed to the cumulative impulse response), using the methods explored in Alvarez and Lippi (2019). We believe that using the tools developed in the present paper can be applied to other setups, such as search models of money with non-degenerate distributions of cash holdings. This should apply for a variety of setups with idiosyncratic states and costly adjustment.

[19] Summarizing, the generalized hazard rate Λ is identified either by the distribution of price changes Q and one temporal statistic (the frequency), or by the distribution of durations S and one statistic on the size of price changes (the variance).

References

Alexandrov, Andrey. 2020. "The Effects of Trend Inflation on Aggregate Dynamics and Monetary Stabilization." Mimeo, University of Mannheim.

Alvarez, Fernando, Martin Beraja, Martin Gonzalez-Rozada, and Pablo Andres Neumeyer. 2019. "From Hyperinflation to Stable Prices: Argentinas Evidence on Menu Cost Modles." *The Quarterly Journal of Economics* 143 (1):451–505.

Alvarez, Fernando and Francesco Lippi. 2020. "Temporary Price Changes, Inflation Regimes, and the Propagation of Monetary Shocks." *American Economic Journal: Macroeconomics* 12 (1):104–52.

Alvarez, Fernando E., Herve Le Bihan, and Francesco Lippi. 2016. "The real effects of monetary shocks in sticky price models: a sufficient statistic approach." *The American Economic Review* 106 (10):2817–2851.

Alvarez, Fernando E. and Francesco Lippi. 2014. "Price setting with menu costs for multi product firms." *Econometrica* 82 (1):89–135.

———. 2019. "The Analytic Theory of a Monetary Shock." Working paper, EIEF.

Alvarez, Fernando E., Francesco Lippi, and Luigi Paciello. 2016. "Monetary Shocks in Models with Inattentive Producers." *Review of Economic Studies* 83:421–459.

Baley, Isaac and Andres Blanco. 2019. "Aggregate Dynamics in Lumpy Economies." Working paper, Universitat Pompeu Fabra.

Barro, Robert J. 1972. "A Theory of Monopolistic Price Adjustment." *Review of Economic Studies* 39 (1):17–26.

Berger, David and Joseph Vavra. 2018. "Dynamics of the U.S. price distribution." *European Economic Review* 103:60 – 82.

Bils, Mark and Peter J. Klenow. 2004. "Some Evidence on the Importance of Sticky Prices." *Journal of Political Economy* 112 (5):947–985.

Caballero, Ricardo J. and Eduardo M. R. A. Engel. 1993a. "Microeconomic Adjustment Hazards and Aggregate Dynamics." *The Quarterly Journal of Economics* 108 (2):359–383.

———. 1999. "Explaining Investment Dynamics in U.S. Manufacturing: A Generalized (S, s) Approach." *Econometrica* 67 (4):783–826.

Caballero, Ricardo J. and Eduardo M.R.A. Engel. 1993b. "Heterogeneity and output fluctuations in a dynamic menu-cost economy." *The Review of Economic Studies* 60 (1):95.

———. 2007. "Price stickiness in Ss models: New interpretations of old results." *Journal of Monetary Economics* 54 (Supplement):100–121.

Calvo, Guillermo A. 1983. "Staggered prices in a utility-maximizing framework." *Journal of Monetary Economics* 12 (3):383–398.

Caplin, Andrew S and Daniel F Spulber. 1987. “Menu Costs and the Neutrality of Money.” *The Quarterly Journal of Economics* 102 (4):703–25.

Carvalho, Carlos. 2006. “Heterogeneity in Price Stickiness and the Real Effects of Monetary Shocks.” *The B.E. Journal of Macroeconomics* 6 (3):1–58.

Carvalho, Carlos and Oleksiy Kryvtsov. 2018. “Price Selection.” Staff Working Papers 18-44, Bank of Canada.

Casella, George and Roger L Berger. 2002. *Statistical inference*, vol. 2. Duxbury Pacific Grove, CA.

Cavallo, Alberto. 2015. “Scraped Data and Sticky Prices.” NBER Working Papers 21490, National Bureau of Economic Research, Inc.

Cavallo, Alberto and Roberto Rigobon. 2016. “The Billion Prices Project: Using online prices for measurement and research .” *Journal of Economic Perspectives* 30 (2):151–178.

Costain, James and Anton Nakov. 2011a. “Distributional dynamics under smoothly state-dependent pricing.” *Journal of Monetary Economics* 58 (6):646 – 665.

———. 2011b. “Price Adjustments in a General Model of State–Dependent Pricing.” *Journal of Money, Credit and Banking* 43:385–406.

Dixit, Avinash. 1991. “Analytical Approximations in Models of Hysteresis.” *Review of Economic Studies* 58 (1):141–51.

Dotsey, Michael, Robert G. King, and Alexander L. Wolman. 1999. “State-Dependent Pricing And The General Equilibrium Dynamics Of Money And Output.” *The Quarterly Journal of Economics* 114 (2):655–690.

Dotsey, Michael and Alexander L. Wolman. 2020. “Investigating nonneutrality in a state-dependent pricing model with firm-level productivity shocks.” *International Economic Review* 61 (1):159–188.

Eichenbaum, Martin, Nir Jaimovich, and Sergio Rebelo. 2011. “Reference Prices, Costs, and Nominal Rigidities.” *American Economic Review* 101 (1):234–62.

Gautier, Erwan and Ronan Le Saout. 2015. “The Dynamics of Gasoline Prices: Evidence from Daily French Micro Data.” *Journal of Money, Credit and Banking* 47 (6):1063–1089.

Golosov, Mikhail and Robert E. Jr. Lucas. 2007. “Menu Costs and Phillips Curves.” *Journal of Political Economy* 115:171–199.

Klenow, J. Peter and Benjamin Malin. 2010. “Microeconomic Evidence on Price-Setting.” *Handbook of monetary economics* 3:231–284.

Kotlarski, Ignacy. 1967. “On characterizing the gamma and the normal distribution.” *Pacific J. Math.* 20 (1):69–76.

Lian, Hairong, Peiguang Wang, and Weigao Ge. 2009. “Unbounded upper and lower solutions method for Sturm?Liouville boundary value problem on infinite intervals.” *Nonlinear Analysis: Theory, Methods & Applications* 70 (7):2627–2633.

Midrigan, Virgiliu. 2011. “Menu Costs, Multi-Product Firms, and Aggregate Fluctuations.” *Econometrica,* 79 (4):1139–1180.

Nakamura, Emi and Jon Steinsson. 2010. “Monetary Non-Neutrality in a Multisector Menu Cost Model.” *The Quarterly Journal of Economics* 125 (3):961–1013.

Petrella, Ivan, Emiliano Santoro, and Lasse de la Porte Simonsen. 2018. “Time-varying Price Flexibility and Inflation Dynamics.” Discussion Paper 13027, CEPR.

Reis, Ricardo. 2006. “Inattentive producers.” *Review of Economic Studies* 73 (3):793–821.

Sheremirov, Viacheslav. 2019. “Price dispersion and inflation: New facts and theoretical implications.” *Journal of Monetary Economics* .

Sheshinski, Eytan and Yoram Weiss. 1977. “Inflation and Costs of Price Adjustment.” *Review of Economic Studies* 44 (2):287–303.

Woodford, Michael. 2009. “Information-Constrained State-Dependent Pricing.” *Journal of Monetary Economics* 56:s100–s124.

Appendix:

The Macroeconomics of Sticky Prices with Generalized Hazard Functions

Fernando Alvarez, Francesco Lippi, and Aleksei Oskolkov

A Proofs

Proof. (of Lemma 1). Define the function $U(x) \equiv v(x) - v(0)$ and rewrite equation (5) as

$$rU(x) = Bx^2 + \frac{\sigma^2}{2}(U''(x) - v''(0)) - \kappa \int_0^{U(x)} G(\psi)d\psi \qquad \text{for } x \in [0, X] \tag{49}$$

with boundary conditions $U'(X) = 0$ and $U(X) = \Psi$. Note that by definition $U(0) = 0$. To obtain equation (49) we used integration by parts on the right hand side of equation (5):

$$\begin{aligned}\int_0^{U(x)} [\psi - U(x)]G'(\psi)d\psi &= G(\psi)\psi\Big|_0^{U(x)} - \int_0^{U(x)} G(\psi)d\psi - U(x)\int_0^{U(x)} G'(\psi)d\psi \\ &= G(\psi)\psi\Big|_0^{U(x)} - \int_0^{U(x)} G(\psi)d\psi - U(x)\left[G(U(x)) - G(0)\right] \\ &= -\int_0^{U(x)} G(\psi)d\psi + U(x)G(0)\end{aligned}$$

Next differentiate both sides of equation (49) with respect to x to obtain:

$$\left[r + \kappa G(U(x))\right] U'(x) = 2Bx + \frac{\sigma^2}{2} U'''(x) \;\; \text{for } x \in [0, X] \tag{50}$$

with boundary conditions given by: $U'(X) = 0$ and $U'(0) = 0$. The first boundary condition is smooth pasting. Note that if $X = \infty$ we do not have smooth pasting, but since v is bounded above so is U, then it must be that $\lim_{x\to\infty} U'(x) = 0$, and hence the analogous boundary condition holds in the case where X is unbounded. The second boundary is implied by the symmetry and differentiability of $v(\cdot)$, and hence of $U(\cdot)$, around $x = 0$. Thus, solving for the value function in equation (5) is equivalent to solving for $U(\cdot)$ in equation (50) with its corresponding boundary conditions.

Now define $u(x) \equiv U'(x)$ and rewrite equation (50) using that $\Lambda(x) = \kappa G(U(x))$, by equation (4). This gives the o.d.e. in equation (6). The boundary conditions described above in terms of U' thus become $u(X) = u(0) = 0$.

Uniqueness and invertibility. Note that equation (6) is a linear second order ordinary differential equation of the Sturm-Liouville type with two Dirichlet boundary conditions, where we write: $L(u)(x) \equiv [r + \Lambda(x)]\, u(x) - \frac{\sigma^2}{2} u''(x)$ and thus the equation above can be written as $L(u)(x) = 2Bx$.

The function $\Lambda(\cdot)$ defining the operator L is continuous, so it has a unique solution $u(\cdot)$. To see this let $L(u)(x) = 2Bx$ and let $\{\theta_j, \varphi_j\}$ be the eigenvalues and orthonormal eigenfunctions of L satisfying the Dirichlet boundary conditions, i.e. solving $L(\varphi_j) = \theta_j \varphi_j$ and with $\varphi_j(0) = \varphi_j(X) = 0$. By linearity we have $L(\sum_j \alpha_j \varphi_j) = \sum_j \theta_j \alpha_j \varphi_j$ for any square integrable sequence $\{\varphi_j(\cdot)\}$. Then we can choose $\{\alpha_j\}$ so that $u(x) = \sum_j \theta_j \alpha_j \varphi_j(x)$, with the equality in the $\mathbf{L}^2$ sense. In particular we can set $\alpha_j = \langle \varphi_j, u \rangle / \theta_j$. Again, the case of $X = \infty$, requires a slightly different argument for the existence of its solution. In particular, the existence of a solution is guaranteed by Theorem 3.1 in Lian, Wang, and Ge (2009). By the Maximum principle then $u(x) > 0$ since $2Bx > 0$ in $(0, X)$. Since $u > 0$ then U is increasing and thus it is invertible.

Value function. We construct $v(\cdot)$ as follows. Recall $u = U'$ and $U(0) = 0$, we have

$$U(x) = \int_0^x u(z)dz \text{ for all } x \in [0, X] \quad \text{and} \quad \Psi = U(X) \ .$$

From the definition of $U(x) = v(x) - v(0)$ and equation (5) we have

$$v''(0) = U''(0) = u'(0) \text{ and } rv(0) = v''(0)\frac{\sigma^2}{2} \text{ so } v(0) = u'(0)\frac{\sigma^2}{2r}$$

which gives equation (7) in the lemma. Note that $v(\cdot)$ is increasing because $u(x) > 0$ on $(0, X)$ as established above. $\square$

Proof. (of Theorem 1). We now construct the fixed cost Ψ, the Poisson arrival rate κ, the value of $G(0)$ and the density $G'(\cdot)$ that rationalize the generalized hazard rate $\Lambda(\cdot)$ using the function $u(\cdot)$. We use equation (4), $\Lambda(x) = \kappa G(U(x))$ for all $x \in [0, X]$, which evaluated at $x = 0$ implies $\Lambda(0) = \kappa G(0)$. Denote by $w(\cdot) \equiv U^{-1}(\cdot)$, the inverse function of $U(\cdot)$, mapping $[0, \Psi]$ onto $[0, X]$. Set κ to be $\kappa = \Lambda(X)$ to ensure that $G(\Psi) = 1$. Differentiating the expression above with respect to x, we have $G'(U(x))\, U'(x) = \frac{\Lambda'(x)}{\Lambda(X)}$ for all $x \in (0, X)$ and thus

$$G'(\psi) = G'(U(w(\psi))) = \frac{\Lambda'(w(\psi))}{u(w(\psi))\Lambda(X)} = \frac{\Lambda'(w(\psi))}{u(w(\psi))\Lambda(X)} \text{ for all } \psi \in (0, \Psi)$$

which gives the density of G' in terms of the function u defined in Lemma 1. $\square$

Proof. (of Theorem 2) Without loss of generality, given the assumed symmetry, let $q(\cdot)$ be the density of minus price changes, so that $q(x)N_a = \Lambda(x)f(x)$. Denote the minus price changes by Δp. We will use four equations for $x > 0$:

$$f''(x) = \frac{2}{\sigma^2} q(x) N_a \tag{51}$$

$$f'(x) = f'(X) - \int_x^X f''(t)dt \tag{52}$$

$$f(x) = -\int_x^X f'(t)dt \tag{53}$$

$$\sigma^2 = N_a Var(\Delta p) \tag{54}$$

where we have used that $f(X)=0$. Combining the first and the second equation we have ,

$$f'(x) = f'(X) - \frac{2}{\sigma^2}N_a\int_x^X q(x)dx = f'(X) - \frac{2}{\sigma^2}N_a\left(1 + f'(X)\frac{\sigma^2}{2N_a} - Q(x)\right) \tag{55}$$

$$= \frac{2}{\sigma^2}N_a(Q(x)-1) \tag{56}$$

where we have used that $\lim Q(x) \to 1 + f'(X)\frac{\sigma^2}{2N_a}$ as $x \to X$. Integrating further,

$$f(x) = \frac{2}{\sigma^2}N_a\int_x^\infty (1-Q(t))dt \tag{57}$$

Now using the last equation,

$$f(x) = \frac{2}{Var(\Delta p)}\int_x^\infty (1-Q(t))dt \tag{58}$$

Incurring the identity $q(x)N_a = \Lambda(x)\bar{p}(x)$ once again,

$$\Lambda(x) = \frac{N_a Var(\Delta p)}{2}\frac{q(x)}{\int_x^\infty (1-Q(t))dt} \tag{59}$$

Finally, we check whether $\Lambda(X)=\kappa<\infty$. If $X<\infty$, then using L'Hopital we get

$$\Lambda(X) = \frac{N_a Var(\Delta p)}{2}\frac{q'(X)}{-f'(X)\frac{\sigma^2}{2}} < \infty \tag{60}$$

If $X=\infty$, we apply L'Hopital rule twice, since $q(x)\to 0$ and $Q(x)\to 1$ as $x\to\infty$. We obtain:

$$\Lambda(X) = \frac{N_a Var(\Delta p)}{2}\lim_{x\to\infty}\frac{q''(x)}{q(x)} \tag{61}$$

which is finite given our assumption on the tail of q. This completes the proof. □

Proof. (of Proposition 5) Under the identification assumptions,

$$\frac{\mathbb{E}[(\Delta p_{it})^j(\Delta p_{is})^k]}{\mathbb{E}[(\Delta p_{it})^{j'}(\Delta p_{is})^{k'}]} = \frac{\mathbb{E}[b_i^{j+k}(\Delta\tilde{p}_t)^j(\Delta\tilde{p}_s)^k]}{\mathbb{E}[b_i^{j'+k'}(\Delta\tilde{p}_t)^{j'}(\Delta\tilde{p}_s)^{k'}]} = \frac{\mathbb{E}[(b_i)^{j+k}]\mathbb{E}[(\Delta\tilde{p}_t)^j]\mathbb{E}[(\Delta\tilde{p}_s)^k]}{\mathbb{E}[(b_i)^{j'+k'}]\mathbb{E}[(\Delta\tilde{p}_t)^{j'}]\mathbb{E}[(\Delta\tilde{p}_s)^{k'}]} \tag{62}$$

$$= \frac{\mathbb{E}[(\Delta\tilde{p}_t)^j]\mathbb{E}[(\Delta\tilde{p}_t)^k]}{\mathbb{E}[(\Delta\tilde{p}_t)^{j'}]\mathbb{E}[(\Delta\tilde{p}_t)^{k'}]} \tag{63}$$

The first equality uses $\Delta p_{it} = b_i\Delta\tilde{p}_t$. The second one uses mutual independence of b_i, $\Delta\tilde{p}_t$, and $\Delta\tilde{p}_s$. The last one uses the fact that $\Delta\tilde{p}_t$ and $\Delta\tilde{p}_s$ are identically distributed. □

Proof. (of Theorem 3) Since

$$\sigma^2 = N_a\frac{\int_{-\infty}^\infty x^2\Lambda(x)f(x)dx}{\int_{-\infty}^\infty \Lambda(x)f(x)dx} \text{ or } N_a = \sigma^2\frac{\int_{-\infty}^\infty \Lambda(x)f(x)dx}{\int_{-\infty}^\infty x^2\Lambda(x)f(x)dx} \tag{64}$$

we can write the formula for kurtosis over $6N_a$ as:

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx \int_{-\infty}^{\infty} \Lambda(x) f(x) dx}{6N_a \left[\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx\right]^2} = \frac{\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx}{6\sigma^2 \int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx} \tag{65}$$

Using the Kolmogorov forward equation,

$$\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx = \frac{\sigma^2}{2} \int_{-\infty}^{\infty} x^4 f''(x) dx \tag{66}$$

Integrating by parts twice,

$$\int_{-\infty}^{\infty} x^4 \Lambda(x) f(x) dx = 6\sigma^2 \int_{-\infty}^{\infty} x^2 f(x) dx \tag{67}$$

This allows us to write

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{\int_{-\infty}^{\infty} x^2 f(x) dx}{\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx} \tag{68}$$

Now we work on the denominator: using again the Kolmogorov Forward equation we have:

$$\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx = \frac{\sigma^2}{2} \int_{-\infty}^{\infty} x^2 f''(x) dx \tag{69}$$

Integrating by parts twice, using that f is a density, we have:

$$\int_{-\infty}^{\infty} x^2 \Lambda(x) f(x) dx = \sigma^2 \tag{70}$$

Thus we can write:

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{\int_{-\infty}^{\infty} x^2 f(x) dx}{\sigma^2} \tag{71}$$

Recall that we have a system of two equations:

$$\Lambda(x) f(x) = \frac{\sigma^2}{2} f''(x) \tag{72}$$

$$\Lambda(x) m(x) = \frac{\sigma^2}{2} m''(x) - x \tag{73}$$

Eliminate Λ:

$$\frac{\sigma^2}{2} \frac{m(x) f''(x)}{f(x)} = -x + \frac{\sigma^2}{2} m''(x) \tag{74}$$

Multiply both sides by $f(x)x$ and rearrange to get

$$\frac{\sigma^2}{2}[m(x)f''(x) - m''(x)f(x)]x = -x^2 f(x) \tag{75}$$

Integrate both sides from 0 to ∞:

$$\frac{\sigma^2}{2}\int_0^\infty [m(x)f''(x) - m''(x)f(x)]x dx = -\int_0^\infty x^2 f(x) dx \tag{76}$$

Perform integration by parts in the left-hand side using the fact that $[m(x)f'(x) - m'(x)f(x)]' = m(x)f''(x) - m''(x)f(x)$:

$$\begin{aligned}\frac{\sigma^2}{2}\int_0^\infty [m(x)f''(x) - m''(x)f(x)]x dx =& \frac{\sigma^2}{2}[m(x)f'(x) - m'(x)f(x)]x \Big|_0^\infty \\ &- \frac{\sigma^2}{2}\int_0^\infty [m(x)f'(x) - m'(x)f(x)] dx \\ =& -\sigma^2 \int_0^\infty m(x)f'(x) dx\end{aligned} \tag{77}$$

The last equality is just integration by parts again. We used $\mathbb{E}[m(x)] < \infty$ and $m(\cdot)$ being almost linear at infinity to justify setting $f'(x)m(x)x$ and $f(x)m'(x)x$ at infinity to 0. Hence, we have

$$\sigma^2 \int_0^\infty m(x)f'(x) dx = \int_0^\infty x^2 f(x) dx \tag{78}$$

As we showed above, this is equivalent to

$$\mathcal{M}'(0) = \frac{Kurt(\Delta p)}{6N_a} \tag{79}$$

This completes the proof. $\square$

Proof. (of Proposition 8) Let the price gap distributions that correspond to Λ_1 and Λ_2 be f_1 and f_2. Recall that for a fixed N_a and σ^2 we have $f_1'(0) = f_2'(0)$ and it is sufficient to compare

$$\int_0^\infty f_1(x)x^2 dx \text{ against } \int_0^\infty f_2(x)x^2 dx \tag{80}$$

(1) We first claim that the graph of the function $f_1(x) - f_2(x)$ cannot cross the $x-$axis from above. That is, there is no segment $[a, b]$ such that $f_1(x) - f_2(x) = 0$ on this segment, $f_1(x) - f_2(x) > 0$ to the left of a, and $f_1(x) - f_2(x) > 0$ to the right of b. Note that this nests the case when $a = b$ and hence $[a, b]$ is a single point. Suppose such a segment exists. Then one of the two statements is true: either $\Lambda_1(x) \geq \Lambda_2(x)$ for all $x \leq a$ or $\Lambda_1(x) \leq \Lambda_2(x)$ for all $x \geq b$.

In the first case, the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the left of a. If it does cross it at some $c < a$, on (c, a) we have $f_1(x) > f_2(x)$ and hence $\Lambda_1(x)f_1(x) > \Lambda_2(x)f_2(x)$, implying $f_1''(x) > f_2''(x)$. But this contradicts $f_1'(c) - f_2'(c) \geq 0$ and $f_1'(a) - f_2'(a) \leq 0$ holding simultaneously. Hence, for all $x < a$ we have $f_1(x) > f_2(x)$, implying $\Lambda_1(x)f_1(x) > \Lambda_2(x)f_2(x)$ and $f_1''(x) > f_2''(x)$ on $(0, a)$. But since $f_1'(a) \leq f_2'(a)$, in this region we have $f_1'(x) < f_2'(x)$, which contradicts $f_1'(0) = f_2'(0)$.

In the second case, the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the right of b. If it does cross it at some $d > b$, on (b, d) we have $f_1(x) < f_2(x)$ and hence $\Lambda_1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$. But this contradicts $f_1'(b) - f_2'(b) \leq 0$ and $f_1'(d) - f_2'(d) \geq 0$ holding simultaneously. Hence the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis again to the right of b, which already rules out $X_1 > X_2$. Moreover, if $X_1 = X_2 \leq \infty$, it must hold that $f_1'(X_1) \geq f_2'(X_1)$, which contradicted by $f_1'(x) < f_2'(x)$ for $x > b$. The latter follows from $f_1'(b) - f_2'(b) \leq 0$ and $f_1''(x) < f_2''(x)$ for $x > b$.

(2) Since the graph of the function $f_1(x) - f_2(x)$ cannot cross the $x-$axis from above, it can only cross the $x-$axis from below. We know that there must be at least one crossing, because f_1 and f_2 are continuous and both integrate to one. Hence, the function $f_1(x) - f_2(x)$ is non-positive until some point ang non-negative after some point until X_1. Morover, there are segments of strict positivity ang strict negativity. Hence,

$$\int_0^{X_1} (f_1(x) - f_2(x))x^2 dx > 0 \tag{81}$$

This completes the proof. □

B Solution for the firm's alternative setup of Section 2.2

The first order condition for choice of ℓ in equation (15) are:

$$c'_-(\ell^*(x)) \leq v(x) - v(0) \leq c'_+(\ell^*(x)) \text{ for all } x$$

where $\ell^*(x)$ denotes the optimal policy, and where $c'_-(\cdot)$ and $c'_+(\cdot)$ denote the right and left derivatives of c. As in the previous case, we have that if $\Psi < \infty$ there is a barrier $X < \infty$ for which: $v(X) = v(0) + \Psi$ and $v'(X) = 0$. Finally, by the same reasons as before, we have symmetry, i.e. $v(x) = v(-x)$, and $\ell^*(x) = \ell^*(-x)$. As before we can summarize the decision rule of the firm for $x \in (-X, X)$ with a generalized hazard function:

$$\Lambda(x) = \ell^*(x) \text{ for all } x$$

To simplify the discussion, next we describe the case of a cost c that is continuously differentiable and strictly convex, where we simply have:

$$c'(\ell^*(x)) = v(x) - v(0) \text{ and } \Lambda(x) = (c')^{-1}(v(x) - v(0)) \text{ for all } x$$

We note that since $v(x)$ is strictly increasing in x for $x \in (0, X)$, and $c(\ell)$ is convex, then $\Lambda(x)$ must also be increasing in x for $x \in (0, X)$.

Replacing ℓ^* into the HBJ equation we obtain:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \ell^*(x)(v(0) - v(x)) + c(\ell^*(x)), r(\Psi + v(0))\right\}$$

Let us assume that the cost function c has a continuous derivative. Defining, as before $U(x) = v(x) - v(0)$, with $u = U' = v'$, we can differentiate the HBJ equation in $x \in (0, X)$, and use the envelope to obtain:

$$[r + \Lambda(x)]\, u(x) = 2Bx + \frac{\sigma^2}{2} u''(x) \; .$$

Using the boundaries $u(0) = u(X) = 0$, and the logic used in the proof of Theorem 1 it is then straightforward to solve for ℓ^*.

C Kurtosis of a mixture

The next proposition shows that if we have a sample with mixed N different type of products all with the same kurtosis but with different variance, then the kurtosis of the price changes of such a mixture is higher than the kurtosis for each of them.

Proposition 10. Assume that Δp is a mixture of N distributions, with weights $\{\omega_j\}_{j=1}^N$. Assume that for each distribution j, price changes have the same kurtosis K, but they may have different variance V_j. Then

$$Kurt(\Delta p) = \frac{\sum_j \omega_j K\, V_j^2}{\left[\sum_j \omega_j V_j\right]^2} = K \frac{\sum_j \omega_j V_j^2}{\left[\sum_j \omega_j V_j\right]^2} = K \frac{\sum_j J(V_j)\omega_j}{J\left(\sum_j V_j \omega_j\right)} \geq K \tag{82}$$

with strict inequality if the distribution of $\{V_j\}_{j=1}^N$ is not degenerate, since $J(V) = V^2$ is a strictly convex function.

D Estimation

In this appendix we present our estimation algorithm and some additional results. First, we plot the symmetrized histograms with fitted densities for two data cleaning procedures: the one that eliminates price changes smaller then 2 cents in absolute value, and the one eliminating those smaller than 1 cent in absolute value. The distributions are very close, with immaterial differences in the bars around zero.

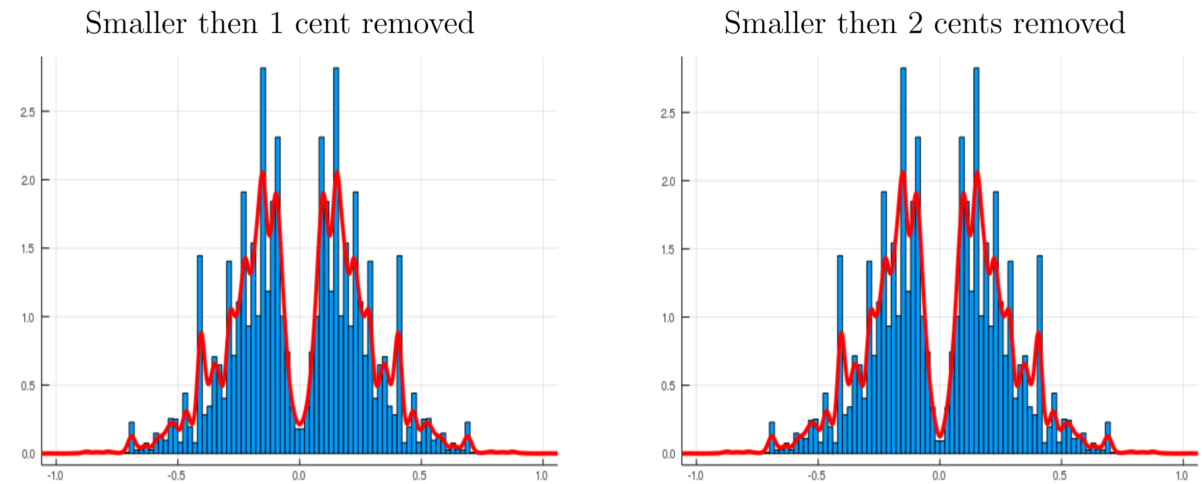
We use the method of moments to estimate the mixture of two Gamma distributions with the parameters ω (the weight), (α_1, β_1) and (α_2, β_2). The moments of $|\Delta \tilde{p}_t|$ we use are denoted by $\gamma_{j,k}$:

$$\gamma_{j,k} = \frac{\mathbb{E}[|\Delta \tilde{p}_t|^{j+k}]}{\mathbb{E}[|\Delta \tilde{p}_t|^j]\mathbb{E}[|\Delta \tilde{p}_t|^k} \tag{83}$$

For a mixture of two Gamma distributions with the weight ξ on the first one,

$$\gamma_{j,k} = \frac{\left[\beta_2^{j+k}\omega \frac{\Gamma(\alpha_1 + j + k)}{\Gamma(\alpha_1)} + \beta_1^{j+k}(1-\omega)\frac{\Gamma(\alpha_2 + j + k)}{\Gamma(\alpha_2)}\right]}{\left[\beta_2^{j}\omega \frac{\Gamma(\alpha_1 + j)}{\Gamma(\alpha_1)} + \beta_1^{j}(1-\omega)\frac{\Gamma(\alpha_2 + j)}{\Gamma(\alpha_2)}\right]\left[\beta_2^{k}\omega \frac{\Gamma(\alpha_1 + k)}{\Gamma(\alpha_1)} + \beta_1^{k}(1-\omega)\frac{\Gamma(\alpha_2 + k)}{\Gamma(\alpha_2)}\right]} \tag{84}$$

Figure 7: Distribution of price changes in a narrow category



Pooling all products for category 561 "Non-durable household goods"

Using these moments allows us to recover ω, α_1, α_2, and the ratio β_1/β_2. The exact values of β_1 and β_2 are pinned down by the normalization $\mathbb{E}[|\Delta\tilde{p}_t|] = 1$. To estimate $\gamma_{j,k}$, we rely on Proposition 5:

$$\frac{\mathbb{E}[|\Delta\tilde{p}_t|^{j+k}]}{\mathbb{E}[|\Delta\tilde{p}_t|^j]\mathbb{E}[|\Delta\tilde{p}_t|^k} = \frac{\mathbb{E}[|\Delta p_{it}|^{j+k}]}{\mathbb{E}[|\Delta p_{it}|^j|\Delta p_{is}|^k]} \tag{85}$$

For all seven product categories, we get four moments ($\hat{\gamma}_{1,1}$, $\hat{\gamma}_{2,1}$, $\hat{\gamma}_{3,1}$, and $\hat{\gamma}_{3,2}$) from the data and solve the system of four analogs of equation (84). We minimize the sum of deivations squared with equal weights. The results are presented in Table 2.

Category	$\hat{\gamma}_{11}$	$\hat{\gamma}_{21}$	$\hat{\gamma}_{31}$	$\hat{\gamma}_{32}$	$\hat{\alpha}_1$	$\hat{\alpha}_2$	$\hat{\beta}_1/\hat{\beta}_2$	$\hat{\omega}$	$\hat{\alpha}_{22}$
111	1.248	1.406	1.507	1.787	2.099	12.190	228.677	0.161	4.248
119	1.282	1.507	1.702	2.381	1.058	6.012	91.439	0.109	3.747
1212	1.242	1.476	1.786	2.9230	0.599	3.873	73.414	0.000	4.151
122	1.243	1.397	1.508	1.903	1.848	9.779	173.048	0.131	4.460
118	1.289	1.539	1.777	2.552	3.123	9.836	0.628	0.580	3.610
117	1.281	1.511	1.721	2.484	0.967	5.442	84.154	0.089	3.801
561	1.216	1.394	1.586	2.271	0.998	5.783	103.470	0.031	4.782

Table 2: Moments taken from the data and the estimated parameters

Specializing to the case with a single Gamma distribution $\omega = 1$ allows us to recover the expressions for α in closed form. Consider $\gamma_{j,1}$ for some j:

$$\gamma_{j,1} = \frac{\Gamma(\alpha+j+1)\Gamma(\alpha)}{\Gamma(\alpha+j)\Gamma(\alpha+1)} = 1 + \frac{j}{\alpha} \tag{86}$$

Hence,

$$\alpha = \frac{j}{\gamma_{j,1} - 1} \tag{87}$$

Since we attach particula importance to the kurtosis, we would also like to use $\gamma_{2,2}$:

$$\gamma_{j,2} = \frac{\Gamma(\alpha + j + 2)\Gamma(\alpha)}{\Gamma(\alpha + j)\Gamma(\alpha + 2)} = \frac{(\alpha + j + 1)(\alpha + j)}{(\alpha + 1)\alpha} = \left(1 + \frac{j+1}{\alpha}\right)\frac{\gamma_{j,1}}{\gamma_{1,1}} \tag{88}$$

This leads to

$$\alpha = \frac{(j+1)\gamma_{j,2}}{\gamma_{j,2}\gamma_{1,1} - \gamma_{j,1}} \tag{89}$$

Notice that β, the scale of the distribution, drops out from these expressions, because $\gamma_{j,k}$ are dimensionless moments. We use a linear combinations of expressions in equation (87) and equation (89) with $\hat{\gamma}_{j,1}$ for $j \in \{1, 2\}$ and $\hat{\gamma}_{22}$ as estimators of α. Consistency requires the weights of the combinations to sum to one, and we make them inversely proportional to the bootstrapped variance of the estimators of summands. The stimates are presented in the last column of Table 2: the estimate $\hat{\alpha}_{22}$ is contructed from $\hat{\gamma}_{11}$, $\hat{\gamma}_{21}$, and $\hat{\gamma}_{22}$.

In Table 3 we present some additional statistics. First, we tabulate skewness of the distribution of price changes to show that the distributions are close to symmetric. Then, we contrast the estimates of the Kurtosis with the full sample and with the first two price changes only. The difference between them is suggestive of a strong correlation between consecutive price changes (squared), and of a weaker correlation between distant price changes. As can be seen from equation (33), how much the underlying Kurtosis is different from that of the pooled distribution (without accounting for product heterogeneity) increases with this correlation. The implied correlation and the coefficient of variation (present in equation (33) as well) are tabulated in the remaining two columns.

Now we present the estimation procedure to recover the flow cost function from Section 2.2. The model in this section permits Λ to be unbounded. We take advatage of that and work with a power hazard $\Lambda(x) = \kappa x^{\nu}$. This form of Λ gives rise to a specifica functional form of Q. We compute the moments of Q as functions of (κ, ν) and then estimate them using the mothod of moments.

Suppose $\Lambda(x) = \kappa x^{\nu}$. Denote $\rho = 2\kappa/\sigma^2$. The corresponding density of price gaps has to obey a Kolmogorov forward equation that has the form

$$\rho x^{\nu} f(x) = f''(x) \tag{90}$$

With $X = \infty$, the solution is

$$f(x) = \frac{x^{1/2}\mathcal{K}_{1/(\nu+2)}\left(\frac{2\sqrt{\rho}}{\nu+2}x^{(\nu+2)/2}\right)}{2\int\limits_0^{\infty} x^{1/2}\mathcal{K}_{1/(\nu+2)}\left(\frac{2\sqrt{\rho}}{\nu+2}x^{(\nu+2)/2}\right)dx} \tag{91}$$

Category	Skewness	Kurtosis	Kurtosis ($t = 1, 2$)	Implied Correlation	$CV(\Delta\tilde{p}_t)$
111	-0.121	1.656	1.426	0.440	1.555
		(0.065)	(0.071)		
119	0.011	1.955	1.288	0.339	1.683
		(0.050)	(0.042)		
1212	-0.020	2.051	1.710	0.284	1.589
		(0.162)	(0.186)		
122	-0.025	1.677	1.189	0.390	1.398
		(0.051)	(0.019)		
118	-0.012	2.044	1.663	0.295	1.620
		(0.118)	(0.150)		
117	-0.004	1.989	1.422	0.303	1.577
		(0.047)	(0.089)		
561	-0.006	1.778	1.403	0.374	1.524
		(0.133)	(0.066)		

Table 3: Additional statistics

The distribution of price changes is then given by

$$q(-x) = \frac{\kappa x^\nu f(x)}{N_a} = \frac{\kappa x^{\nu+1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right)}{2N_a \int\limits_0^\infty x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx} \tag{92}$$

Since $\mathbb{V}[\Delta\tilde{p}_t] = 1$, we have $\sigma^2 = N_a$, so

$$q(-x) = \frac{\kappa x^\nu f(x)}{N_a} = \frac{\rho x^{\nu+1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right)}{4 \int\limits_0^\infty x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx} \tag{93}$$

This has to be a probability distribution, so it integrates two one. We also have the moment condition $\mathbb{E}[(\Delta\tilde{p}_t)^4] = Kurt(\Delta\tilde{p}_t)$. Writing the two restrictions in a convenient form,

$$\int\limits_0^\infty (\rho x^\nu - 2) x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx = 0 \tag{94}$$

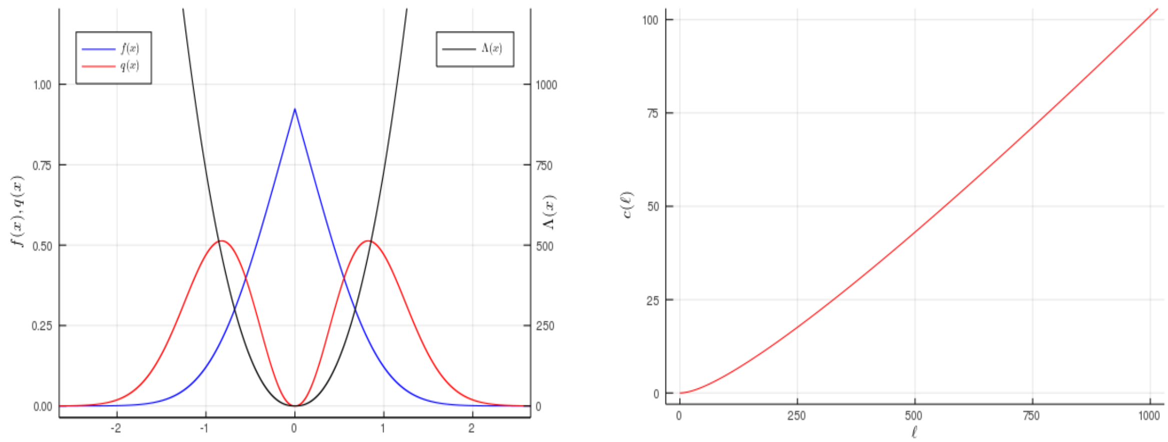
$$\int\limits_0^\infty (\rho x^{\nu+4} - 2Kurt(\Delta\tilde{p}_t)) x^{1/2} \mathcal{K}_{1/(\nu+2)} \left(\frac{2\sqrt{\rho}}{\nu+2} x^{(\nu+2)/2} \right) dx = 0 \tag{95}$$

From these two relations we can get $(\hat{\rho}, \hat{\nu})$. After that, using $\sigma^2 = N_a$, we can recover $\hat{\kappa}$:

$$\hat{\kappa} = \frac{\hat{\rho} N_a}{2} \tag{96}$$

Figure 8: Estimated distribution of price changes and implied cost function

Estimated $q(\cdot)$ and $f(\cdot)$, assumed $\Lambda(\cdot)$ — Recovered flow cost function



The system of two restrictions can be solved exactly, and the model is just identified. The results for the category 561 (”non-durable household goods”) are presented on Figure 8. The estimated parameters are $\hat{\nu} = 2.285$ and $\hat{\kappa} = 30.747$, corresponding to the Kurtosis 1.64, slightly below the quadratic case.

E Duration Analysis and Generalized Hazard Rate

In this section we consider the Survival and the Hazard Rate as functions of the *duration* of the price spells. Duration-based functions are often used in sticky price models. It is interesting to know whether the information encoded in them is different from that encoded in the size-distribution of price changes used above. We establish conditions for a non-trivial equivalence result: the distribution of durations and the variance of price changes together contain the same information about the fundamentals of the model as the distribution of price changes and frequency of adjustment. The distribution of spells with one statistic on the size of changes (the variance) is as informative as the size-distribution of changes and one temporal statistic (the frequency).

Denote by $S(t)$ the Survival function, the probability that a price spell lasts at least t units of time. We will show that, when $X = \infty$, an analytical Survival Function S uniquely identifies an analytical Generalized Hazard Rate function Λ. When $X = \infty$, the Survival function is given by

$$S(t) = \mathbb{E}\left[e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \text{ for all } t \geq 0 \tag{97}$$

where the expectation is taken with respect to the paths of the drift-less Brownian motion x with variance per unit of time equal to σ^2. The value of $S(t)$ is the Feynman-Kac formula evaluated at $x = 0$. The hazard rate $h(t) = -S'(t)/S(t)$ measures the probability per unit of time of a price spell ending conditional on lasting at least t. For example, the Survival function and its associated hazard rate for the case of a quadratic generalized hazard rate $\Lambda(x) = \Lambda(0) + \kappa x^2$ are:

$$S(t) = \frac{e^{-t\Lambda(0)}}{\left(\cosh\left(t\sqrt{2\kappa\sigma^2}\right)\right)^{\frac{1}{2}}} \text{ and } h(t) = \Lambda(0) + \sqrt{\kappa\frac{\sigma^2}{2}} \tanh\left(t\sqrt{2\kappa\sigma^2}\right) \text{ for all } t \geq 0 \tag{98}$$

This was obtained by Kac in his seminal study of what we now know as the Kac formula. The next lemma gives the main technical result to establish the link between the Survival function, which can in principle be measured in the data, and the generalized hazard function $\Lambda(x)$.

LEMMA 2. Fix a value of $\sigma^2 > 0$, and assume that $X = \infty$. Assume that S is related to Λ by equation (97). The derivatives of the Survival function S a time $t = 0$ and the derivatives of Λ at $x = 0$ are related by the recursively generated functions $\{F_n\}$ as follows:

$$\frac{\partial^n S(t)}{\partial t^n}\Big|_{t=0} = F_n(0) \text{ and all } n = 1, 2, \ldots \text{ where } F_n(\cdot) \text{ are given by} \tag{99}$$

$$F_{n+1}(x) = \frac{\sigma^2}{2}\frac{\partial^2 F_n(x)}{\partial x^2} - \Lambda(x)F_n(x) \text{ for all } x \in \mathbb{R} \text{ and } n = 1, 2, \ldots \text{ and} \tag{100}$$

$$F_1(x) = -\Lambda(x) \text{ for all } x \in \mathbb{R} \tag{101}$$

Lemma 2 is the base of an algorithm to compute the derivatives of S at $t = 0$ given Λ and the derivatives of Λ at $x = 0$ given S. Using this lemma, we obtain the main result of this section:

PROPOSITION 11. Assume that $\sigma^2 > 0$, $X = \infty$, and Λ satisfies Assumption 1. Let S be the Survival function of Λ, as in equation (97). If the generalized hazard function Λ is analytical, then the Survival function S uniquely identifies Λ. Likewise, if the Survival function S is analytical, then the generalized hazard function Λ uniquely identifies S.

As remarked before, Lemma 2 gives an algorithm to recursively compute an expansion of S based on the derivatives of Λ, or an expansion of Λ based on the derivatives of S. An implication of Lemma 2 and Proposition 11 is that the hazard rate and its first three derivatives at zero duration ($t = 0$) are given by particularly simple expressions involving the level and first two even derivatives of the generalized hazard function evaluated at zero price gap, i.e. $x = 0$:

$$h(0) = \Lambda(0) \geq 0\,,\ \frac{\partial h(t)}{\partial t}|_{t=0} = \frac{\sigma^2}{2}\frac{\partial^2\Lambda(x)}{\partial x^2}|_{x=0}\,,\ \frac{\partial^2 h(t)}{\partial t^2}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^2\frac{\partial^4\Lambda(x)}{\partial x^4}|_{x=0}\,,$$
$$\text{and } \frac{\partial^3 h(t)}{\partial t^3}|_{t=0} = \left(\frac{\sigma^2}{2}\right)^3\frac{\partial^6\Lambda(x)}{\partial x^6}|_{x=0} - 4\left(\frac{\sigma^2}{2}\frac{\partial^2\Lambda(x)}{\partial x^2}|_{x=0}\right)^2$$

These formulas give a simple connection between the local behavior of Λ around $x = 0$ and h around $t = 0$. Note that if $\Lambda(x)$ is, in addition of being symmetric and differentiable in x, increasing in $|x|$ around $x = 0$, then $\Lambda''(0) > 0$, and hence the hazard rate as function of duration, $h(t)$, must be increasing in duration, at least for small durations t. Likewise, if $\Lambda(x)$ were decreasing in $|x|$ around $x = 0$, then $\Lambda''(0) < 0$ and hence $h(t)$ must be locally decreasing in duration.

Comparing with the case of Theorem 2, in this case we use much more restrictive conditions for Λ, and obtain a more cumbersome representation — an infinite expansion instead of a closed-form expression involving an integral. In spite of this Theorem 2 and Proposition 11 have the same flavor: they show that if Λ is analytical and $X = \infty$, then Λ can be fully identified either using the information contained in the Survival function, i.e duration on price changes, and σ^2, which can be recovered from N_a and the variance of price changes with equation (19). Of course, this also means that the information on the survival function and the size distribution of price changes can be used as an over-identifying test of the model.

Finally, we can also estimate $\mathcal{C} \equiv \Lambda(0)/N_a$, the fraction of price changes independent of the state, by using duration data. Given the results above, $\mathcal{C}$ can be estimated as $h(0)/N_a$. This can be an alternative to the estimates presented in Table 1 using the size distribution of price changes. As in Section 4, a correction of unobserved heterogeneity may be important.

F Flexibility Index: scope and limitations

Caballero and Engel (2007) introduced the concept of the Flexibility Index ($\mathcal{F}$), subsequently used in several studies such as Berger and Vavra (2018), as an inverse measure of monetary nonneutrality. We show below that $\mathcal{F}$ measures the slope of the impulse response of prices right after a small once-and-for-all monetary shock. Below we define $\mathcal{F}$ in terms of the model, using Caballero and Engel's (2007) formula, and study the extent to which it is an accurate summary of the model's non-neutrality. We show that for models with barriers, where $X < \infty$, the flexibility index is always infinite. This prompts us to focus on the cases without barriers, $X = \infty$, where $\mathcal{F}$ is finite. In such cases we can compare $\mathcal{F}$ with the summary measure given by the cumulative impulse response defined in equation (39). We display non-pathological simple examples where the $\mathcal{F}$ is *not* an accurate summary of the effect on output, neither of its cumulative response, nor of its short term response.

The IRF of the aggregate price level after a shock δ can be written as

$$\mathcal{P}(t,\delta) = \Omega(\delta) + \int_0^t \omega(s,\delta)\, ds \tag{102}$$

where $\omega(s,\delta)$ is the flow contribution to the IRF at time $s > 0$, and $\Omega(\delta)$ is the time $t = 0$ jump in the price level. By definition $\frac{\partial}{\partial t}\mathcal{P}(t,\delta) = \omega(t,\delta)$. The flow value of the IRF of the aggregate price level at time $t > 0$ is given by

$$\omega(t,\delta) = -\int_{-X}^{X} x\Lambda(x) f(x,t) dx + X\sigma^2 \left[f'(-X,t) - f'(X,t)\right]$$

where $f(x,t)$ is the distribution of the price gaps among the firms that have not adjusted prices t units of time after the monetary shock. The first term is the change of prices across the distribution of price gaps at time t, with $f(x,t)$ solving the time dependent Kolmogorov Forward Equation:

$$\partial_t f(x,t) = -\Lambda(x) f(x,t) + \frac{\sigma^2}{2}\partial_{xx} f(x,t) \text{ for all } x \in [-X, X] \text{ and } t \geq 0, \tag{103}$$

$$f(X,t) = f(-X,t) = 0 \text{ for all } t > 0, \text{ and } f(x,0) = f_0(x) \text{ for all } x \in [-X, X] \tag{104}$$

The initial jump is given by

$$\Omega(\delta) = \int_{-X}^{-X+\delta} (-x + \delta)\ f_0(x)\, dx \tag{105}$$

The initial distribution f_0 that we consider is a uniform shift by δ of some distribution $\hat{f}$:

Assumption 2. The initial condition is $f_0(x) = \hat{f}(x+\delta)$, where $\hat{f}$ i) equals zero at the bounds, $0 = \hat{f}(-\bar{X}) = \hat{f}(\bar{X})$, ii) increases close to the lower bound, $0 < \hat{f}'(-\bar{X}) < \infty$, and iii) is differentiable on $(-\bar{X}, 0)$.

We write $f_0(x) = \hat{f}'(x)\delta + o(\delta)$ and consider the case of small δ. Note that the assumptions allow $\hat{f}$ to be the invariant distribution corresponding to $\{X, \Lambda, \sigma^2\}$, but they do not require it. In particular, $\hat{f}$ can be any distribution that has for any strictly positive time evolved according to equation (103) and equation (104). The Flexibility index is defined as $\mathcal{F} \equiv \frac{\partial}{\partial \delta}\omega(0,\delta)|_{\delta=0}$, which is equivalent to the definition in equation (17) in Caballero and Engel (2007).

Proposition 12. Let Ω and ω be the jump and flow values of the IRF of prices at $t=0$. Let $X < \infty$, let Λ satisfy Assumption 1, and assume that the initial distribution f_0 satisfies Assumption 2. Then $\Omega(0) = \Omega'(\delta)|_{\delta=0} = 0$. Moreover, $\partial_\delta \omega(0,\delta)|_{\delta=0} = \infty$ and $\omega(0,0) = 0$. Thus, if $X < \infty$, the flexibility index is infinite for any Λ.

Because of this result we will move to analyze the flexibility index for models with $X = \infty$, where it is finite. We will will do so for a family of hazard functions which is a slight generalization of the one treated in Section 5.1.

F.1 Power plus family of generalized hazard functions

We consider a simple four parameter family of models where $\Lambda(x) = \Lambda(0) + \kappa x^\nu$. We label this case as power-plus, because it adds a constant to the power case. Besides $\Lambda(0)$, κ, and ν, the other parameter of the model is σ^2. We introduce the parameter η and let α be the adjusted intercept:

$$\eta = \left(\frac{2\kappa}{\sigma^2}\right)^{\frac{1}{\nu+2}} \quad , \quad \alpha = \frac{\Lambda(0)\eta^\nu}{\kappa} \quad . \tag{106}$$

The quadratic case is $\nu = 2$ and $\alpha = 0$. This adjusted intercept measures the relative magnitude of $\Lambda(0)$ and the slope κ, increasing in the former and decreasing in the latter. We will show that for a fixed power the Kurtosis, adjustment frequency, and the flexibility index only depend on α.

Proposition 13. Fix σ^2 and let $\Lambda(x)$ be a power-plus hazard function parameterized by $(\kappa, \Lambda(0), \nu)$. The adjustment frequency, the kurtosis of price changes, and the flexibility index are

$$N_a = \frac{\eta^2\sigma^2}{2}\tilde{N}(\nu,\alpha) \tag{107}$$

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{1}{\eta^2\sigma^2}\tilde{K}(\nu,\alpha) \tag{108}$$

$$\mathcal{F} = \frac{\eta^2\sigma^2}{2}(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha) \tag{109}$$

where $\tilde{N}(\nu,\alpha)$ and $\tilde{K}(\nu,\alpha)$ only depend on ν and α; $\tilde{N}(0,\alpha) \equiv 1+\alpha$, and $\tilde{K}(0,\alpha) \equiv 2/(1+\alpha)$.

With no intercept, the flexibility index and adjustment frequency are related by a simple formula via the elasticity of the hazard:

$$\mathcal{F} = N_a(1+\nu) \tag{110}$$

If two models have the same (ν, α), the density of price changes in one is a rescaling of that in the other. This implies that kurtosis (and other dimensionless statistics) is the same. If η also coincides in the two models, the distributions of price changes are identical.

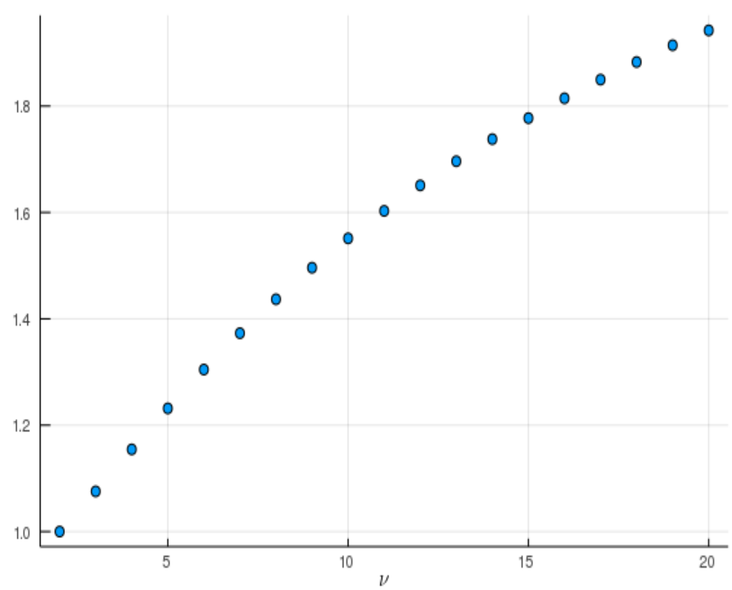
The power-plus parameterization allows us to illustrate substantial disconnect between the CIR and the flexibility index. In one example where we vary one parameter at time: in this case the flexibility index and the cumulative IRF move in the same direction. In the second example we change three parameters at a time and show how for the same flexibility index cumulative IRF can vary substantially, even keeping the adjustment frequency fixed.

Proposition 14. Assume that Λ is given by a power-plus function. Fix (ν, σ^2) and take two different power plus generalized hazard functions Λ_1 and Λ_2. If they generate the same frequency N_a, then $Kurt_1(\Delta p) > Kurt_2(\Delta p)$ if and only if $\mathcal{F}_1 < \mathcal{F}_2$.

This result is not surprising, since we are varying one parameter only. This comparative static exercise is very far away from the idea of a "sufficient statistic", where one finds a statistic that summarizes significant outputs of a class of models. Even the simple power-plus parameterization affords much more flexibility than varying one parameter can offer.

Now we turn to the second case, where we argue that, however intuitive this might be, relying on the flexibility index can be quite misleading. In Figure 9 we display a number of economies with the same adjustment frequency N_a, and with the same Flexibility Index $\mathcal{F}$, but with very different cumulative response to a monetary shock. That is, we vary the parameters in such a way that both $\mathcal{F}$ and N_a stay constant, while $\mathcal{M}'(0)$ varies substantially. This is done by increasing the power parameter ν and finding the pairs $(\Lambda(0), \kappa)$ that keep N_a and $\mathcal{F}$ constant. We solve this problem numerically and find that for the same N_a and $\mathcal{F}$ the Kurtosis of price changes varies by 90% when ν increases from 2 to 20, as plotted on the Figure 9. The slope of the impulse response at $t = 0$ does not capture the area under it quite well.

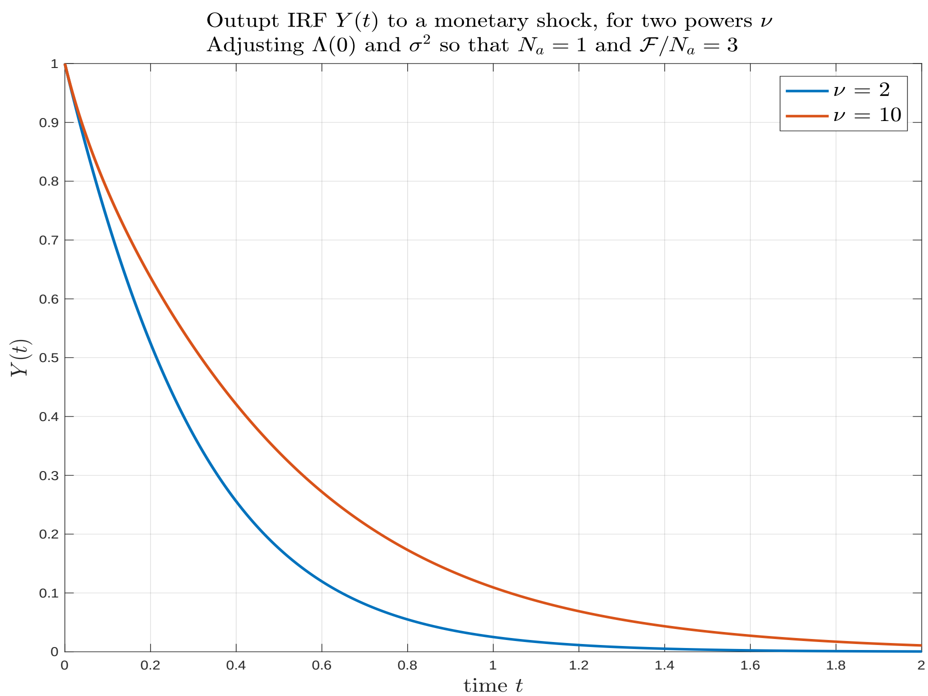
Figure 9: Values of $Kurt(\Delta p)$ or CIR relative to the case of $\nu = 2$, all cases have $\mathcal{F} = 3$, $N_a = 1$



In Figure 10 we take two examples from the previous plot, one with $\nu = 2$ and the other with

$\nu = 10$, and display the entire output impulse response function $Y(t)$ as a function of time t. Thus, both IRF's have the same frequency N_a and flexibility index $\mathcal{F}$. The areas under both IRF's are clearly different, the one for $\nu = 10$ is at least 50% larger than the one for $\nu = 2$, consistent with the values displayed on Figure 9. By construction the slope of $Y(\cdot)$ at $t = 0$ is the same for both cases (i.e. for $\nu = 2$ and $\nu = 10$), since both IRF's have the same Flexibility index $\mathcal{F}$. Yet, the slopes of both impulse responses starts to differ substantially even for low values of t. Since in both cases $N_a = 1$, the values of time in the horizontal axis can be measured in terms of expected adjustment time. For instance, if prices change on average three times a year, meaning $N_a = 3$, then $t = 1$ represents 4 months. The ratio of the two IRF evaluated at $t = 1$ is higher than 4, namely $Y_{10}(1)/Y_2(1) \approx 4.4$. This example shows that even the short run output effect can be substantially different with the same flexibility index.

Figure 10: Impulse Responses for power plus case, both cases with same $\mathcal{F}$ and N_a



Online Appendix:

The Macroeconomics of Sticky Prices with Generalized Hazard Functions

Fernando Alvarez, Francesco Lippi, and Aleksei Oskolkov

A Proofs

Proof. (of Proposition 2) To show this let the density of the invariant distribution be $\tilde{f}(z) = f(z/b)/b$. This function solves the KFE for $\tilde{\Lambda}$ and $\tilde{\sigma}^2$. This can be verified using that f solves the KFE for Λ and σ^2. Since $N_a = -\sigma^2 f'(0)$ and $\tilde{N}_a = -\tilde{\sigma}^2 \tilde{f}'(0)$ then it implies that $\tilde{N}_a = N_a$ for any b. Also we can see that $\tilde{q}(z) = q(z/b)/b$, by using $q(x) = \Lambda(x)f(x)/N_a$ and $\tilde{q}(z) = \tilde{\Lambda}(z)\tilde{f}(z)/\tilde{N}_a$ for all $z \in (-X\,b, X\,b)$. Using the formula for a change on variable, and the relationship between q and $\tilde{q}$ and of Λ and $\tilde{\Lambda}$ we get $\int_{-X}^{X} \Lambda(x)f(x)dx = \int_{-\tilde{X}}^{\tilde{X}} \tilde{\Lambda}(z)\tilde{f}(z)dz$, and thus $\tilde{s} = s$. □

Proof. (of Proposition 3) We start by describing the o.d.e and boundary that f and f_k satisfy. For f we have:

$$\Lambda(x)f(x) = \frac{\sigma^2}{2} f''(x) \text{ for all } x \in (0, X) \tag{111}$$

$$f(X) = 0 \tag{112}$$

$$1/2 = \int_0^X f(x)dx \tag{113}$$

For f_k we have

$$\Lambda(x)f_k(x) = \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (0, X) \tag{114}$$

$$kf(x) = \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (X, \infty) \tag{115}$$

$$1/2 = \int_0^X f_k(x)dx + \int_X^\infty f_k(x)dx \tag{116}$$

and that p_k has a continuous first derivative at $x = X$. We can then solve for f_k for $x > X$, obtaining $f_k(x) = f_k(X)e^{-\eta(x-X)}$ for all $x > X$, where $\eta = \sqrt{2k}/\sigma$. Thus, using the required continuity we can write:

$$\Lambda(x)f_k(x) = \frac{\sigma^2}{2} f_k''(x) \text{ for all } x \in (0, X) \tag{117}$$

$$f_k'(X) = -\eta f_k(X) \tag{118}$$

$$1/2 = \int_0^X f_k(x)dx + f_k(X)/\eta \tag{119}$$

Now consider the solutions of the homogenous second order o.d.e. given by $\sigma^2/2f''(x) = \Lambda(x)f(x)$ for $x \in [0, X]$. Given the assumption that Λ is continuous, we know that the solution is given by linear combinations of two linearly independent functions g_1, g_2 defined $[0, X]$. This functions depend on the interval $(0, X)$, the constant $\sigma > 0$ only. Thus we can write the solution of each of the two o.d.e. above as:

$$f_k(x) = a_k g_1(x) + b_k g_2(x) \tag{120}$$
$$f(x) = a g_1(x) + b g_2(x) \tag{121}$$

for all $x \in [0, X]$. The coefficients a_k, b_k, a, b can be chosen to satisfy the two boundary conditions written for f and f_k. We can use the homogeneity of the boundary conditions and preliminary set $a_k = a = 1$, drop the boundary conditions given by the integral equation for each system, use $\bar{b}, b_k$ to solve the remaining boundary conditions at X, and then find a, a_k and rescale b, b_k to satisfy the two integral equations. To do so, let $\hat{b} = b/a$ and $\hat{b}_k = a_k/b_k$. Thus we write the remaining boundary conditions:

$$f(X) = 0 \text{ becomes } 0 = g_1(X) + \hat{b} g_2(X) \tag{122}$$
$$f_k'(X) = -\eta f_k(X) \text{ becomes } g_1'(X) + \hat{b}_k g_2'(X) = -\eta \left[g_1(X) + \hat{b}_k g_2(X) \right] \tag{123}$$

equivalently we can write:

$$\hat{b} = -\frac{g_1(X)}{g_2(X)} \text{ and } \hat{b}_k = -\frac{\eta g_1(X) + g_1'(X)}{\eta g_2(X) + g_2'(X)} \tag{124}$$

Furthermore let $I_i \equiv \int_0^X g_i(x)dx$ for $i = 1, 2$ so that we can write the remaining boundary conditions as:

$$1/2 = aI_1 + bI_2 \implies a = \frac{1}{2\left(I_1 + I_2 \hat{b}\right)}$$
$$1/2 = a_k I_1 + b_k I_2 + a_k \frac{g_1(X)}{\eta} + b_k \frac{g_2(X)}{\eta} \implies a_k = \left(I_1 + \hat{b}_k I_2 + \frac{g_1(X)}{\eta} + \hat{b}_k \frac{g_2(X)}{\eta} \right) /2$$

Note that, given the expression for η, taking $k \to \infty$ it is equivalent to take $\eta \to \infty$. Then, using L'Hopital in the second equation we obtain that $\hat{b}_k \to \hat{b}$, which them implies that $a_k \to a$ and finally $b_k \to b$. Now we can compare f_k and f to obtain:

$$\begin{aligned} |f_k(x) - f(x)| &= |(a_k - a)|g_1(x) + (b_k - b) g_2(x)| \\ &\leq |a_k - a||g_1(x)| + |b_k - b||g_2(x)| \text{ for all } x \in [0, X] \end{aligned}$$

Since g_1 and g_2 are continuous in x, then they are bounded in $[0, X]$. Thus as $k \to \infty$ we have that f_k converges uniformly to f. $\square$

Proof. (of Proposition 4) Absolute continuity of $Q(\cdot)$ follows from continuity of $f(\cdot)$ on $(-X, X)/\{0\}$ and boundedness of $\Lambda(\cdot)$ on $(-X, X)$. Symmetry of $q(\cdot)$ follows from both $f(\cdot)$ and $\Lambda(\cdot)$ being symmetric, and its continuity follows from the continuity of $f(\cdot)$.

That $Q(\cdot)$ is fully identified by all its moment requires either $X < \infty$ or the existence of its moment generating function in some neighborhood of zero when $X = \infty$. This is Theorem 2.3.11

in Casella and Berger (2002). Take the case $X = \infty$. We will show the existence of the moment generating function in a neighborhood of zero, which amounts to convergence of a series

$$\sum_{n=0}^{\infty} \frac{(ia)^n \mathbb{E}[x^n]}{n!} \tag{125}$$

for some $a > 0$. Due to symmetry, all odd moments are zero, so we will prove that the even moments grow no faster than the factorial.

Consider an even moment $\mathbb{E}[x^{2k+2}]$:

$$\mathbb{E}\left[x^{2k+2}\right] = \int_{-\infty}^{\infty} x^{2k+2} q(x)dx = \frac{2}{N_a}\int_{0}^{\infty} x^{2k+2}\Lambda(x)f(x)dx = \frac{\sigma^2}{N_a}\int_{0}^{\infty} x^{2k+2} f''(x)dx \tag{126}$$

This uses the definition of and symmetry $q(\cdot)$ and the KFE. Integrate the right-hand side by parts twice:

$$\frac{\sigma^2}{N_a}\int_{0}^{\infty} x^{2k+2} f''(x)dx = \frac{\sigma^2(2k+2)(2k+1)}{N_a}\int_{0}^{\infty} x^{2k} f(x)dx \tag{127}$$

Here we used the fact that, due to Assumption 1, $\Lambda(\cdot)$ is bounded away from zero for $x > x^H$, so the decay rate of $q(\cdot)$ is no slower than exponential. This drives the intermediate terms from integration by parts to zero.

Now we will prove that

$$\int_{0}^{\infty} x^{2k} f(x)dx \leq \xi \int_{0}^{\infty} x^{2k}\Lambda(x)f(x)dx \tag{128}$$

for some number ξ that does not depend on k. Two cases are interesting. First is when there is a number $\lambda_1 > 0$ such that $\Lambda(x) > \lambda$ with probability one with respect to the measure defined by $f(\cdot)$. In this case,

$$\int_{0}^{\infty} x^{2k} f(x)dx \int_{0}^{\infty} \frac{1}{\Lambda(x)} x^{2k}\Lambda(x)f(x)dx < \frac{1}{\lambda}\int_{0}^{\infty} x^{2k}\Lambda(x)f(x)dx \tag{129}$$

and we are done. Now assume, on the contrary, for any positive number λ there is a positive measure (corresponding to $f(\cdot)$) of x such that $\Lambda(x) < \lambda$. Recall that, by Assumption 1, there exist $x^H > 0$ and $\lambda > 0$ such that $\Lambda(x) > \lambda$ for $x > x^H$. The there exists a pair of numbers (λ_2, x_2) with and two sets A_1 and A_2 such that $A_1 = \{x : \Lambda(x) < \lambda_2\}$, $A_2 = [x_2, \infty)$, the measures of A_1 and A_2 associated with $f(\cdot)$ are equal to $F > 0$, and

$$\int_{A_1}\left(F\Lambda(x) - \int_{A_1\cup A_2}\Lambda(x)f(x)dx\right)f(x)dx = -\int_{A_2}\left(F\Lambda(x) - \int_{A_1\cup A_2}\Lambda(x)f(x)dx\right)f(x)dx \tag{130}$$

To see why these sets exist, take first $x_2 = x^H$. If there is no $\lambda_2 < \lambda$ such that the measure of

$\{x : \Lambda(x) < \lambda_2\}$ is equal to $[x_2, \infty)$, increase x_2 until there is. Since $X = \infty$, the measure of $[x_2, \infty)$ decreases continuously as x_2 increases, so for any $\lambda_1 < \lambda$ the value of $x_2 \geq x^H$ such that the measures of A_2 and A_1 are equal exists.

Now consider the difference

$$F \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \int\limits_{A_1 \cup A_2} x^{2k}f(x)dx$$
$$= \int\limits_{A_1 \cup A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) x^{2k}f(x)dx$$
$$= \int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) x^{2k}f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) x^{2k}f(x)dx$$

Consider the last line. We know from equation (130) that the expression in brackets under the first integral is negative, and that under the second integral is positive. This is because they are the sum to zero, and $\Lambda(x)$ is greater on A_2 then on A_1. We also know that $x \leq x^H$ on A_1 and $x \geq x^H$ on A_2. Hence,

$$\begin{aligned} &\int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) x^{2k}f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) x^{2k}f(x)dx \\ &\geq \left(x^H\right)^{2k} \left[\int\limits_{A_1} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) f(x)dx + \int\limits_{A_2} \left(F\Lambda(x) - \int\limits_{A_1 \cup A_2} \Lambda(x)f(x)dx \right) f(x)dx \right] \\ &= 0 \end{aligned} \tag{131}$$

This insures

$$\int\limits_{A_1 \cup A_2} x^{2k}f(x)dx \leq \frac{F}{\int_{A_1 \cup A_2} \Lambda(x)f(x)dx} \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx = \xi_1 \int\limits_{A_1 \cup A_2} x^{2k}\Lambda(x)f(x)dx \tag{132}$$

At the same time,

$$\int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}f(x)dx \leq \frac{1}{\lambda_2} \int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}\Lambda(x)f(x)dx = \xi_2 \int\limits_{\mathbb{R}_+/\{A_1 \cup A_2\}} x^{2k}\Lambda(x)f(x)dx \tag{133}$$

Hence,

$$\int\limits_0^\infty x^{2k}f(x)dx \leq \max\{\xi_1, \xi_2\} \int\limits_0^\infty x^{2k}\Lambda(x)f(x)dx = \max\{\xi_1, \xi_2\}\mathbb{E}\left[x^{2k}\right] \tag{134}$$

Pluggin this to what was obtained before,

$$\mathbb{E}\left[x^{2k+2}\right] \leq \frac{\sigma^2(2k+2)(2k+1)\max\{\xi_1,\xi_2\}}{N_a}\mathbb{E}\left[x^{2k}\right] \tag{135}$$

This implies that the series in question converges, and thus the moment generating function exists, at least in the circle of the radius $\sqrt{N_a/(\sigma^2\max\{\xi_1,\xi_2\})}$. $\square$

Proof. (of Proposition 6) Start with $Q(x)$:

$$Q(x) = \mathbb{P}\{\Delta p_{it} \leq x\} = \int_0^\infty \mathbb{P}\left\{\Delta\tilde{p}_t \leq \frac{x}{b_i}\Big| b_i\right\} dH(b_i) = \int_0^\infty \mathbb{P}\left\{\Delta\tilde{p}_t \leq \frac{x}{b_i}\right\} dH(b_i) \tag{136}$$

The last equality uses the mutual independence of $\Delta\tilde{p}_t$ and b_i. Differentiate with respect to x:

$$q(x) = \partial_x\mathbb{P}\{\Delta p_{it} \leq x\} = \int_0^\infty \frac{1}{b_i}\partial_x\mathbb{P}\left\{\Delta\tilde{p}_t \leq \frac{x}{b_i}\right\} dH(b_i) \tag{137}$$

Evaluate at $x = 0$:

$$q(0) = \int_0^\infty \frac{1}{b_i}\tilde{q}(0)dH(b_i) = \mathbb{E}[b_i^{-1}]\ \tilde{q}(0) \tag{138}$$

Now turn to $\mathcal{C}_{pooled}$:

$$\mathcal{C}_{pooled} = \frac{q(0)}{2}\frac{Var(\Delta p_{it})}{\mathbb{E}[|\Delta p_{it}|]} = \frac{\tilde{q}(0)}{2}\frac{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}{\mathbb{E}[b_i]}\frac{Var(\Delta\tilde{p}_t)}{\mathbb{E}[|\Delta\tilde{p}_t|]} = \mathcal{C}\ \frac{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}{\mathbb{E}[b_i]} \tag{139}$$

Hence,

$$\mathcal{C} = \mathcal{C}\ \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \mathcal{C}_{pooled}\left(1 + \frac{Cov(b_i^{-1}, b_i^2)}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\right) < \mathcal{C}_{pooled} \tag{140}$$

That the correction multiplier is smaller then one follows from the correlation between $1/b_i$ and b_i^2 being negative. Next we find the expression for the correction as a function of the moments:

$$\frac{\mathbb{E}[|\Delta p_{it}|]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|]} = \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\frac{\mathbb{E}[|\Delta\tilde{p}_t|]}{\mathbb{E}[|\Delta\tilde{p}_t|^{-1}]\mathbb{E}[|\Delta\tilde{p}_t|^2]} = \frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]}\frac{\mathbb{E}[|\Delta p_{it}|]}{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]} \tag{141}$$

Hence,

$$\frac{\mathbb{E}[b_i]}{\mathbb{E}[b_i^{-1}]\mathbb{E}[b_i^2]} = \frac{\mathbb{E}[|\Delta p_{it}|^{-1}|\Delta p_{is}|^2]}{\mathbb{E}[|\Delta p_{it}|^{-1}]\mathbb{E}[|\Delta p_{it}|]} \tag{142}$$

This completes the proof. $\square$

Proof. (of Lemma 2) Denote $S^n(t) \equiv \frac{\partial^n}{\partial t^n} S(t)$. We will derive the following recursion:

$$S^{(n)}(t) = \mathbb{E}\left[F_n(x(t))e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \text{ for all } t \geq 0 \text{ and all } n = 1, 2, \ldots \tag{143}$$

for a sequence of functions $F_n : \mathbb{R} \to \mathbb{R}$. For $n = 1$ it follows from differentiating equation (97) with respect to t:

$$S^{(1)}(t) = -\mathbb{E}\left[\Lambda(x(t))e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right] \tag{144}$$

thus $F_1(x) = -\Lambda(x)$. For the induction step, assume that equation (143) hold and we will differentiate it with respect to t. To do this, since $F_n(x(t))$ is an Ito's process, and thus not differentiable with respect to time, we use Ito's lemma for the product of two Ito's process, namely $F_n(x(t))$ and $Z(t) \equiv e^{-\int_0^t \Lambda(x(s))ds}$, the second one being a degenerate one, since it has bounded variation. We then use that $dF_n(x(t)) = \partial_{xx} F_n(x(t))\frac{\sigma^2}{2}dt + \partial_x F_n(x(t)\sigma dW$, since x has no drift, and $dZ(t) = -\Lambda(x(t))Z(t)dt$. Thus,

$$\begin{aligned}
S^{(n+1)}(t) &\equiv \lim_{\Delta \downarrow 0} \frac{S^{(n)}(t+\Delta) - S^{(n)}(t)}{\Delta} \\
&= \lim_{\Delta \downarrow 0} \frac{1}{\Delta}\mathbb{E}\left[F_n(x(t+\Delta))Z(t+\Delta) - F_n(x(t))Z(t) \,|\, x(0) = 0\right] \\
&= \mathbb{E}\left[\left(\frac{\sigma^2}{2}\partial_{xx}F_n(x(t)) - \Lambda(x(t))F_n(x(t))\right) Z(t) \,|\, x(0) = 0\right] \\
&= \mathbb{E}\left[\left(\frac{\sigma^2}{2}\partial_{xx}F_n(x(t)) - \Lambda(x(t))F_n(x(t))\right) e^{-\int_0^t \Lambda(x(s))ds} \,|\, x(0) = 0\right]
\end{aligned}$$

which give us a recursion for F_n:

$$F_{n+1}(x) = \frac{\sigma^2}{2}\partial_{xx}F_n(x) - \Lambda(x)F_n(x) \text{ for all } x \tag{145}$$

Finally, evaluating the n^{th} derivatives of S at $t = 0$ we have:

$$S^{(n)}(0) = F_n(0) \text{ and all } n = 1, 2, \ldots \tag{146}$$

This completes the proof. □

Proof. (of Proposition 7) In the text. □

Proof. (of Corollary 2) Fix X and let $\Lambda_1(x) \equiv \lambda_1$ on $(0, X)$ correspond to the Calvo$^+$ model. The other hazard function, Λ_2, is at least somewhere strictly increasing. We claim it cannot be that $\Lambda_2(x) \geq \lambda_1$ for all x. Assume toward a contradiction that this is the case.

Then it cannot be that the graph of $f_2(x) - f_1(x)$ crosses the $x-$axis from below on $(0, X)$. If it does, there is a segment $[a, b]$ such that $f_2(x) - f_1(x)$ is positive to the right of b. But then the graph of $f_2(x) - f_1(x)$ never crosses the $x-$axis on $(b, X]$ again, because if it did cross it at some $d > b$, we would have $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (b, d), implying $f_2''(x) > f_1''(x)$ on (b, d), which contradicts $f_2'(b) \geq f_1'(b)$ and $f_2'(d) \leq f_1'(d)$ holding simultaneously. But we know that $f_1(X) = f_2(X) = 0$, which yields a contradiction.

Neither can it be that the graph of $f_2(x) - f_1(x)$ crosses the $x-$axis from above on $(0, X)$. If it does, there is a segment $[a, b]$ such that $f_2(x) - f_1(x)$ is positive to the left of a. But then the graph of $f_2(x) - f_1(x)$ never crosses the $x-$axis on $[0, a)$ again, because if it did cross it at some $c < a$, we would have $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (c, a), implying $f_2''(x) > f_1''(x)$ on (c, a), which contradicts $f_2'(a) \leq f_1'(a)$ and $f_2'(c) \geq f_1'(c)$ holding simultaneously. Hence, $\Lambda_2(x)f_2(x) > \Lambda_1(x)f_1(x)$ on (c, a), implying $f_2''(x) > f_1''(x)$ on (c, a). But together with $f_2'(a) \leq f_1'(a)$ this contradicts $f_1'(0) = f_2'(0)$.

Hence, the graph of $f_2(x) - f_1(x)$ does not cross the $x-$axis from above or below on $(0, X)$. But Λ_2 is not identically equal to λ_1, so f_2 cannot coincide with f_1 everywhere. This yields the contradiction. Now we know that $\Lambda_2(x) < \lambda_1$ for some x. Since Λ_2 is non-decreasing, the conditions of Proposition 8 are satisfy, and Λ_1 generates a higher kurtosis of price changes. This completes the proof. □

Proof. (of Corollary 3) Let $X_1 > X_2$ and let Λ_1 and Λ_2 be constants λ_1 and λ_2 on their intervals. We claim that $\lambda_1 > \lambda_2$. Assume toward the contradiction $\lambda_1 \leq \lambda_2$. We know that the graph of the function $f_1(x) - f_2(x)$ must cross the $x-$axis from below at some point, because $f_1(X_2) > 0$, $f_2(X_2) = 0$, and both f_1 and f_2 integrate to one. Hence, there is a point a such that $f_1(x) < f_2(x)$ to the left of a. Then the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis on $(0, a)$ again, since if it did there would be a point $c < a$ such that on (c, a) we have $f_1(x) < f_2(x)$ and hence $\Lambda 1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$ everywhere on (c, a). The latter contradicts $f_1'(a) \geq f_2'(a)$ and $f_1'(c) \leq f_2'(c)$ holding simultaneously.

But that the graph of $f_1(x) - f_2(x)$ never crosses the $x-$axis on $(0, a)$ again means that $f_1(x) < f_2(x)$ and hence $\Lambda 1(x)f_1(x) < \Lambda_2(x)f_2(x)$, implying $f_1''(x) < f_2''(x)$ everywhere on $(0, a)$. Together with $f_1'(a) \geq f_2'(a)$ this contradicts $f_1'(0) = f_2'(0)$. Hence, $\lambda_1 > \lambda_2$. The pair Λ_1 and Λ_2 thus qualify for the Proposition 8, and Λ_1 generates a higher kurtosis of price changes. Hence, within the space of constant hazard functions with barriers higher X generate higher Kurtoses. By Proposition 3, the kurtosis for $X = \infty$ is the limit of any sequence generated by constant hazard functions with $X_k \to \infty$. Without loss of generality, the sequence can be constructed as monotone, so the kurtosis for $X = \infty$ is higher then any its element. But the kurtosis for an arbitrary Λ is majorized by that corresponding to a constant $\tilde{\Lambda}$ with the same barrier. Hence, the kurtosis for a constant Λ and $X = \infty$ is the highest possible one. This completes the proof. □

Proof. (of Corollary 4) If the two hazard functions have the same curvature $k(x)$, it means that

$$\Lambda_1(x) = \Lambda_1(0) + \Lambda_1'(0) \int_0^x e^{\int_0^z \frac{k(w)}{w} dw} dz \tag{147}$$

$$\Lambda_2(x) = \Lambda_2(0) + \Lambda_2'(0) \int_0^x e^{\int_0^z \frac{k(w)}{w} dw} dz \tag{148}$$

We have $\mathcal{C}_1 > \mathcal{C}_2$ if and only if $\Lambda_1(0) > \Lambda_2(0)$. Using the same method as in the proof of Corollary 2, we can show that, since the frequency of adjustment is the same, there exists a $z < X$ such that $\Lambda_1(z) < \Lambda_2(z)$. Hence, $\Lambda_1'(0) < \Lambda_2'(0)$, and $\Lambda_1(x) - \Lambda_2(x)$ is a decreasing function. The graphs of $\Lambda_1(\cdot)$ and $\Lambda_2(\cdot)$ thus only cross once, so they qualify for Proposition 8, and $Kurt_1(\Delta p) > Kurt_2(\Delta p)$. □

Proof. (of Proposition 9) Fix $\nu \geq 0$. In Lemma 3, we know that s increases in ρ, so it is sufficient to show that $Kurt(\Delta p)$ also does. For this purpose, take some $\rho_1 = 2\kappa_1 X_1^2/\sigma_1^2$. They generate

$f_1(\cdot)$ with

$$\kappa_1 \left(\frac{x}{X_1}\right)^\nu f_1(x) = \frac{\sigma_1^2}{2} f_1''(x) \tag{149}$$

Now we want to increase ρ_1 to some $\rho_2 > \rho_1$. This can induce multiple $f_2(\cdot)$, since the distribution of price gaps also depends on X and σ^2. But the kurtosis of price changes only depends on ρ, so it suffices to show that one of the densities $f_2(\cdot)$ corresponding to ρ_2 generates a higher $Kurt(\Delta p)$. Let the new ρ_2 and the density $f_2(\cdot)$ be such that $\sigma_1^2 = \sigma_2^2$ and $f_1'(0) = f_2'(0)$. To compare the Kurtosis in this case it is enough to evaluate the sign of

$$\int_0^{X_2} f_2(x)x^2dx - \int_0^{X_1} f_1(x)x^2dx = \int_0^{X_2} (f_2(x) - f_1(x))x^2dx \tag{150}$$

First, from the proof of Lemma 3 we know that $\hat{p}_2'(0) < \hat{p}_1'(0)$, which implies $X_2 > X_1$ because $f_2'(0) = f_1'(0)$. This, in turn, implies that $f_2(x) - f_1(x)$ is positive on (a, X_2) for some $a < X_1$. Since $f_1(\cdot)$ and $f_2(\cdot)$ integrate to the same number over their supports, there must be a crossing b, to the left of which $f_1(x) > f_2(x)$. At this crossing, $f_1'(b) \geq f_2'(b)$. Now we will argue that there is no other crossing $c < b$.

Suppose, by way of contradiction, such a crossing exists. We have $f_2'(c) \leq f_1'(c)$ Subtract one Kolmogorov forward equations from the other:

$$x^\nu \left[\frac{\kappa_2}{X_2^\nu} f_2(x) - \frac{\kappa_1}{X_1^\nu} f_1(x)\right] = \frac{\sigma^2}{2}[f_2(x) - f_1(x)]'' \tag{151}$$

Now there are two options: $\kappa_2/X_2^\nu \geq \kappa_1/X_1^\nu$ or $\kappa_2/X_2^\nu < \kappa_1/X_1^\nu$. In the first case, since $f_2'(c) \leq f_1'(c)$ and $f_2(x) > f_1(x)$ to the left of c, from equation (151) we can conclude that $f_2''(x) > f_1''(x)$ for $x < c$, and hence $f_2'(x) - f_1'(x)$ only increases as x decreases. But this contradicts $f_1'(0) = f_2'(0)$. In the second case, since $f_2'(c) \leq f_1'(c)$ and $f_2(x) < f_1(x)$ to the right of c, from equation (151) we can conclude that $f_2''(x) < f_1''(x)$ for $x > c$, and hence $f_2'(x) - f_1'(x)$ only decreases as x decreases. But this contradicts $f_2'(b) > f_1'(b)$. Hence, there is no crossing to the left of b.

This means that $f_2(x) - f_1(x)$ is negative on $[0, b)$ and positive on (b, X_2). Since it integrates to zero over this whole interval, its integral with any positive increasing function (such as x^2) is positive. Hence, the kurtosis is higher for $\rho_2 > \rho_1$. □

Proof. (of Lemma 3) By the definition of $\hat{f}(\cdot)$, we have

$$\hat{f}(z) = Xf(zX) \tag{152}$$
$$\hat{f}'(z) = X^2 f'(zX) \tag{153}$$

The function $\hat{f}(\cdot)$ itself is derived from

$$\rho\hat{\Lambda}(z)\hat{f}(z) = \hat{f}''(z) \text{ with } \hat{f}(1) = 0 \text{ and } \int_0^1 \hat{f}(z)dz = \frac{1}{2} \tag{154}$$

Computing the Kurtosis,

$$Kurt(\Delta p) = \frac{12N_a}{\sigma^2}\int_0^X f(x)x^2dx = -12f'(0)\int_0^X f(x)x^2dx = -12\hat{f}'(0)\int_0^1 \hat{f}(z)z^2dz \tag{155}$$

Since $\hat{f}(\cdot)$ is completely determined by ρ and $\hat{\Lambda}(\cdot)$, this quantity does not depend on other parameters. The share of adjustment between the boundaries is

$$s = 1 - \frac{f'(X)}{f'(0)} = 1 - \frac{\hat{f}'(1)}{\hat{f}'(0)} \tag{156}$$

It also only depends on $\hat{\Lambda}(\cdot)$ and ρ. The frequency of price changes is given by

$$N_a = -\,\sigma^2 f'(0) = -\frac{\sigma^2}{X^2}\hat{f}'(0) \tag{157}$$

From this we have $\hat{n}(\rho) = -\hat{f}'(0)$, so $\hat{n}(\rho)$ only depends on $\hat{\Lambda}$ and ρ. In the case when $\rho = 0$ the Kolmogorov forward equation is solved by a linear $\hat{f}(\cdot)$, and the slope is -1 from the boundary condition and the normalization. Hence, $\hat{n}(0) = 1$. Now take the other statistic:

$$\frac{Kurt(\Delta p)}{6N_a} = \frac{2X^2}{\sigma^2}\int_0^1 \hat{f}(z)z^2dz = \frac{X^2}{6\sigma^2}\hat{m}(\rho) \tag{158}$$

Here the function $\hat{m}(\rho)$ is twelve times the integral of $\hat{f}(z)z^2$ which only depends on $\hat{\Lambda}(\cdot)$ and ρ. In the case when $\rho = 0$ we have $\hat{f}(z) = 1 - z$ for $z \in [0, 1]$ and hence $\hat{m}(0) = 1$.

Now fix the shape $\hat{\Lambda}(\cdot)$. Consider two different values of ρ: $\rho_1 > \rho_2$. They generate two distributions $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$. Taking the difference between the Kolmogorov forward equations that define them,

$$\hat{\Lambda}(z)(\rho_1\hat{f}_1(z) - \rho_2\hat{f}_2(z)) = (\hat{f}_1(z) - \hat{f}_2(z))'' \tag{159}$$

It holds that $\hat{f}_1(1) = \hat{f}_2(1)$, so there must be another point $y \in (0, 1)$ where $\hat{f}_1(y) = \hat{f}_2(y)$, because $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$ integrate to the same number. Moreover, this point must be a crossing, meaning that $\hat{f}_1(z) - \hat{f}_2(z)$ has different signs on to the left and to the right of it. Suppose $\hat{f}_1(z) - \hat{f}_2(z)$ is positive to the right of y. This means $\hat{f}_1'(y) - \hat{f}_2'(y) \geq 0$. But then to the right of y it holds that $\hat{f}_1'(z) - \hat{f}_2'(z) > 0$, since the left-hand side of equation (156) is positive. Hence, the difference between $\hat{f}_1(\cdot)$ and $\hat{f}_2(\cdot)$ only increases to the right of y, and they cannot cross again at $z = 1 > y$. This is a contradiction. The crossing is therefore such that $\hat{f}_1'(y) - \hat{f}_2'(y) \leq 0$. But then to the left of y it holds that $\hat{f}_1'(z) - \hat{f}_2'(z) < 0$, since the right-hand side of equation (156) is positive in this region. The difference between $\hat{f}_1(z)$ and $\hat{f}_2(z)$ increases as z decreases, as does he difference between $\hat{f}_1'(z)$ and $\hat{f}_2'(z)$. Hence, the crossing is unique and $\hat{f}_1'(0) < \hat{f}_2'(0)$. Moreover, $\hat{f}_1(z) - \hat{f}_2(z) > 0$ for $z \in [0, y)$ and $\hat{f}_1(z) - \hat{f}_2(z) < 0$ for $z \in (y, 1)$. From the latter fact together with $\hat{f}_1(1) = \hat{f}_2(1)$ it follows that $\hat{f}_1'(1) > \hat{f}_2'(1)$. To summarize:

- there is a unique $y \in (0, 1)$ such that $\hat{f}_1(z) - \hat{f}_2(z) > 0$ for $z \in [0, y)$ and $\hat{f}_1(z) - \hat{f}_2(z) < 0$ for $z \in (y, 1)$;
- $\hat{f}_1'(0) < \hat{f}_2'(0)$

- $\hat{f}_1'(1) > \hat{f}_2'(1)$

From the first bulletpoint it follows that $\hat{m}(\cdot)$ decreases in ρ. This is because $\hat{f}_1(\cdot) - \hat{f}_2(\cdot)$ integrates to zero over $(0,1)$. Since it is positive until some z and negative afterwards, its integral with increasing positive functions (such as z^2) is always negative. From the second bulletpoint it follows that $\hat{n}(\cdot)$ increases in ρ, because $\hat{n}(\rho_i) = -\hat{f}_i'(0)$. From the second and the third bulletpoints combined it follows that s increases in ρ, because $\hat{f}'(1)$ and $\hat{f}'(0)$ are both negative, so their ratio decreases with ρ. This completes the proof. $\square$

Proof. (of Proposition 12) Differentiating Ω

$$\Omega'(\delta) = X\, f(-X+\delta) + \int_{-X}^{-X+\delta} f(x)\, dx$$

taking $\delta \to 0$, since the invariant distribution satisfies $f(-X) = 0$, we have $\Omega'(0) = 0$.

Now we seek to characterize $\lim_{t \downarrow 0} \omega_\delta(t;\delta)$. We will show that $\lim_{t \downarrow 0} \omega_\delta(t;0) = \infty$ if $X < \infty$.

For this case we replace the initial condition by $f(x+\delta)$ by $f(x) + f'(x)\delta$ where f is the density of the invariant distribution. We can ommitt the contribution from the term $f(x)$, since it is equal to zero by virtue of being the invariant distribution.

The KFE gives the following properties:

1. For all $t > 0$, since $-X$ is an exit point, $f(-X,t) = 0$.
2. For all $t > 0$, there exists $\underline{x}(t) > -X$, so that $f(x,t) < f(x,0) = f'(x)\delta > 0$ for all $x \in [-X, \underline{x}(t)]$. This follows because $f(x,t)$ is differentiable in x and $f(-X,t) = 0$.
3. For all $x \in (-X, 0)$ we have: $f(x,t) \to f(x,0)$ as $t \downarrow 0$. This follows since $f(x,t)$ is differentiable in time t for all x.

From these properties we obtain that $f'(-X,t) \to \infty$ as $t \downarrow 0$. Hence, $\omega_\delta(0,0) = \infty$. $\square$

Proof. (of Proposition 13) The frequency of adjustment is given by

$$\begin{aligned} N_a &= \int_{-\infty}^{\infty} f(x)(\Lambda(0) + \kappa x^\nu) dx = \int_{-\infty}^{\infty} \tilde{f}(z) \left(\Lambda(0) + \kappa \left(\frac{z}{\eta}\right)^\nu \right) dz \\ &= \frac{\kappa}{\eta^\nu} \int_{-\infty}^{\infty} p(z)(\alpha + z^\nu) dz = \frac{\kappa}{\eta^\nu} \tilde{N}(\nu,\alpha) = \frac{\beta^2\eta^2}{2} \tilde{N}(\nu,\alpha) \end{aligned} \tag{160}$$

The flexibility index is

$$\begin{aligned} \mathcal{F} &= -\int_{-\infty}^{\infty} x(\Lambda(0) + \kappa x^\nu) f'(x) dx = -\int_{-\infty}^{\infty} z \left(\Lambda(0) + \kappa \left(\frac{z}{\eta}\right)^\nu \right) p'(z) dz \\ &= -\frac{\kappa}{\eta^\nu} \int_{-\infty}^{\infty} z(\alpha + z^\nu) \tilde{f}'(z) dz = \frac{\kappa}{\eta^\nu} \left(\int_{-\infty}^{\infty} \tilde{f}(z)(\alpha + z^\nu) dz + \nu \int_{-\infty}^{\infty} p(z) z^\nu dz \right) && (161) \\ &= \frac{\kappa}{\eta^\nu} \left(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha \right) = \frac{\beta^2\eta^2}{2} \left(\tilde{N}(\nu,\alpha)(1+\nu) - \nu\alpha \right) && (162) \end{aligned}$$

The distribution of price changes is given by

$$q(x) = \frac{f(x)(\Lambda(0) + \kappa x^\nu)}{N_a} = \frac{\eta\tilde{(}\eta x)(\alpha + (\eta x)^\nu)}{\tilde{N}(\nu,\alpha)} \tag{163}$$

To compute the kurtosis, we need the fourth moment and the variance:

$$\mathbb{E}[\Delta p^4] = \int_{-\infty}^{\infty} x^4 q(x)dx = \frac{1}{\eta^4 N(\nu,\alpha)} \int_{-\infty}^{\infty} z^4 p(z)(\alpha + z^\nu)dz \tag{164}$$

$$\mathbb{E}[\Delta p^2] = \int_{-\infty}^{\infty} x^2 q(x)dx = \frac{1}{\eta^2 N(\nu,\alpha)} \int_{-\infty}^{\infty} z^2 p(z)(\alpha + z^\nu)dz \tag{165}$$

These expressions imply that $\mathbb{E}[\Delta p^4]/\mathbb{E}[\Delta p^2]^2$ only depends on (ν, α). □

Proof. (of Proposition 14) Let $f_1(x)$ and $f_2(x)$ be the price gap distributions generated by $\Lambda_1(x)$ and $\Lambda_2(x)$. Assume without loss that $\kappa_1 < \kappa_2$. We will first prove that $\Lambda_1(0) > \Lambda_2(0)$ whenever N_a is the same in the two models. That $Kurt_1(\Delta p) > Kurt_2(\Delta p)$ will then follow from Proposition 8. Finally, we will show that $\mathcal{F}_1 < \mathcal{F}_2$.

(1) Suppose by contradiction that $\Lambda_1(0) \leq \Lambda_2(0)$. Then, $\Lambda_1(x) < \Lambda_2(x)$ for all $x > 0$. Since N_a and σ^2 are the same in the two models, we know that $f_1'(0) = f_2'(0)$.

Suppose there is a point $a > 0$ at which the graph of $f_1(x)$ crosses that of $f_2(x)$ from below. That is, $f_1(a) = f_2(a)$ and $f_1(x) < f_2(x)$ to the left of a. Then the graphs of $f_1(x)$ and $f_2(x)$ never cross again to the left of a. If they did cross at some point $b < a$, we would have $f_1'(a) \geq f_2'(a)$ and $f_1'(b) \leq f_2'(b)$, so that $f_1'(a) - f_1'(b) \geq f_2'(a) - f_2'(b)$, but this is impossible, since $f_1(x) < f_2(x)$ and $\Lambda_1(x) < \Lambda_2(x)$ on (a, b), while $\sigma^2 f_i''(x)/2 = \Lambda_i(x) f_i(x)$ for $i \in \{1, 2\}$. Hence, $f_1(x) < f_2(x)$ for all $x < a$, which contradicts $f_1'(0) = f_2'(0)$ for the same reason.

Suppose there is a point $c > 0$ at which the graph of $f_1(x)$ crosses that of $f_2(x)$ from above. That is, $f_1(c) = f_2(c)$ and $f_1(x) < f_2(x)$ to the right of c. Then the graphs of $f_1(x)$ and $f_2(x)$ never cross again to the right of c. If they did cross at some point $d > c$, we would have $f_1'(d) \geq f_2'(d)$ and $f_1'(c) \leq f_2'(c)$, so that $f_1'(d) - f_1'(c) \geq f_2'(d) - f_2'(c)$, but this is impossible, since $f_1(x) < f_2(x)$ and $\Lambda_1(x) < \Lambda_2(x)$ on (c, d), while $\sigma^2 f_i''(x)/2 = \Lambda_i(x) f_i(x)$ for $i \in \{1, 2\}$. Hence, $f_1(x) < f_2(x)$ for all $x > c$, which contradicts $f_1'(x) - f_2'(x) \longrightarrow 0$ as $x \longrightarrow \infty$ for the same reason.

By what was said above, the graphs of $f_1(x)$ and $f_2(x)$ cannot cross, but they must, since these functions integrate to the same number and have the same limit at infinity. Hence, $\Lambda_1(0) \leq \Lambda_2(0)$ is impossible when σ^2 and N_a are the same in the two models.

(2) Now since $\kappa_1 < \kappa_2$ and $\Lambda_1(0) > \Lambda_2(0)$, the two generalized hazard functions $\Lambda_1(x)$ and $\Lambda_2(x)$ satisfy the conditions of Proposition 8. From this it follows that $Kurt_1(\Delta p) > Kurt_2(\Delta p)$.

(3) The flexibility index for the power-plus case is given by

$$\mathcal{F} = \int_{-\infty}^{\infty} f(x)(\Lambda(x) + \Lambda'(x)x)dx = (1 + \nu)N_a - \nu\Lambda(0) \tag{166}$$

Since the two models deliver the same N_a and ν is fixed, the one with a greater intercept has a smaller $\mathcal{F}$. This completes the proof. □

Proof. (of Proposition 11). We will make two observations, one about Λ and one about F_n required to establish the two main results of the proposition. Then we will use Lemma 2 finish the proof.

The first observation is that the symmetry of Λ around $x = 0$ implies that all the odd numbered derivatives evaluated at $x = 0$ of Λ are equal to zero.

The second observation is a property of the function $F_n(x)$ generated by the recursion in equation (145), which can be written as:

$$F_n(x) = \tilde{F}_n(x) - \left(\frac{\sigma^2}{2}\right)^{n-1} \frac{\partial^{2n-2}\Lambda(x)}{\partial x^{2n-2}}$$

where $\tilde{F}_n(\cdot)$ depends only on the level of $\Lambda(\cdot)$ and at most the first $2n-1$ derivatives of $\Lambda(\cdot)$, evaluated at x. This property can be established by induction. It is true for $F_1(x) = -\Lambda(x)$ for $n = 1$. Now assume it holds for n, and we will show that it holds $n+1$. To do so we compute F_{n+1} according to the recursion. On this computation, the first term is the product of $\sigma^2/2$ times the sum of the second derivative of $\tilde{F}_n(x)$ with respect to x and of the second derivative of $-\left(\sigma^2/2\right)^{n-1} \partial^{2n-2}\Lambda(x)/\partial x^{2n-2}$ with respect to x. The remaining term, $-\Lambda(x)F_n(x)$, involves no derivatives. This finishes the induction step, and thus established the desired result for F_n.

1. If we know the function $\Lambda(x)$, then we can recursively compute $F_n(x)$ from equation (145). Evaluating this expressions at $x = 0$ and using equation (146) we obtain all the derivatives of S evaluated at $t = 0$. In particular, these expressions only use the level and the even derivatives of Λ evaluated at $x = 0$.If S is analytical, the expansion of S at $t = 0$ gives the values everywhere.

2. If we know the function S, we can take all its derivatives at $t = 0$, and by equation (146) we know all the values of $F_n(0)$ for $n \geq 1$. Next we argue that the recursion in equation (145) evaluated at $x = 0$, will give us all the even order derivatives of Λ evaluated at $x = 0$. Since Λ is symmetric, all the derivatives of odd order, evaluated at $x = 0$, so we are only interested in the even derivatives at $x = 0$. Next we argue that, algorithmically, we can recursively recover the derivatives up to order $2n-2$ with $\{F_n(0)\} for j = 1, \ldots, 2n-2$. First we note that $\Lambda(0)$ and $\Lambda''(0)$ are given by $F_1(0)$ and $F_2(0)$. Now assume we know all the derivatives up to order $2n-2$. Then, given the value of $\partial^{n+1}S(0)/\partial t^{n+1} = F_{n+1}(0)$, the known values for $\Lambda(0)$, $F_n(0)$, and $\sigma^2/$, using the recursion we obtain the implied value for $\partial_{xx}F_n(0)$. Using that F_n depend at most on $2n-2$ derivatives of Λ, as well as the particular expression derived above, we obtain the value of $\partial^{2n}\Lambda(0)/\partial t^{2n}$. This completes the induction step, and hence establishes the desired property, and hence the level and all the derivatives of Λ at $x = -0$ have been recovered. Finally, since Λ is assumed to be analytical, an expansion around $x = 0$ gives its value at any other x.

This completes the proof. □

B Properties of Distribution of Menu Cost

In this appendix we note that the posited behavior of Λ in a neighbourhood of $x = 0$ or $x = |X|$ determines whether the underlying density g is bounded. It is shown in equation (4) that the hazard function inherits the shape of the value function because of the underlying optimization: when the firm draws a fixed cost, what matters is how the value of the draw compares to the gains from adjustment. Taking a first order derivative of equation (4) gives

$$\Lambda'(x) = \kappa \; g(v(x) - v(0)) \; v'(x) \tag{167}$$

A bounded density g would make $\Lambda'(x)$ have zero limits at $x = 0$ and $x = X$ because of the smooth-pasting conditions on $v(x)$ at these points. Thus, if the hazard function of the inverse problem (the one that solves for g given Λ) is not flat at 0 or Ψ, then the density g must be diverging. We formalize this observation next:

Corollary 5. Let $\varepsilon > 0$ and suppose $\Lambda'(x)$ is bounded away from zero for $x \in (0, \varepsilon)$. Then $g(\psi)$ is unbounded on any $(0, \psi)$. Likewise, if $\Lambda'(x)$ is bounded away from zero for $x \in (X - \varepsilon, X)$ then $g(\psi)$ is unbounded on any (ψ, Ψ).[20]

We can also characterize the behavior of the density g around $\psi = 0$ for different forms of Λ arounf $x = 0$. Take the limiting elasticity of the hazard

$$\nu = \lim_{x \downarrow 0} \frac{x\Lambda'(x)}{\Lambda(x) - \Lambda(0)} \tag{168}$$

If Λ is symmetric and smooth, it admits a quadratic approximation close to zero, and $\nu = 2$. Interestingly, deviations from $\nu = 2$ imply irregular behavior of g. Theorem 1 states that

$$g(x) = \frac{\Lambda'(x)}{\kappa u(x)} \tag{169}$$

But $u(x)$ converges to zero as $x \to 0$, so the limit is tricky. To resolve the indeterminacy, notice that $u(x)$ goes to zero linearly, since $u''(0) = 0$ (immediate from the equation (6) defining $u(x)$ in Lemma 1). Thus whether the limit is (i) zero, (ii) positive and finite, or (iii) infinite, depends respectively on whether $\Lambda'(x)$ goes to zero (i) faster than a linear rate ($\nu > 2$), (ii) at a linear rate ($\nu = 2$), (iii) slower than a linear rate ($\nu < 2$). We can formalize this:

Corollary 6. Suppose that $\Lambda'(x)$ and $g(\psi)$ both have (possibly infinite) right limits at zero. Then $\lim_{\psi \downarrow 0} g(\psi) = \infty$ for $\nu < 2$, $0 < \lim_{\psi \downarrow 0} g(\psi) < \infty$ for $\nu = 2$, and $\lim_{\psi \downarrow 0} g(\psi) = 0$ for $\nu > 2$.

This corollary states that a quadratic hazard function implies a density of ψ that is positive and finite around $\psi = 0$. If the leading term in $\Lambda(x)$ is higher than quadratic ($\nu > 2$) then the density must be zero, meaning that G is flat close to $\psi = 0$. A hazard function with a leading term $\nu < 2$ implies a distribution of ψ with density that is diverging around $\psi = 0$.

C Alternative Normalization

We consider an alternative normalization to one used in Proposition 2. This normalization requires that $X < \infty$. For a triplet $\{\sigma^2, X, \Lambda\}$ we can define a new problem represented by pair $\{\rho, \hat{\Lambda}\}$ where $\hat{\Lambda} : (-1, 1) \to \mathbb{R}_+$ and where ρ is a scalar defined as follows:

$$\hat{\Lambda}(z) = \frac{\Lambda(zX)}{\kappa} \text{ for all } z \in [-1, 1] \text{ and } \rho = \frac{2\kappa X^2}{\sigma^2} \tag{170}$$

Note that this is the normalization used in Proposition 2 with $b = 1/X$. This is a slight generalization of Proposition 2, in that it allows to have some comparative static with respect to κ.

Given the triplet $\{\sigma^2, X, \Lambda\}$ we can solve for f as indicated in equation (16). And given the

[20] Since $\Lambda(x)$ is symmetric, to be smooth at zero it has to have $\Lambda'(0) = 0$. The proof is done by standard analysis.

pair $\{\rho, \hat{\Lambda}\}$ we can solve for the probability density $\hat{f}$, using a change of variables:

$$\hat{f}(z) \equiv f(zX)\, X \text{ for all } z \in [-1, 1] \tag{171}$$

We note that $\hat{f}$ satisfies the

$$\hat{\Lambda}(z)\, \rho\, \hat{f}(z) = \hat{f}''(z) \text{ for all } z \in [-1, 1] \text{ and } z \notin \mathbb{Z} \tag{172}$$

where $z \in \mathbb{Z}$ if $z = x/X$ and $x \in \mathbb{J}$. Moreover, the density $\hat{f}$ must satisfy

$$\hat{f}(1) = \hat{f}(-1) = 0 \text{ and } \int_{-1}^{1} \hat{f}(z) dz = 1 \tag{173}$$

LEMMA 3. Consider two triplets $\{\sigma, X, \Lambda\}$ such that both generate the function $\hat{\Lambda}(\cdot)$ and the parameter ρ by using equation (170). The two triplets have the same Kurtosis of price changes $Kurt(\Delta p)$ and the same share of adjustment in the interior s. Furthermore,

$$N_a = \frac{\sigma^2}{X^2} \hat{n}(\rho) \tag{174}$$

$$\frac{Kurt(\Delta p)}{6 N_a} = \frac{X^2}{\sigma^2} \frac{\hat{m}(\rho)}{6} \tag{175}$$

$$s = \hat{s}(\rho) \tag{176}$$

where $\hat{n}(\rho)$, $\hat{m}(\rho)$ and $\hat{s}(\rho)$ only depend on $\hat{\Lambda}(\cdot)$ and ρ. Moreover, $\hat{n}(\cdot)$ is increasing in ρ, $\hat{m}(\cdot)$ is decreasing in ρ, $\hat{s}(\cdot)$ is increasing in ρ, and $\hat{n}(0) = \hat{m}(0) = \hat{s}(0) = 1$.

D Functional forms of $\langle f(x), m(x), \mathcal{T}(x) \rangle$ for integer ν

The invariant density f has to be symmetric around $x = 0$, and has to satisfy:

$$\Lambda(x) f(x) = \frac{\sigma^2}{2} f''(x) \text{ for all } x \in [0, X]\,, \tag{177}$$

$$\frac{1}{2} = \int_0^X f(x) dx \text{ and } f(X) = 0\,. \tag{178}$$

The contribution of an individual firm to the IRF is antisymmetric around $x = 0$ and satisfies the following:

$$\Lambda(x) m(x) = -x + \frac{\sigma^2}{2} m''(x) \text{ for all } x \in [0, X]\,, \tag{179}$$

$$m(0) = m(X) = 0\,. \tag{180}$$

Fianlly, $\mathcal{T}(x)$ is symmetric around $x = 0$ and satisfies

$$\Lambda(x)\mathcal{T}(x) = 1 + \frac{\sigma^2}{2}\mathcal{T}(x) \text{ for all } x \in [0, X]\,, \tag{181}$$

$$\mathcal{T}(X) = 0 \text{ and } \mathcal{T}'(0) = 0\,. \tag{182}$$

The latter equality is a consequence of $\mathcal{T}(\cdot)$ being continuously differentiable ay zero and antisymmetric.

Denote $y = \sigma^2/2a$. We will assume that the functions of interest are analytical, so we can write them as:

$$f(x) = \sum_{k=0}^{\infty} \alpha_k x^k \text{ for } x \in [0, X] \tag{183}$$

$$m(x) = \sum_{k=0}^{\infty} \beta_k x^k \text{ for } x \in [0, X] \tag{184}$$

$$\mathcal{T}(x) = \sum_{k=0}^{\infty} \gamma_k x^k \text{ for } x \in [0, X] \tag{185}$$

so that, in particular, $\gamma_0 = \mathcal{T}(0)$. Inserting these expressions into the equations above and using the functional form for $\Lambda(\cdot)$, we obtain:

$$a\sum_{k=0}^{\infty} \alpha_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \alpha_k k(k-1)x^{k-2} \qquad \text{for } x \in [0, X] \tag{186}$$

$$a\sum_{k=0}^{\infty} \beta_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \beta_k k(k-1)x^{k-2} - x \qquad \text{for } x \in [0, X] \tag{187}$$

$$a\sum_{k=0}^{\infty} \gamma_k x^{k+\nu} = \frac{\sigma^2}{2}\sum_{k=2}^{\infty} \gamma_k k(k-1)x^{k-2} + 1 \qquad \text{for } x \in [0, X] \tag{188}$$

Matching the coefficient of each of the powers of x we have

$$\alpha_k = y(k+\nu+2)(k+\nu+1)\alpha_{k+\nu+2} \text{ for } k \geq 0 \tag{189}$$

$$\beta_k = y(k+\nu+2)(k+\nu+1)\beta_{k+\nu+2} \text{ for } k \geq 0 \tag{190}$$

$$\gamma_k = y(k+\nu+2)(k+\nu+1)\gamma_{k+\nu+2} \text{ for } k \geq 0 \tag{191}$$

The symmetry and smoothness properties also lead to

$$\beta_0 = \beta_2 = \gamma_1 = 0 \tag{192}$$

Relabelling the coefficients, we can write the sums as

$$f(x) = \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} y^{-j} x^{j(\nu+2)}\right) + \alpha_1 x \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} y^{-j} x^{j(\nu+2)}\right) \tag{193}$$

$$m(x) = \beta_1 x \left(1 + \sum_{j=1}^{\infty} \xi_{m,j} y^{-j} x^{j(\nu+2)}\right) + \beta_3 x^3 \left(1 + \sum_{j=1}^{\infty} \eta_{m,j} y^{-j} x^{j(\nu+2)}\right) \tag{194}$$

$$\mathcal{T}(x) = \gamma_0 \left(1 + \sum_{j=1}^{\infty} \xi_{t,j} y^{-j} x^{j(\nu+2)}\right) + \gamma_2 x^2 \left(1 + \sum_{j=1}^{\infty} \eta_{t,j} y^{-j} x^{j(\nu+2)}\right) \tag{195}$$

Here the coefficients $\xi_{\cdot,j}$ and $\eta_{\cdot,j}$ are given by

$$\xi_{p,j} = \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)-1)} \qquad \eta_{p,j} = \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)+1)} \tag{196}$$

$$\xi_{m,j} = \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)+1)} \qquad \eta_{m,j} = \prod_{i=1}^{j} \frac{1}{(i(\nu+2)+2)(i(\nu+2)+3)} \tag{197}$$

$$\xi_{t,j} = \prod_{i=1}^{j} \frac{1}{i(\nu+2)(i(\nu+2)-1)} \qquad \eta_{t,j} = \prod_{i=1}^{j} \frac{1}{(i(\nu+2)+1)(i(\nu+2)+2)} \tag{198}$$

$$\tag{199}$$

Now define the following parameter:

$$Z = \frac{X^{\nu+2}}{y} = 2aX^{\nu}\frac{X^2}{\sigma^2} = 2\kappa\mathcal{T}_0 \tag{200}$$

It will be useful in pinning down the coefficients. Here $\tilde{\Lambda}$ is the left limit of the hazard rate when x approaches X, and $\mathcal{T}_0$ is the expected time to adjustment when $a = 0$.

Consider first $f(\cdot)$. The boundary condition is

$$0 = f(X) = \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} y^{-j} X^{j(\nu+2)}\right) + \alpha_1 x \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} y^{-j} X^{j(\nu+2)}\right) \tag{201}$$

$$= \alpha_0 \left(1 + \sum_{j=1}^{\infty} \xi_{p,j} Z^j\right) + \alpha_1 X \left(1 + \sum_{j=1}^{\infty} \eta_{p,j} Z^j\right) \tag{202}$$

Define additionally $\xi_{\cdot,0} = \eta_{\cdot,0} = 1$. The condition that $f(\cdot)$ is a density states

$$\frac{1}{2} = \int_0^X f(x)dx = \alpha_0 X \left(1 + \sum_{j=1}^{\infty} \frac{\xi_{p,j} Z^j}{j(\nu+2)+1}\right) + \alpha_1 X^2 \left(\frac{1}{2} + \sum_{j=1}^{\infty} \frac{\eta_{p,j} Z^j}{j(\nu+2)+2}\right) \tag{203}$$

$$= \alpha_0 X \sum_{j=0}^{\infty} \frac{\xi_{p,j} Z^j}{j(\nu+2)+1} + \alpha_1 X^2 \sum_{j=0}^{\infty} \frac{\eta_{p,j} Z^j}{j(\nu+2)+2} \tag{204}$$

This leads to

$$\alpha_1 = \frac{1}{2X^2}\frac{\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)}{\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)-\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}$$
$$= \frac{1}{2X^2}\hat{\alpha}_1(\nu,Z) \tag{205}$$

$$\alpha_0 = -\frac{1}{2X}\frac{\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}{\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^{\infty}\xi_{p,j}Z^j\right)-\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^{\infty}\eta_{p,j}Z^j\right)}$$
$$= \frac{1}{2X}\hat{\alpha}_0(\nu,Z) \tag{206}$$

Now observe that the integral of $f(x)x^2$ is in fact proportional to X^2 for a fixed Z:

$$\int_0^X f(x)x^2dx = \alpha_0X^3\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+3}+\alpha_1X^4\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+4} \tag{207}$$
$$= \frac{X^2}{2}\left[\hat{\alpha}_0(\nu,Z)\sum_{j=0}^{\infty}\frac{\xi_{p,j}Z^j}{j(\nu+2)+3}+\hat{\alpha}_1(n,Z)\sum_{j=0}^{\infty}\frac{\eta_{p,j}Z^j}{j(\nu+2)+4}\right] \tag{208}$$

To determine $m(\cdot)$ and $\mathcal{T}(\cdot)$, it is useful to consider separately the cases $\nu \geq 1$ and $\nu = 0$. Start with $\nu \geq 1$. In this case, in addition to equation (192), we know that

$$3\sigma^2\beta_3 = 1 \text{ and } \sigma^2\gamma_2 = -1 \tag{209}$$

The boundary conditions are $m(X) = \mathcal{T}(X) = 0$, so

$$-\beta_1 = \frac{X^2}{3\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{m,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{m,j}Z^j}\right) \tag{210}$$
$$\gamma_0 = \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right) \tag{211}$$

The functional forms are then

$$m(x) = -\frac{xX^2}{3\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{m,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{m,j}Z^j}\right)\sum_{j=0}^{\infty}\xi_{m,j}y^{-j}x^{j(\nu+2)}+\frac{x^3}{3\sigma^2}\sum_{j=0}^{\infty}\eta_{m,j}y^{-j}x^{j(\nu+2)} \tag{212}$$
$$\mathcal{T}(x) = \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right)\sum_{j=0}^{\infty}\xi_{t,j}y^{-j}x^{j(\nu+2)}-\frac{x^2}{\sigma^2}\sum_{j=0}^{\infty}\eta_{t,j}y^{-j}x^{j(\nu+2)} \tag{213}$$

Observe that for $\mathcal{T}(0)$ we have

$$\mathcal{T}(0) = \frac{X^2}{\sigma^2}\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}Z^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}Z^j}\right) = \mathcal{T}_0\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}{1+\sum_{j=1}^{\infty}\xi_{t,j}(2\kappa\mathcal{T}_0)^j}\right) \tag{214}$$

At $a = 0$ or, equivalently, $\kappa = 0$, we have $\mathcal{T}(0) = \mathcal{T}_0$.

Now consider the case $\nu = 0$. Here the conditions we add to equation (192) are

$$a\beta_1 = 3\sigma^2\beta_3 - 1 \text{ and } a\gamma_0 = \sigma^2\gamma_2 + 1 \tag{215}$$

Plugging them into the boundary conditions $m(X) = \mathcal{T}(X) = 0$,

$$-\beta_1 = \frac{X^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j} \tag{216}$$

$$\beta_3 = \frac{\sum_{j=0}^{\infty}\xi_{m,j}Z^j}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j} \tag{217}$$

$$\gamma_0 = \frac{X^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j} \tag{218}$$

$$-\gamma_2 = \frac{\sum_{j=0}^{\infty}\xi_{t,j}Z^j}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j} \tag{219}$$

The functional forms in this case are

$$\begin{aligned} m(x) = &-x\frac{X^2\left(\sum_{j=0}^{\infty}\eta_{m,j}Z^j\right)\left(\sum_{j=0}^{\infty}\xi_{m,j}y^{-j}x^{j(\nu+2)}\right)}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j} \\ &+ x^3\frac{\left(\sum_{j=0}^{\infty}\xi_{m,j}Z^j\right)\left(\sum_{j=0}^{\infty}\eta_{m,j}y^{-j}x^{j(\nu+2)}\right)}{3\sigma^2\sum_{j=0}^{\infty}\xi_{m,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{m,j}Z^j} \end{aligned} \tag{220}$$

$$\begin{aligned} \mathcal{T}(x) = &\frac{X^2\left(\sum_{j=0}^{\infty}\eta_{t,j}Z^j\right)\left(\sum_{j=0}^{\infty}\xi_{t,j}y^{-j}x^{j(\nu+2)}\right)}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j} \\ &- x^2\frac{\left(\sum_{j=0}^{\infty}\xi_{t,j}Z^j\right)\left(\sum_{j=0}^{\infty}\eta_{t,j}y^{-j}x^{j(\nu+2)}\right)}{\sigma^2\sum_{j=0}^{\infty}\xi_{t,j}Z^j + aX^2\sum_{j=0}^{\infty}\eta_{t,j}Z^j} \end{aligned} \tag{221}$$

Observe that in this case for $\mathcal{T}(0)$ we have

$$\begin{aligned} \mathcal{T}(0) &= \frac{X^2}{\sigma^2}\left(\frac{\sum_{j=0}^{\infty}\eta_{t,j}Z^j}{\sum_{j=0}^{\infty}\xi_{t,j}Z^j + \frac{aX^2}{\sigma^2}\sum_{j=0}^{\infty}\eta_{t,j}Z^j}\right) \\ &= \mathcal{T}_0\left(\frac{1+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}{1+\kappa\mathcal{T}_0+\sum_{j=1}^{\infty}\xi_{t,j}(2\kappa\mathcal{T}_0)^j+\sum_{j=1}^{\infty}\eta_{t,j}(2\kappa\mathcal{T}_0)^j}\right) \end{aligned} \tag{222}$$

When $\kappa = 0$, we have $\mathcal{T}(0) = \mathcal{T}_0$.

We know that the adjustment frequency is given by

$$N_a = \frac{1}{\mathcal{T}(0)} \tag{223}$$

Hence, the adjustment frequency can be represented as a function of κ and $\mathcal{T}_0$. The same is true for the kurtosis of price changes. From equation (23),

$$\begin{aligned}
Kurt(\Delta p) &= \frac{2\left[\int_0^X x^4\Lambda(x)f(x)dx - X^4\frac{\sigma^2}{2}f'(X)\right]}{N_a}\frac{1}{[Var(\Delta p)]^2} \\
&= \frac{2N_a\left[\int_0^X x^4\Lambda(x)f(x)dx - X^4\frac{\sigma^2}{2}f'(X)\right]}{\sigma^4} = \frac{12N_a}{\sigma^2}\int_0^X f(x)x^2dx \\
&= 6N_a\frac{X^2}{\sigma^2}\frac{\sum_{j=0}^\infty \frac{\eta_{p,j}Z^j}{j(\nu+2)+4}\left(\sum_{j=0}^\infty \xi_{p,j}Z^j\right) - \sum_{j=0}^\infty \frac{\xi_{p,j}Z^j}{j(\nu+2)+3}\left(\sum_{j=0}^\infty \eta_{p,j}Z^j\right)}{\sum_{j=0}^\infty \frac{\eta_{p,j}Z^j}{j(\nu+2)+2}\left(\sum_{j=0}^\infty \xi_{p,j}Z^j\right) - \sum_{j=0}^\infty \frac{\xi_{p,j}Z^j}{j(\nu+2)+1}\left(\sum_{j=0}^\infty \eta_{p,j}Z^j\right)} \\
&= 6N_a\mathcal{T}_0\frac{\sum_{j=0}^\infty \varphi_{K,j}(2\kappa\mathcal{T}_0)^j}{\sum_{j=0}^\infty \chi_{K,j}(2\kappa\mathcal{T}_0)^j}
\end{aligned} \tag{224}$$

Here the coefficients $\{\varphi_{K,j}, \chi_{K,j}\}_{j=0}^\infty$ are given by

$$\varphi_{K,j} = \sum_{i=0}^{j}\left(\frac{\xi_{p,j-i}\eta_{p,i}}{i(\nu+2)+4} - \frac{\eta_{p,j-i}\xi_{p,i}}{i(\nu+2)+3}\right) \tag{225}$$

$$\chi_{K,j} = \sum_{i=0}^{j}\left(\frac{\xi_{p,j-i}\eta_{p,i}}{i(\nu+2)+2} - \frac{\eta_{p,j-i}\xi_{p,i}}{i(\nu+2)+1}\right) \tag{226}$$

As expected, when $\kappa = 0$ we have $N_a = 1/\mathcal{T}_0$ and

$$Kurt(\Delta p) = 1. \tag{227}$$

The coefficients $\{\varphi_{N,j}, \chi_{N,j}\}$ for N_a are taken from the corresponding formula for $\mathcal{T}(0)$ in the cases $\nu = 0$ and $\nu \geq 1$. In both cases $\varphi_{N,0} = \chi_{N,0} = 1$. To verify $\varphi_{K,0} = -1/12$ and $\psi_{K,0} = 1/2$, plug $\xi_{p,0} = \eta_{p,0} = 1$. For $\varphi_{K,1}$ and $\chi_{K,1}$, recall that

$$\xi_{p,1} = \frac{1}{(\nu+2)(\nu+1)} \text{ and } \eta_{p,1} = \frac{1}{(\nu+2)(\nu+3)} \tag{228}$$

The first derivative of $Kurt(\Delta p)/(6N_a)$ evaluated at $\kappa = 0$ is

$$\frac{\partial}{\partial\kappa}\left(\frac{Kurt(\Delta p)}{6N_a}\right)\Big|_{\kappa=0} = \mathcal{T}_0\frac{\chi_{K,0}\varphi_{K,1} - \varphi_{K,0}\chi_{K,1}}{\chi_{K,0}^2} = -C(6\varphi_{K,1} - \chi_{K,1}) \tag{229}$$

for some positive constant C. Plugging the terms,

$$\varphi_{K,1} = -\frac{1}{12(\nu+5)(\nu+6)} \tag{230}$$

$$\chi_{K,1} = -\frac{1}{2(\nu+3)(\nu+4)} \tag{231}$$

Hence,

$$\frac{\partial}{\partial\kappa}\left(\frac{Kurt(\Delta p)}{6N_a}\right)\Big|_{\kappa=0} = \frac{C}{2}\left(\frac{1}{(\nu+5)(\nu+6)} - \frac{1}{(\nu+3)(\nu+4)}\right) < 0 \tag{232}$$

This proves the fact that $Kurt(\Delta p)/(6N_a)$ decreases for small κ.

E Special Cases of Interest

E.1 m and f in the discrete, unbounded case

We assume that we can divide $[0,\infty)$ into N segments, each one where $\Lambda(x)$ is constant at the value $\rho_k > 0$ and with thresholds $\{\bar{x}_k\}_{k=0}^N$ as follows. The values of $\{\bar{x}_k\}$ and $\{\rho_k\}$ are given. We let

$$0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_{N-1} < \bar{x}_N = \infty$$

The function $\Lambda(x)$ takes N different strictly positive values denoted by $\{\rho_k\}_{k=1}^N$, so that:

$$\Lambda(x) = \rho_k \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k) \text{ for } k = 1, 2, \ldots, N$$
$$0 < \rho_1 < \rho_2 < \cdots < \rho_N .$$

Since $m(\cdot)$ and $f(\cdot)$ solve Kolmogorov equations (backward for $m(\cdot)$ and forward for $f(\cdot)$), on each segment they can parametrized by a pair of unknown constants:

$$m(x) = M_k(x) = -\frac{x}{\rho_k} + u_k e^{\eta_k x} + v_k e^{-\eta_k x} \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k] \tag{233}$$

$$f(x) = \bar{P}_k(x) = p_k e^{\eta_k x} + q_k e^{-\eta_k x} \text{ for } x \in [\bar{x}_{k-1}, \bar{x}_k] \tag{234}$$

$$\eta_k = \sqrt{\frac{2\rho_k}{\sigma^2}} \tag{235}$$

for $k = 1, 2, \ldots, N$. We require that $f(\cdot)$ and $m(\cdot)$ be continuously differentiable on $(0,\infty)$. This implies that

$$M_k(\bar{x}_k) = M_{k+1}(\bar{x}_k) \text{ and } M_k'(\bar{x}_k) = M_{k+1}'(\bar{x}_k) \text{ for all } k = 1, 2, \ldots, N-1 \tag{236}$$

$$\bar{P}_k(\bar{x}_k) = \bar{P}_{k+1}(\bar{x}_k) \text{ and } \bar{P}_k'(\bar{x}_k) = \bar{P}_{k+1}'(\bar{x}_k) \text{ for all } k = 1, 2, \ldots, N-1 \tag{237}$$

In addition we have the following conditions. Since m is antisymmetric around zero we require $m(0) = 0$. Since f is a density, it must integrate to one, and since it symmetric it must integrate to one half over positive x. Finally, both m and f should converge to $-x/\rho_N$ and 0 as $x \to \infty$.

These conditions are sometimes referred as no-bubble conditions. Hence:

$$M_1(0) = 0,\ \frac{1}{2} = \int_0^\infty f(x)dx = \sum_{k=1}^{N} \int_{\bar{x}_{k-1}}^{\bar{x}_k} \bar{P}_k(x)dx,\ \text{and } p_N = u_N = 0 \tag{238}$$

Overall, we have $2N$ unknowns, namely $\{u_k, v_k\}_{k=1}^N$, and $2N$ linear equations for $m(\cdot)$, namely $2(N-1)$ from equation (236), that $m(0) = 0$, and the no-bubble condition. Likewise for $f(\cdot)$. We can write these equations and solve for the constants. Once we have them we can evaluate:

$$\int_0^\infty x^2 f(x)dx = \sum_{k=1}^{N} \int_{\bar{x}_{k-1}}^{\bar{x}_k} x^2 \bar{P}_k(x)dx \text{ and} \tag{239}$$

$$\int_0^\infty m(x)f(x)dx = \sum_{k=1}^{N} \int_{\bar{x}_{k-1}}^{\bar{x}_k} M_k'(x)\bar{P}_k(x)dx\,. \tag{240}$$

and check if:

$$\sum_{k=1}^{N} \int_{\bar{x}_{k-1}}^{\bar{x}_k} x^2 \bar{P}_k(x)dx = -\sigma^2 \sum_{k=1}^{N} \int_{\bar{x}_{k-1}}^{\bar{x}_k} M_k'(x)\bar{P}_k(x)dx\,. \tag{241}$$

Now we will determine the coefficients $\{p_k, q_k\}_{k=1}^N$ and $\{u_k, v_k\}_{k=1}^N$. Start with the ones for $\bar{p}(\cdot)$. Combining the continuity and differentiability conditions, we can write the coefficients recursively for $k = 1, 2...N-1$:

$$p_k = \frac{1}{2}\left(1 + \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_{k+1}-\eta_k)x_k} p_{k+1} + \frac{1}{2}\left(1 - \frac{\eta_{k+1}}{\eta_k}\right) e^{-(\eta_{k+1}+\eta_k)x_k} q_{k+1} \tag{242}$$

$$q_k = \frac{1}{2}\left(1 + \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_k-\eta_{k+1})x_k} q_{k+1} + \frac{1}{2}\left(1 - \frac{\eta_{k+1}}{\eta_k}\right) e^{(\eta_{k+1}+\eta_k)x_k} p_{k+1} \tag{243}$$

We also have the terminal condition $p_N = 0$ and the normalization (the density must integrate to one half over positives). Observe that the coefficients are in fact linear in q_N, so q_N can easily be found from the normalization. The integral is

$$\frac{1}{2} = \int_0^\infty f(x)dx = \sum_{k=0}^{N-1} p_{k+1} \frac{e^{\eta_{k+1}x_{k+1}} - e^{\eta_{k+1}x_k}}{\eta_{k+1}} - \sum_{k=0}^{N-1} q_{k+1} \frac{e^{-\eta_{k+1}x_{k+1}} - e^{-\eta_{k+1}x_k}}{\eta_{k+1}} \tag{244}$$

We can use linearity: letting $p_k = \hat{p}_k q_N$ and $q_k = \hat{q}_k q_N$ and plugging this into the normalization, we can write

$$\frac{1}{2} = \sum_{k=0}^{N-1} \left(\hat{p}_{k+1} \frac{e^{\eta_{k+1}x_{k+1}} - e^{\eta_{k+1}x_k}}{\eta_{k+1}} - \hat{q}_{k+1} \frac{e^{-\eta_{k+1}x_{k+1}} - e^{-\eta_{k+1}x_k}}{\eta_{k+1}} \right) q_N \tag{245}$$

The numbers $\{\hat{p}_k, \hat{q}_k\}_{k=1}^{N-1}$ are easily obtained from $\{p_k, q_k\}_{k=1}^{N-1}$ computed recursively for some presupposed value of q_N. Knowing them, we can recover the real q_N from equation (245) and recompute the real $\{p_k, q_k\}_{k=1}^{N-1}$.

Now we will determine the coefficients for $m(\cdot)$. The continuity and differentiability conditions lead to the following recursivve representation:

$$u_k = \frac{1}{2}\left(1+\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_{k+1}-\eta_k)x_k}u_{k+1}+\frac{1}{2}\left(1-\frac{\eta_{k+1}}{\eta_k}\right)e^{-(\eta_{k+1}+\eta_k)x_k}v_{k+1} + \frac{1}{2}\left(x+\frac{1}{\eta_k}\right)\left(\frac{1}{\rho_k}-\frac{1}{\rho_{k+1}}\right)e^{-\eta_k x_k} \tag{246}$$

$$v_k = \frac{1}{2}\left(1+\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_k-\eta_{k+1})x_k}v_{k+1}+\frac{1}{2}\left(1-\frac{\eta_{k+1}}{\eta_k}\right)e^{(\eta_{k+1}+\eta_k)x_k}u_{k+1} + \frac{1}{2}\left(x-\frac{1}{\eta_k}\right)\left(\frac{1}{\rho_k}-\frac{1}{\rho_{k+1}}\right)e^{-\eta_k x_k} \tag{247}$$

We also have the terminal condition $u_N = 0$ and the antisymmetry condition $m(0) = 0$. The latter one reduces to $u_1 + v_1 = 0$. Now we can observe that all u_k and v_k are in fact affine in v_N: $u_k = \hat{u}_k v_N + \tilde{u}_k$ and $v_k = \hat{v}_k v_N + \tilde{v}_k$. The condition $m(0) = 0$ can be written as

$$0 = u_1 + v_1 = (\hat{u}_1 + \hat{v}_1)v_N + (\tilde{u}_1 + \tilde{v}_1) \tag{248}$$

The coefficients $\{\hat{u}_k, \hat{v}_k\}_{k=1}^{N-1}$ and $\{\tilde{u}_k, \tilde{v}_k\}_{k=1}^{N-1}$ can be found from $\{u_k, v_k\}_{k=1}^{N-1}$ computed recursively for two different presupposed values of v_N (we need two because the functions are affine, not linear). After that, we can recover the real v_N from equation (248) and recompute the real $\{u_k, v_k\}_{k=1}^{N-1}$.

F Discrete Distribution of Fixed Costs

Let $g_i > 0$ be the probability of drawing a fixed cost ψ_i for $i = 0, 1, \ldots, n-1$, conditional of drawing a low adjustment cost opportunity. We have $0 = \psi_0 < \psi_1 < \cdots < \psi_{n-1}$. A firm can always pay a fixed cost $\Psi \equiv \psi_n$ and change prices, with $\psi_n > \psi_{n-1}$. At all points x where v is twice differentiable we have:

$$rv(x) = \min\left\{Bx^2 + \frac{\sigma^2}{2}v''(x) + \kappa\sum_{j=0}^{n-1}\min\left\{\psi_j + v(0) - v(x)\,,\,0\right\}g_j\,,\, r\left(\psi_n + v(0)\right)\right\}$$

The optimal decision rule can be described by $n+1$ thresholds $0 = \bar{x}_0 < \bar{x}_1 < \bar{x}_2 < \cdots < \bar{x}_n \equiv X$. The optimal decision rule is that conditional on drawing the adjustment cost ψ_j an adjustment takes place if $|x| \geq \bar{x}_j$ for $j = 0, 1, \ldots, n$. Note that this implies that:

$$v(\bar{x}_j) + \psi_j = v(0) \text{ for } j = 0, 1, 2, \ldots, n.$$

To simplify the notation we let:

$$\lambda_j = \kappa g_j \text{ for } j = 0, \ldots, n-1 \text{ and } \Lambda(x) = \sum_{k=0}^{n-1}\lambda_k\, 1_{\{x \geq x_k\}}$$

To summarize the firm's problem is defined by parameters $r, B, \sigma^2, \{\lambda_j\}_{j=0}^{n-1}, \{\psi_j\}_{j=1}^{n}$, and the two normalized values $\psi_0 = 0$ and $\bar{x}_0 = 0$. The solution is given by a set of thresholds $\{\bar{x}_j\}_{j=0}^{n}$ with $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_n$.

We can write the value function for each segment $j = 1, 2, \ldots, n$:

$$\left(r + \sum_{k=0}^{j-1} \lambda_k\right) v(x) = Bx^2 + \frac{\sigma^2}{2} v''(x) + \sum_{k=0}^{j-1} \left[v(0) + \psi_k\right] \lambda_k \text{ for } x \in (\bar{x}_{j-1}\,,\, \bar{x}_j] \tag{249}$$

The value function v must be differentiable at all $x \in \mathbb{R}$, and twice differentiable for all $x \in \mathbb{R}$, except $x = \bar{x}_j$ for $j = 1, \ldots, n$. Thus we have the boundary conditions:

$$v'(\bar{x}_0) = v'(\bar{x}_n) = 0 \tag{250}$$

F.1 Value function for discrete ψ distribution

The solution of the value function v is characterized by coefficients $\{a_j, b_j, c_j\}_{j=0}^{n}$, roots $\{\eta_j\}_{j=1}^{n}$ and thresholds $\{\bar{x}_j\}_{j=0}^{n}$. In particular, given the thresholds $\{\bar{x}_j\}_{j=0}^{n}$ we write a linear o.d.e. for each segment $[\bar{x}_{j-1}, \bar{x}_j]$ for $j = 1, \ldots, n$. This o.d.e. is parametrized by three constants a_j, b_j, c_j as follows:

$$v_j(x) = a_j + b_j x^2 + c_j \left(e^{\eta_j x} + e^{-\eta_j x}\right) \text{ for } x \in [\bar{x}_{j-1}, \bar{x}_j] \text{ and } j = 1, \ldots, n \tag{251}$$

where η_j is given by:

$$\eta_j = \sqrt{\frac{(r + \sum_{k=0}^{j-1} \lambda_j)}{\sigma^2/2}} \tag{252}$$

Replacing the non-homogenous solution $a_j + b_j x^2$ into the o.d.e. in each segment we have:

$$\left(r + \sum_{k=0}^{j-1} \lambda_k\right) (a_j + b_j x^2) = Bx^2 + \frac{\sigma^2}{2} 2b_j + \sum_{k=0}^{j-1} \left[v(0) + \psi_k\right] \lambda_k \text{ for } x \in [\bar{x}_{j-1}, \bar{x}_j] \text{ and } j = 1, \ldots, n \tag{253}$$

Matching the terms quadratic in x, and using that $v(0) = a_1 + 2c_1$, we get:

$$\left(r + \sum_{k=0}^{j-1} \lambda_k\right) b_j = B \text{ for } j = 1, \ldots, n \tag{254}$$

Matching the constant we have:

$$\left(r + \sum_{k=0}^{j-1} \lambda_k\right) a_j = \sigma^2 b_j + \sum_{k=0}^{j-1} \left[a_1 + 2c_1 + \psi_k\right] \lambda_k \text{ for } j = 1, \ldots, n \tag{255}$$

The continuity and (once) differentiability at $x = \bar{x}_j$ for $j = 1, \ldots, n-1$ gives:

$$a_{j+1} + b_{j+1} \left(\bar{x}_j\right)^2 + c_{j+1} \left(e^{\eta_{j+1} \bar{x}_j} + e^{-\eta_{j+1} \bar{x}_j}\right) = a_j + b_j \left(\bar{x}_j\right)^2 + c_j \left(e^{\eta_j \bar{x}_j} + e^{-\eta_j \bar{x}_j}\right) \text{ for } j = 1, \ldots, n-1 \tag{256}$$

and

$$b_{j+1} 2\bar{x}_j + c_{j+1} \eta_{j+1} \left(e^{\eta_{j+1} \bar{x}_j} - e^{-\eta_{j+1} \bar{x}_j}\right) = 2b_j \bar{x}_j + c_j \eta_j \left(e^{\eta_j \bar{x}_j} - e^{-\eta_j \bar{x}_j}\right) \text{ for } j = 1, \ldots, n-1 \tag{257}$$

value matching and smooth pasting at $\bar{x}_n$ gives:

$$\psi_n + a_1 + 2c_1 = a_n + b_n (\bar{x}_n)^2 + c_n \left(e^{\eta_n \bar{x}_n} + e^{-\eta_n \bar{x}_n}\right) \tag{258}$$

$$0 = 2b_n \bar{x}_n + c_n \eta_n \left(e^{\eta_n \bar{x}_n} - e^{-\eta_n \bar{x}_n}\right) \tag{259}$$

The optimal return point conditions, $v'(0) = 0$, is automatically satisfied.

Thus we have $4 \times n$ unknowns, namely $\{\bar{x}_j, a_j, b_j, c_j\}_{j=1}^n$, and $4 \times n$ equations, namely n equations matching quadratic terms, i.e. equations (254), n equations matching constants, i.e. equations (255), $n-1$ equations enforcing continuity, i.e. equations (256), $n-1$ equations enforcing differentiability, i.e. equations (257), value matching, i.e. equation (258), and smooth pasting, i.e. equation (259).

F.2 Inverse problem: recovering the cost function

We now solve an inverse problem, namely how to recover the menu cost values ψ_j that underlie a given observed hazard function $\Lambda(x)$ at given thresholds $\{\bar{x}_j\}$. The main result is summarized by the next proposition:

Proposition 15. Fix a discount rate, curvature and variance $r, B, \sigma^2 > 0$, and a step function Λ giving the probability per unit of time of a price adjustment for $|x| < x_n$. The function Λ is described by a set of probability rates for costs $\{\lambda_j\}_{j=1}^{n-1} \in \mathbb{R}_+^n$ for $n \geq 1$, and a set of n thresholds $\{\bar{x}_j\}_{j=1}^n$ with $0 = \bar{x}_0 < \bar{x}_1 < \cdots < \bar{x}_n$. Then there is a unique set of n fixed costs $0 = \psi_0 < \psi_1 < \cdots < \psi_n$ so that the n thresholds $\{\bar{x}_j\}_{j=1}^n$ solve the firm's problem defined by $r, B, \sigma^2, \{\lambda_j\}_{j=0}^{n-1}, \{\psi_j\}_{j=1}^n$. Moreover, the fixed costs $\{\psi_j\}_{j=1}^n$ and the coefficients of the value function $\{a_j, b_j, c_j\}_{j=1}^n$ solve a system of linear equations.

Proof. (of Proposition 15) We first solve for each of the coefficients b_j using equation (254) for each $j = 1, \ldots, n$.

We note that the thresholds $\{\bar{x}_j\}_{j=1}^n$ are given and that roots $\{\eta_j\}_{j=1}^n$ can be computed as functions of given parameters.

Using the coefficients $\{b_j\}_{j=1}^n$, we solve for the coefficients $\{c_j\}_{j=1}^n$. First we solve for c_n enforcing smooth pasting at $\bar{x}_n$ given by equation (259). Using c_n we recursively use c_{j+1} to solve for c_j imposing differentiability between adjecent segments, i.e. equations (257) for $j = n-1, n-2, \ldots, 1$.

Next we solve for the $\{a_j\}_{j=1}^n$, given $\{b_j, c_j\}_{j=1}^n$. First, use $rv(0) = \frac{\sigma^2}{2} v''(0) = \frac{\sigma^2}{2}\left(2b_1 + (\eta_1)^2 2c_1\right)$ and $v(0) = a_1 + 2c_1$ to solve for a_1, namely $a_1 = \frac{\sigma^2}{r}\left(b_1 + \eta_1^2 c_1\right) - 2c_1$. Next, use equations (256) to solve recursively for $\{a_j\}_{j=2}^n$.

Finally, we solve for the fixed costs $\{\psi_j\}_{j=1}^n$ using value matching and the values of $\{a_j, b_j, c_j\}_{j=1}^n$. They give:

$$\psi_j = v(\bar{x}_j) - v(0) = a_j + b_j (\bar{x}_j)^2 + c_j \left(e^{\eta_j \bar{x}_j} + e^{-\eta_j \bar{x}_j}\right) - a_1 - 2c_1 \tag{260}$$

for $j = 1, \ldots, n$. $\square$