Agenda-manipulation in ranking*

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Abstract

A committee ranks a set of alternatives by sequentially voting on pairs, in an order chosen by the committee's chair. Although the chair has no knowledge of voters' preferences, we show that she can do as well as if she had perfect information. We characterise strategies with this 'regret-freeness' property in two ways: (1) they are *efficient*, and (2) they avoid two intuitive errors. One regret-free strategy is a sorting algorithm called *insertion sort*. We show that it is characterised by a lexicographic property, and is outcome-equivalent to a recursive variant of the much-studied *amendment procedure*.

1 Introduction

A committee is tasked with ranking a set of alternatives. This could be a hiring committee deciding to which candidate a job should be offered, to whom next if the first candidate declines, and so on. Or it could be the leadership of a political party drawing up its party list: if the party wins K parliamentary seats in the subsequent election, these will go to the K highest-ranked candidates.

One pair of alternatives is considered at a time, with a majority vote determining which is to be ranked higher. Transitivity is enforced: once the

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committee has ranked x above y and y above z, x is considered ranked above z.[1] Pairwise voting continues until the alternatives are fully ranked.

The order in which pairs of alternatives are considered—the *agenda*—is chosen by the committee's chair. The chair has a (strict) preference $\succ$ over alternatives, and we assume that she prefers rankings that are unambiguously more aligned with $\succ$.[2] In the hiring application, with uncertainty about which candidates would accept an offer, a more aligned ranking is exactly one that hires a weakly $\succ$-better candidate in every state of the world.[3] The chair has no knowledge of how voters will behave. How should she set the agenda, and how much influence can she exert?

Fix how each voter votes on each pair of alternatives. Write $x\ W\ y$ if a majority favour x in a vote on $\{x, y\}$; we call W the *general will.* Since the general will is all that matters to the chair, we work with W directly. This makes our analysis invariant to institutional details: extensions like super-majority rules or abstentions merely alter W. The 'general will' need not even arise from voting: another interpretation is that the chair is an advisor to a leader with expressed will W.

A *strategy* specifies what pair of alternatives to offer a vote on after each history.[4] Call a ranking of alternatives *W-feasible* if some strategy achieves it under general will W, and *W-unimprovable* if there is no other ranking that is both W-feasible and unambiguously more aligned with $\succ$. The chair could do no better than a W-unimprovable ranking even if she had perfect knowledge of the general will W.

The chair does not know the general will. A *regret-free* strategy is one that reaches a W-unimprovable ranking under every W. Were a regret-free strategy to exist, it would allow the chair to reach ex-post unimprovable outcomes. Surprisingly, regret-free strategies do exist: we show in Theorem 1 that a sorting algorithm called *insertion sort* is regret-free.

What other strategies are regret-free? We provide two characterisations. For the first, call a ranking *W-efficient* if it ranks x above y whenever both $x \succ y$ and $x\ W\ y$ (i.e. the chair and the general will agree). Theorem 2 asserts that regret-free strategies are precisely those that reach a W-efficient

[1] To turn the (possibly cyclic) majority will into a (transitive) ranking, some pairs of alternatives must be ranked by fiat. The described rule is the only one satisfying *committee sovereignty:* each alternative is ranked above those that it beat in a vote.

[2] A ranking R is *(unambiguously) more aligned* with $\succ$ than another ranking R' if for any pair of alternatives $x \succ y$, if R' ranks x above y, then so does R.

[3] Thus the chair prefers more aligned rankings if she has e.g. expected-utility preferences. Similarly in the party lists application, for every $k \leq K$, the k^{th} $\succ$-best candidate who gets a seat is weakly $\succ$-better under a more aligned ranking.

[4] A *history* records what pairs were voted on, and which alternative won each vote.

ranking under each general will W.[5]

Our second characterisation delineates two intuitive errors that regret-free strategies avoid. For the first error, consider three alternatives $x \succ y \succ z$, such that y has already been ranked above z. If the chair offers $\{x, y\}$ for a vote and it goes her way ($x \; W \; y$), then she gets x ranked above z 'for free' as a consequence of transitivity. Thus offering a vote on $\{x, z\}$ *misses an opportunity* for a 'favourable imposition of transitivity'.[6]

For the second error, consider three alternatives x, y, z such that the chair prefers x to both y and z, and y has already been ranked below z. Offering $\{x, y\}$ *takes a risk* in that if the vote goes badly ($y \; W \; x$), then not only will x be ranked below y, but x will be ranked below z—an 'unfavourable imposition of transitivity'.

Theorem 3 states that regret-free strategies are precisely those that never miss an opportunity or take a risk. Our advice to the chair is thus to, (separately) in each period, offer any pair that does not constitute an error of either kind.[7] Insertion sort is one way of doing this.

What is special about insertion sort? We show in Theorem 4 that it is characterised by a 'lexicographic' property: among all strategies, it optimises the position of the chair's favourite alternative; among such strategies, it optimises the position of the her second-favourite alternative; and so on.

The insertion-sort strategy can also be described in terms of the *amendment procedure*, which is used by many legislatures and has been extensively studied in the literature (see §1.1 below for references). This procedure offers a vote on the chair's two least-favourite alternatives, then pits the winner against her third-least favourite, then pits the winner against her fourth-least favourite, and so on.[8] The *recursive-amendment procedure* runs the amendment procedure to obtain a top-ranked alternative y_1, then runs the amendment procedure on all alternatives but y_1 to obtain a second-ranked alternative y_2, then again on all alternatives but y_1 and y_2, and so on. We establish in Proposition 3 that recursive amendment offers the same votes as insertion sort (albeit in a different order), so that the two produce the same final ranking under each general will W.

[5] This characterisation is tight: for any general will W, every W-feasible W-efficient ranking is reached by some regret-free strategy (Proposition 1).

[6] This can hurt: if $z \; W \; x \; W \; y \; W \; z$, then missing this opportunity leads x to be ranked below z, whereas it would have been ranked above had $\{x, y\}$ been offered.

[7] When no errors have been committed, there is always a non-error pair that can be offered next (Proposition 2).

[8] In the literature, the amendment procedure is defined given some ordering of the alternatives. In our definition, that ordering is the chair's preference $\succ$.

Our analysis assumes history-invariant voting, so that whether $x \, W \, y$ or $y \, W \, x$ is independent of what has happened so far. This assumption is satisfied if voting is expressive or if voters are unsophisticated. When voting is strategic, we provide a rationale for *sincere* (a fortiori history-invariant) voting in Proposition 4: the sincere-voting strategy is uniquely *dominant*, meaning that any other strategy fails to be obviously better against every profile of strategies of the chair and other voters, and is obviously worse against some profile. By 'obviously better', we mean that the final ranking is unambiguously more aligned with the voter's preference over alternatives.

1.1 Related literature

We contribute to the agenda-manipulation literature initiated by Black (1958) and Farquharson (1969), which asks how a committee's choice can be influenced by varying the order in which binary questions are voted on (the *agenda*). Two classes of agenda are emphasised: the *amendment procedure* used in Anglo-Saxon and Scandinavian legislatures, and the *successive procedure* widely used in continental Europe. Under complete information, for both sincere and strategic voting, Miller (1977) and Banks (1985) characterise which alternatives an agenda-setter can induce a committee to choose using (i) amendment agendas, (ii) successive agendas, and (iii) arbitrary agendas.[9,10] Extensions include super- and sub-majority voting rules (Barberà & Gerber, 2017) and random agendas (Roessler, Shelegia & Strulovici, 2018).

Specifically, this paper belongs to the literature on agenda-setting under incomplete information about voters' preferences. This literature long consisted of a single pioneering paper (Ordeshook & Palfrey, 1988), but has recently received interest from Kleiner and Moldovanu (2017), Gershkov, Moldovanu and Shi (2017, 2019a, 2019b) and Gershkov, Kleiner, Moldovanu and Shi (2019). We depart from these papers by considering a committee tasked not with choosing a single alternative, but rather with *ranking* all of the alternatives. We establish a link with the older literature by showing that the amendment procedure forms the basis of a regret-free strategy.

Also related is the social choice literature on *ranking methods*, meaning maps that assign to each (possibly cyclic) general will a (transitive) ranking. 'Impossibility' results such as Arrow's (1950, 1951, 1963) theorem assert that

[9] Part (i) under strategic voting due to Banks (1985); the rest are from Miller (1977). See Myerson (1991, §4.10) for a nice textbook treatment of (iii) under strategic voting.

[10] Related work by Apesteguia, Ballester and Masatlioglu (2014) and Horan (2020) axiomatises the decision rules (which specify a choice for each choice set and preference profile) defined by various agendas under strategic voting.

certain normatively appealing properties are inconsistent. The literature beginning with Zermelo (1929), Wei (1952) and Kendall (1955) studies ranking methods with at least *some* attractive normative properties.[11] Our chair's problem can be formulated as a choice among ranking methods. But there is a feasibility constraint: only ranking methods induced by some strategy are available.[12] Furthermore, the objective captures the chair's self-interest, not any normative notion. This suggests that the solutions to our chair's problem will bear little relation to the normatively-motivated ranking methods studied in the literature. We confirm this in supplemental appendix M.

We assume that the chair prefers one ranking to another whenever the former is *more aligned* with her preference over alternatives. This is an instance of *single-crossing dominance,* a general way of comparing rankings. In a separate paper (Curello & Sinander, 2019), we study its lattice structure.

Regret-based criteria for evaluating strategies appear in decision theory, including minimax regret (Savage, 1951) and 'regret theory' (Bell, 1982; Loomes & Sugden, 1982; Fishburn, 1983). The online learning literature (Gordon, 1999; Zinkevich, 2003) studies (asymptotic) regret-freeness.

1.2 Roadmap

We describe the environment and basic concepts in §2. In §3, we introduce efficiency and show that it implies regret-freeness. We next (§4) define the insertion-sort strategy, and show that it is efficient (Theorem 1), hence regret-free. In §5, we characterise regret-freeness in terms of efficiency (Theorem 2) and error-avoidance (Theorem 3), and show that both characterisations are tight (Propositions 1 and 2). In §6, we show that insertion sort is characterised by a lexicographic property (Theorem 4) and that it is outcome-equivalent to recursive amendment (Proposition 3). Finally, we establish in §7 that sincere voting is dominant (Proposition 4).

[11] For example, Copeland's (1951) method (Rubinstein, 1980), the Kemeny–Slater method (Kemeny, 1959; Slater, 1961; Young & Levenglick, 1978; Young, 1986, 1988) and the fair-bets method (Daniels, 1969; Moon & Pullman, 1970; Slutzki & Volij, 2005). See Charon and Hudry (2010) and González-Díaz, Hendrick and Lohmann (2014) for an overview.

[12] The ranking method induced by a strategy is the one that assigns to each W the outcome (final ranking) of that strategy under W.

2 Environment

There is a finite set $\mathcal{X}$ of alternatives. We are concerned with *rankings* of alternatives, formalised as follows:

Definition 1. A *proto-ranking* is an irreflexive and transitive binary relation on $\mathcal{X}$. A *ranking* is a proto-ranking that is total.[13]

Given a (proto-)ranking R and alternatives $x, y \in \mathcal{X}$, $x \; R \; y$ reads 'x is ranked above y according to R'. Proto-rankings are to be thought of as incomplete rankings.

There is a committee of voters $i \in \{1, \ldots, I\}$, where $I \in \mathbf{N}$ is odd, and the committee's *chair*. The committee meets to determine a ranking, with the chair in charge of the agenda. Three applications are as follows.

Hiring. The alternatives $\mathcal{X}$ are candidates for a job. Only some unknown subset $X \subseteq \mathcal{X}$ of candidates would accept the job if offered it. The hiring committee decides the order in which offers should be made (a ranking): the job is offered to the first candidate, then to the second if the first declined, and so on. It suffices for the committee to meet once, as is natural if meetings are costly or impractical. Alternatively, the committee could re-convene every time a candidate turns down an offer, taking just enough decisions in each meeting to identify the next candidate to whom an offer will be made.[14]

Relabelling, we may instead think of the alternatives as investment projects of unknown viability. A firm's board (or a lower-level committee) ranks the projects, whereupon a manager evaluates the first project (e.g. by commissioning market research) and implements it if viable, and otherwise evaluates the second project and implements that if viable, and so on.

Similarly, the committee could be a policy-making body, such as a ministerial cabinet or a parliamentary committee. Any given policy may turn out to be infeasible, for example due to a court ruling or political opposition. The committee therefore ranks the policies and tasks a bureaucrat with implementing the first policy if feasible, the second if not, and so on.

Party lists. A political party's leadership committee must draw up a *party list*, meaning a ranking of the party's parliamentary candidates $\mathcal{X}$. The K top-ranked candidates will earn parliamentary seats, where K is the (uncertain) number of seats won by the party in a subsequent election.

[13]Definitions of some order-theoretic terms are collected in appendix A.

[14]This is natural when the committee wishes to decide as little as possible prior to each job offer. The recursive amendment procedure studied in §6.3 is attractive in this case: it quickly determines what candidate is ranked first (allowing an offer to be made), then (if necessary) quickly identifies the second-highest-ranked, and so on.

Electoral systems of this type, called *party-list proportional representation*, are used in most of the world's democracies. More precisely, we described the *closed-list* variant that gives voters no sway over party lists; this system is used in several dozen states.[15] Other countries allow the electorate to influence party lists.[16] In many of these, voters exert little influence on party lists in practice,[17] making the closed-list system a reasonable idealisation.

We may re-interpret this as a firm planning for downsizing. The firm will have to fire an uncertain number K of its workers $\mathcal{X}$. The board (or a lower-level committee) plans ahead by drawing up an order in which employees will (if necessary) be let go.

2.1 Interaction

The committee collectively determines a ranking R as follows. Initially, no alternatives are ranked. In each period, the chair offers a vote on a pair of alternatives. The committee votes, and whichever alternative garners more votes wins. (The chair does not have a vote.) The winning alternative is ranked above the losing one. In addition, transitivity is imposed: if x is ranked above y and y above z, then x is considered ranked above z. The chair offers votes until all alternatives are ranked.

More formally, the interaction is as follows. For each period $t \in \{0, 1, \dots, \}$, a proto-ranking R_t describes what has been decided by the end of period t: $x \mathrel{R_t} y$ if x has been ranked above y, and $x \not\mathrel{R_t} y$ otherwise. Initially, nothing has been decided: $R_0 = \varnothing$. In each period $t \in \mathbf{N}$, unless R_{t-1} is already total,

(1) The chair offers for a vote an unranked pair of alternatives: namely distinct $x, y \in \mathcal{X}$ with $x \not\mathrel{R_{t-1}} y \not\mathrel{R_{t-1}} x$.

(2) Each voter $i \in \{1, \dots, I\}$ votes for either x or y. Whichever alternative garners more (fewer) votes is said to *win* (*lose*).

(3) The winner is ranked above the loser, and transitivity is imposed: if (say) x won, then R_t is the transitive closure of $R_{t-1} \cup \{(x, y)\}$.[18]

[15] For example, Argentina, Germany, Japan, Russia, South Africa and Turkey.

[16] E.g. Brazil, Indonesia, Iraq, the Netherlands and Ukraine.

[17] In many countries, such as Indonesia and the Netherlands, the party's list is only altered if a candidate receives a large number of personal votes. Furthermore, voting for an individual candidate is typically optional, and most voters do not. (In Swedish parliamentary elections, only about a quarter do (Oscarsson, 2019).) Of course there are exceptions: voting for individual candidates is mandatory and important in Finnish parliamentary elections, for example.

[18] Equivalently, $z \mathrel{R_t} w$ iff either (1) $z \mathrel{R_{t-1}} w$, or (2) both (a) $z = x$ or $z \mathrel{R_{t-1}} x$ and (b) $y = w$ or $y \mathrel{R_{t-1}} w$. See appendix D.1 (Observation 3, p. 31) for a proof of equivalence.

The interaction ends when R_t is total, hence a ranking. We often write T for the period in which this occurs, and $R = R_T$ for the final ranking.

To understand why we assume that transitivity is imposed in this manner, observe that the purpose of the interaction is to turn the will of the majority, which may contain (Condorcet) cycles, into a (by definition transitive) ranking. Some pairs will thus necessarily be ranked by fiat. We require that transitivity be imposed immediately after each vote because this is necessary and sufficient for *committee sovereignty*, the requirement that x be ranked above y if x beat y in a vote.[19] Indeed, we contend that the protocol described above is the only natural one, given that the interaction must end with a ranking: as shown in supplemental appendix H, any other protocol must violate either committee sovereignty or *democratic legitimacy*, the requirement that the chair offer enough votes to give the committee a fair say.

A *history* is a sequence of pairs offered for a vote and a winner of each vote. A *strategy* specifies what pair to offer for a vote after each history. We give formal versions of these definitions in appendix B.1.

Remark 1. A history records only which alternative in each pair wins, rather than the full vote tally. Our definition of a strategy therefore rules out conditioning on who voted how in the past.[20] This is merely to avoid uninteresting complications: we show in supplemental appendix K that our results hold for 'extended strategies' that can condition on past votes cast.

2.2 The general will

The chair need not keep track of individual votes: all that matters for each pair of alternatives is which one wins. This essential information is captured by the binary relation W such that $x\ W\ y$ iff a majority of voters vote for x over y when the pair $\{x, y\}$ is offered. We call W the *general will*.[21]

By using the general will, we implicitly assume history-invariance: $x\ W\ y$ means that x garners a majority over y at any history at which $\{x, y\}$ is offered. We must therefore suppose that voters' behaviour is (at least approximately) history-invariant. This is reasonable if voters are non-strategic: if they are unsophisticated, say, or vote 'expressively' (to please their constituents, for example). Empirically, non-strategic voting appears to be the norm in

[19] Sufficiency is obvious. For necessity, suppose the chair were allowed to offer $\{x, z\}$ even though $x\ R_{t-1}\ y\ R_{t-1}\ z$; then committee sovereignty is violated whenever z beats x.

[20] Of course, this is only a restriction if the chair can observe individual votes.

[21] Our use of this term is inspired by Rousseau (1755a, 1755b, 1762). Although there is a long tradition of using the term in this way, it probably isn't quite what Rousseau meant.

many important institutions, such as the US Congress.[22] Even if voters behave strategically, we argue in §7 that it is not unreasonable to assume history-invariant voting.

Definition 2. A *tournament* is a total and asymmetric relation on $\mathcal{X}$.

Only tournaments can be general wills, since for any pair of alternatives, one wins and the other loses. In fact, all and only tournaments are the general will of some committee, as we show in appendix B.2. We therefore study the chair's problem for an arbitrary tournament W.

Remark 2. An alternative interpretation of our model is that there is a single individual with (expressed) preference W. We can think of this individual as a leader, and of the chair as his crafty advisor.[23] Intransitivities in W reflect inconsistencies in the leader's judgement. The advisor manipulates the leader by asking him to make pairwise comparisons in a well-chosen order.

Remark 3. Varying the committee's rules merely gives rise to a different general will, leaving the analysis below applicable without modification. For example, we can accommodate a super-majority voting rule (with the size of the required super-majority possibly varying across pairs of alternatives), an even electorate $I \in \mathbf{N}$, and abstentions. We need only specify which alternative wins in case of an indecisive vote (meaning one in which neither wins the required (super-)majority), for example by assuming that there is a status quo ranking that prevails in such cases.

A more substantial variation is to permit the chair (sometimes) to decide how $\{x, y\}$ are to be ranked following an indecisive vote on them. This happens if the chair has a vote, for example. We extend all of our results to allow for this in supplemental appendix J.

The *outcome* of a strategy under a tournament W is the ranking that results. If a history is visited by a strategy σ under some tournament, we say that it belongs to the *path* of σ. We give formal definitions of outcomes and paths in appendix B.1.

2.3 W-feasible rankings

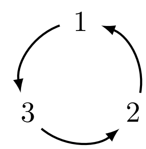
A W-feasible ranking is one that is reachable ex post: had the chair known that the general will was W, then she could have attained such a ranking by offering some sequence of pairs.

[22] See Landha (1994), Poole and Rosenthal (1997) and Wilkerson (1999), as well as the survey by Groseclose and Milyo (2010).

[23] For example, Henry VIII and Cardinal Wolsey, or Louis XIII and Cardinal Richelieu.

Definition 3. Given a tournament W, a ranking is *W-feasible* iff it is the outcome under W of some strategy of the chair.

Example 1. There are three alternatives $\mathcal{X} = \{1, 2, 3\}$, and the general will satisfies $1 \; W \; 3 \; W \; 2 \; W \; 1$. Graphically:



The ranking $2 \; R \; 1 \; R \; 3$ is W-feasible, achieved by offering $\{1, 2\}$ and $\{1, 3\}$. Similarly, $1 \; R' \; 3 \; R' \; 2$ and $3 \; R'' \; 2 \; R'' \; 1$ are W-feasible.

2.4 The chair's preferences

The chair has a strict preference (formally: a ranking) over the alternatives $\mathcal{X}$, denoted $\succ$. We do not fully specify the chair's preferences over rankings. We assume only that she prefers a ranking over another whenever the former is *more aligned* with her own preference over alternatives.

Definition 4. For rankings $\succ$, R and R', we say that R is *more aligned with $\succ$ than R'* iff for any pair $x, y \in \mathcal{X}$ of alternatives with $x \succ y$, if $x \; R' \; y$ then also $x \; R \; y$.

In words, whenever a pair $x, y \in \mathcal{X}$ is ranked 'right' by R' (viz. $x \succ y$ and $x \; R' \; y$), it is also ranked 'right' by R (i.e. $x \; R \; y$). The definition is illustrated in Figure 1 for the case of three alternatives.

We assume that if R is more aligned with $\succ$ than R', then the chair weakly prefers R to R'. We view this as a minimal notion of consistency between the chair's preference over alternatives and her preference over rankings of alternatives. It has natural meaning in the applications:

Hiring (continued)**.** Recall that the top-ranked candidate in X will be hired, where $X \subseteq \mathcal{X}$ is unknown. A ranking is more aligned with $\succ$ exactly if it hires a weakly $\succ$-better candidate at every realisation of the uncertainty X, as we show in appendix B.4. Thus any reasonable preferences over rankings favour more aligned rankings. In particular, R is preferred to R' by all expected-utility preferences consistent with $\succ$ iff R is more aligned with $\succ$.

Party lists (continued)**.** Recall that the K highest-ranked candidates earn parliamentary seats, where K is uncertain. Say that a ranking R *dominates* another ranking R' iff for any K and $k \leq K$, the k^{th} $\succ$-best candidate who

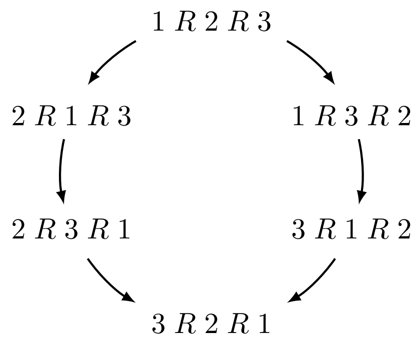


Figure 1 – 'More aligned with $\succ$ than' for three alternatives $\mathcal{X} = \{1, 2, 3\}$, where $1 \succ 2 \succ 3$. In this ('Hasse') diagram, there is a directed path from one ranking to another iff the former is more aligned with $\succ$.

wins a seat under R is weakly $\succ$-better than the k^{th} $\succ$-best under R'. Clearly any reasonable preferences over rankings will favour a dominating ranking. We show in appendix B.4 that 'more aligned with $\succ$ than' implies dominance, and that it is also necessary provided there is (at least some) uncertainty about which candidates can take up seats.[24]

2.5 Regret-free strategies

Suppose the chair were to know the general will W. Then she could reach all and only W-feasible rankings, so would choose one that is W-*unimprovable:*

Definition 5. Given a tournament W, a ranking is W-*unimprovable* iff there is no other W-feasible ranking that is more aligned with $\succ$.

Remark 4. A ranking R is W-unimprovable precisely if for any W-feasible ranking $R' \neq R$, there is some pair of alternatives $x, y \in \mathcal{X}$ that R ranks 'right' ($x \succ y$ and $x \; R \; y$) and that R' ranks 'wrong' ($y \; R' \; x$).

W-unimprovability is the strongest optimality concept available without further assumptions about the chair's preference over rankings. It can therefore be thought of as optimality for a chair who is unable to make fine comparisons between rankings (which are complicated objects). Were we to fully specify the chair's preference over rankings, we could still break

[24] A party cannot alter its list after submitting it, but circumstances may render some of its candidates ineligible for parliamentary seats. For example, many countries disqualify candidates convicted of a serious crime, and all disqualify the deceased.

her problem under full information about W into two parts: first, reach the frontier (W-unimprovability), then choose among the frontier rankings.

Hiring (continued)**.** A ranking R is W-unimprovable exactly if any W-feasible ranking $R' \neq R$ hires a strictly $\succ$-worse candidate at some realisation $X \subseteq \mathcal{X}$ of uncertainty.

Remark 5. W-unimprovability is a non-trivial property only if there are W-feasible rankings that are *not* W-unimprovable. We show in supplemental appendix I that this is the case whenever W contains a (Condorcet) cycle, and that for a typical W, most W-feasible rankings are not W-unimprovable.

The chair does not know the general will. One would therefore expect her to face trade-offs: a strategy that does well against W may have a regrettable outcome under some W'. A *regret-free* strategy is one that has no such downside: its outcome under any general will is unimprovable ex post.

Definition 6. A strategy is *regret-free* iff for any tournament W, its outcome under W is W-unimprovable.

Regret-freeness is a highly demanding optimality property. Surprisingly, however, regret-free strategies exist: we show in §4 that a strategy called *insertion sort* is regret-free (Theorem 1). We then provide two characterisations of regret-freeness (§5, Theorems 2 and 3). To develop these results, we require a new concept: *efficiency*.

3 Efficiency

An *efficient* strategy is one that never wastes a consensus: whenever the chair and the general will agree on a pair of alternatives, they are ranked accordingly. In this section, we introduce efficiency and show that it implies regret-freeness. We shall use this result in §4 to prove that insertion sort is regret-free (Theorem 1). We obtain the converse (regret-freeness implies efficiency) in §5 (Theorem 2).

3.1 W-efficient rankings

A W-efficient ranking is one that, whenever the chair and committee agree on a pair of alternatives, ranks them accordingly:

Definition 7. Given a tournament W, a ranking R is *W-efficient* iff for any pair $x, y \in \mathcal{X}$ of alternatives with $x \succ y$ and $x \; W \; y$, we have $x \; R \; y$.

W-efficient rankings exist trivially, since the chair's preference $\succ$ is a W-efficient ranking for any tournament W.

Example 1 (continued)**.** Recall the details from p. 10, and let the chair's preference be $1 \succ 2 \succ 3$. W-efficiency requires precisely that 1 be ranked above 3. There are three such rankings: $\succ$ itself, $2\ R\ 1\ R\ 3$ and $1\ R'\ 3\ R'\ 2$.

In Example 1, the W-efficient ranking $\succ$ is not W-feasible. But the other two W-efficient rankings, R and R', are both W-feasible. In fact, any tournament W admits at least one W-feasible W-efficient ranking.[25]

3.2 W-efficiency and W-unimprovability

W-efficiency is a useful property because it implies W-unimprovability:

Lemma 1. For any tournament W, a W-efficient ranking is W-unimprovable.

Proof. Fix a tournament W, and let R be a W-efficient ranking. To establish that R is W-unimprovable, assume toward a contradiction that some W-feasible ranking $R' \neq R$ is more aligned with $\succ$ than R.

Since $R' \neq R$, there are alternatives $x, y \in \mathcal{X}$ such that $x\ R'\ y$ and $y\ R\ x$. Enumerate the alternatives that R' ranks between x and y as

$$x = z_1\ R'\ z_2\ R' \cdots R'\ z_N = y.$$

Since R' is W-feasible, we must have $z_m\ W\ z_{m+1}$ for each $m < N$. (This follows from Observation 1 in appendix B.3 (p. 27).) There has to be an $n < N$ at which $z_{n+1}\ R\ z_n$, since otherwise we would have $x\ R\ y$ by transitivity of R. And it must be that $z_{n+1} \succ z_n$, because otherwise the W-efficiency of R would require that $z_n\ R\ z_{n+1}$. Thus the pair (z_n, z_{n+1}) is ranked 'right' by R ($z_{n+1} \succ z_n$ and $z_{n+1}\ R\ z_n$) and 'wrong' by R' ($z_n\ R'\ z_{n+1}$), contradicting the hypothesis that R' is more aligned with $\succ$ than R. ■

The converse of Lemma 1 is false whenever there are $|\mathcal{X}| \geq 4$ alternatives: a W-unimprovable ranking can easily fail to be W-efficient. We shall see an example of this in §5.1 (p. 17).

3.3 Efficient strategies

An efficient strategy is one whose outcomes are efficient:

[25] This follows from Theorem 1 in §4.1 below; a direct proof is also possible.

Definition 8. A strategy is *efficient* iff for any tournament W, its outcome under W is W-efficient.

As an immediate consequence of Lemma 1, we obtain:

Corollary 1. Any efficient strategy is regret-free.

Since the converse of Lemma 1 is false, one might expect the converse of Corollary 1 also to be false. But it is true, as we shall see in §5 (Theorem 2).

4 Insertion sort

In this section, we introduce the *insertion-sort strategy*. We show in Theorem 1 (§4.2) that it is efficient, and hence regret-free by Corollary 1. This establishes in particular that regret-free strategies exist.

4.1 The insertion-sort strategy

Insertion-sort algorithm. Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$. First, pit $n-1$ against n. Then set $k = n-2$, and repeat the following while $k \geq 1$:

– Alternatives $\{k+1, \ldots, n\}$ have already been totally ranked. Label them accordingly: $\{x_{k+1}, \ldots, x_n\} \equiv \{k+1, \ldots, n\}$, where $x_{k+1} \; R \cdots R \; x_n$.

– Pit k against x_{k+1}, the highest-ranked alternative in $\{k+1, \ldots, n\}$.

  If k won, then $\{k, \ldots, n\}$ is now totally ranked as $k \; R \; x_{k+1} \; R \cdots x_n$.

– If k lost to x_{k+1}, pit k against x_{k+2}, the second-highest-ranked.

  If k won, then $\{k, \ldots, n\}$ is totally ranked: $x_{k+1} \; R \; k \; R \; x_{k+2} \; R \cdots x_n$.

– If k lost also to x_{k+2}, pit k against x_{k+3}.

  If k won, then $\{k, \ldots, n\}$ is now totally ranked.

  . . .

– Now $\{k, \ldots, n\}$ is totally ranked.

  Decrease k by 1 and repeat.

Insertion sort is illustrated in Figure 2 for the case of three alternatives.

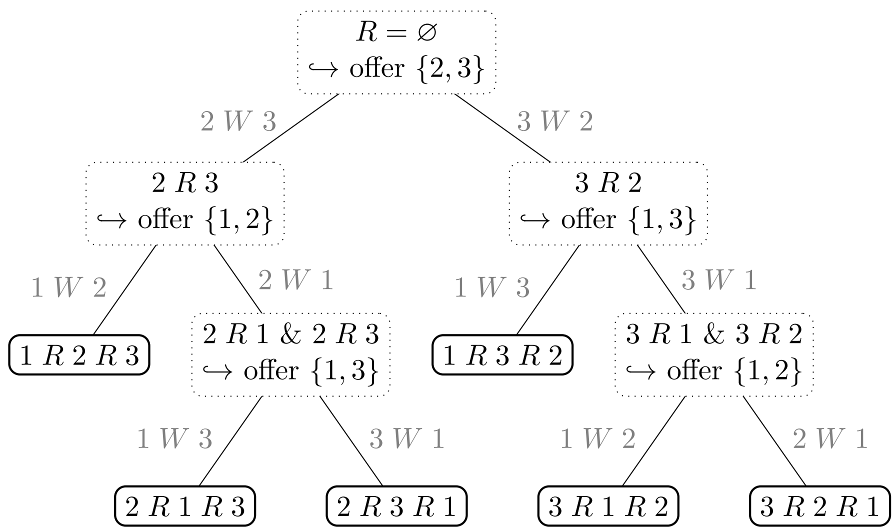


Figure 2 – The insertion-sort algorithm with three alternatives $\mathcal{X} = \{1, 2, 3\}$, where the chair's preference is $1 \succ 2 \succ 3$. R denotes the (evolving) proto-ranking. The path taken depends on the general will W.

Remark 6. Insertion sort is a standard sorting algorithm—see e.g. Knuth (1998, §5.2.1). The relationship with sorting is as follows. By a *list*, we mean exactly a ranking. Call W the 'comparator', and assume that it is transitive. A sorting algorithm takes as input an 'unsorted' list $\succ$, performs pairwise comparisons using W, and outputs a list R that is 'sorted', meaning that $R = W$. The term 'insertion' stems from the fact that for each $k \in \{1, \ldots, n-1\}$, the algorithm first sorts (fully ranks) $\{k+1, \ldots, n\}$, then 'inserts' k into $\{k+1, \ldots, n\}$ by comparing it with the highest-ranked, then the second-highest-ranked, and so on until k's place is found.

To operationalise insertion sort, we use a strategy whose on-path behaviour is described by the algorithm. The details of its definition are not important—all that matters is that it obeys the insertion-sort algorithm.[26]

Definition 9. The *insertion-sort strategy* is the strategy that after each history offers a vote on $\{x, y\}$, where x is the $\succ$-worst alternative unranked

[26]Our definition of strategies requires them to specify behaviour even after histories that never arise. (For example, if $\mathcal{X} = \{1, 2, 3\}$, a strategy that first offers $\{2, 3\}$ must specify behaviour after histories at which only $\{1, 2\}$ are ranked, even though these are never reached.) Any strategy whose on-path behaviour is described by the insertion-sort algorithm will do for our purposes—we use this particular one for concreteness.

with some even $\succ$-worse alternative, and y is the highest-ranked alternative $\succ$-worse than and unranked with x. Equivalently in symbols, writing R for the current proto-ranking and letting

$$L(z) \coloneqq \{w \in \mathcal{X} : z \succ w \text{ and } z \not R\, w \not R\, z\} \quad \text{for each } z \in \mathcal{X},$$

x is the $\succ$-worst x' such that $L(x')$ is non-empty, and y is the R-highest alternative in $L(x)$ (i.e. $y\ R\ z$ for every $z \in L(x) \setminus \{y\}$).[27]

4.2 Insertion sort is efficient

Theorem 1. The insertion-sort strategy is efficient, hence regret-free.

Proof of Theorem 1. Fix a tournament W, and let R be the outcome of the insertion-sort strategy under W. Fix alternatives $x, y \in \mathcal{X}$ with $x \succ y$ and $x\ W\ y$; we must show that $x\ R\ y$.

Enumerate all alternatives $\succ$-worse than x as $z_1\ R\ \cdots\ R\ z_K$, where $z_k = y$ for some $k \leq K$. By definition of insertion sort, x will be pitted against $z_1, z_2, \ldots$ in turn until it wins a vote. If it loses against $z_1, \ldots, z_{k-1}$, then it is pitted against $z_k = y$ and wins since $x\ W\ y$ by hypothesis, so that $x\ R\ y$. If instead x wins against z_ℓ for $\ell < k$, then $x\ R\ z_\ell\ R\ \cdots\ R\ z_k = y$, whence $x\ R\ y$ by transitivity of R. ■

Corollary 2. There exists an efficient, hence regret-free, strategy.

To see intuitively why the insertion-sort strategy does so well, consider again the case of three alternatives depicted in Figure 2 (p. 15). Suppose first that the initial vote on $\{2, 3\}$ went well ($2\ W\ 3$). Offering $\{1, 2\}$ next is a good idea because it affords an opportunity of a favourable imposition of transitivity: if the chair gets lucky ($1\ W\ 2$), then she obtains $1\ R\ 3$ 'for free' from $1\ R\ 2$ and $2\ R\ 3$. (Even though it may be that $3\ W\ 1$.) Offering $\{1, 3\}$ would miss this opportunity.

Suppose instead that the initial vote on $\{2, 3\}$ went badly ($3\ W\ 2$). Offering $\{1, 2\}$ next would risk an unfavourable imposition of transitivity: were the vote to go against her ($2\ W\ 1$), then 3 would be ranked above 1 since $3\ R\ 2$ and $2\ R\ 1$. (Even though it may be that $1\ W\ 3$.)

To summarise, an intuition for why the insertion-sort strategy is regret-free is that it never (1) misses an opportunity for a favourable imposition of transitivity or (2) risks an unfavourable imposition of transitivity. We shall

[27] $L(x)$ has an R-highest element (i.e. y is well-defined) because R is total on $L(x)$. This holds in turn because if $z, w \in L(x)$ were unranked by R, where wlog $z \succ w$, then $L(z)$ would be non-empty, which would contradict the definition of x since $x \succ z$.

see in §5.2 that this intuition is correct: Theorem 3 asserts that regret-free strategies are precisely those that never miss an opportunity or take a risk, in the above sense.

5 Two characterisations of regret-free strategies

Insertion sort is not the only regret-free strategy. In this section, we characterise the class of regret-free strategies twice over. The first characterisation, Theorem 2 (§5.1), is in terms of outcomes: the regret-free strategies are exactly the efficient ones, i.e. those whose outcomes are W-efficient, for each W. This characterisation is tight in the sense that for each W, every W-feasible W-efficient ranking is the outcome under W of some regret-free strategy (Proposition 1).

The second characterisation, Theorem 3 (§5.2), is in terms of behaviour: regret-freeness requires precisely that two intuitive errors be avoided. These errors, *missing an opportunity* and *taking a risk*, formalise the intuition for the regret-freeness of insertion sort given in the previous section. We show in addition that the advice offered by Theorem 3 can be operationalised myopically: avoiding errors ensures that a non-error pair will always be available to be offered next (Proposition 2).

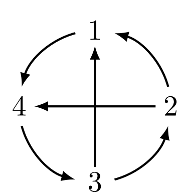
5.1 The outcomes of regret-free strategies

We learned in §3.2 that efficient strategies are regret-free (Corollary 1, p. 14). In fact, the converse is also true:

Theorem 2. A strategy is regret-free iff it is efficient.

The proof, in appendix D, establishes Theorem 2 jointly with Theorem 3 below (§5.2). The 'if' direction was already established in Corollary 1 (p. 14). The following gives a feel for the 'only if' direction:

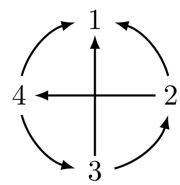
Example 2. Consider alternatives $\mathcal{X} = \{1, 2, 3, 4\}$ with $1 \succ 2 \succ 3 \succ 4$ and the following general will W:[28]



[28]Symbolically, $1 \; W \; 4 \; W \; 3 \; W \; 2 \; W \; 1$, $3 \; W \; 1$ and $2 \; W \; 4$.

There is exactly one W-feasible ranking that ranks 1 above 2, namely $1\ R\ 4\ R\ 3\ R\ 2$.[29] Since no other W-feasible ranking ranks 1 above 2, R is W-unimprovable. But R fails to be W-efficient, for it ranks 4 above 2 despite the fact that $2\ W\ 4$. (Thus the converse of Lemma 1 (p. 13) is false.)

Let σ be a strategy that (i) first offers $\{2, 3\}$, (ii) then, in case 3 won, offers $\{3, 4\}$, and (iii) next, in case 3 and then 4 won, offers $\{1, 4\}$. This strategy has outcome R under W, so fails to be efficient. To see that it is not regret-free, consider a general will W' that differs from W only in that $4\ W'\ 1$:



The outcome of σ under W' is $4\ R'\ 3\ R'\ 2\ R'\ 1$.[30] This ranking fails to be W'-unimprovable since $3\ R''\ 2\ R''\ 4\ R''\ 1$ is W'-feasible and is more aligned with $\succ$ by inspection. Thus σ fails to be regret-free.

The broader insight underlying the 'only if' part of Theorem 2 is that reaching a non-W-efficient outcome necessarily involves a sacrifice. (In the example, σ forgoes the opportunity to rank 2 above 4.) For some general wills, such as W, the sacrifice pays off. (It allows 1 to be ranked above 2, something that no W-feasible W-efficient ranking achieves.) But for other general wills, such as W', no reward materialises, yielding a non-W'-unimprovable outcome.

The characterisation in Theorem 2 is tight, in the following sense:

Proposition 1. For any tournament W and W-feasible W-efficient ranking R, some regret-free strategy has outcome R under W.

Thus no statement sharper than Theorem 2 can be made about the outcomes under W of regret-free strategies: every W-feasible W-efficient ranking is admissible. We give the proof of Proposition 1 in appendix E. A non-trivial argument is required because a given W-feasible W-efficient ranking can be reached in many ways, not all of which form part of a regret-free strategy.[31]

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[29] This can be verified directly. Alternatively, since there is exactly one W-path from 1 to 2 (namely $(1, 4, 3, 2)$), it follows by Observation 1 in appendix B.3 (p. 27).

[30] First $4\ R'\ 3\ R'\ 2$ and $4\ R'\ 1$ are determined. Then $\{1, 2\}$ and $\{1, 3\}$ are offered (in some order that doesn't matter), and 1 loses in both votes.

[31] To see why, return to Example 1 (pp. 10 and 13). The W-efficient ranking $1\ R'\ 3\ R'\ 2$

5.2 The behaviour of regret-free strategies

Recall the intuition provided in §4.2 for why insertion sort is regret-free: it does not miss opportunities for favourable impositions of transitivity, nor does it risk unfavourable impositions of transitivity. Our second characterisation formalises this intuition.

Definition 10. Let R be a non-total proto-ranking, and let $x \succ y$ be unranked alternatives. (I.e. $x, y \in \mathcal{X}$ such that $x \not R\, y \not R\, x$.)

(1) Offering $\{x, y\}$ for a vote *misses an opportunity (at R)* iff there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$ and $y \not R\, z \not R\, x$.

(2) Offering $\{x, y\}$ for a vote *takes a risk (at R)* iff there is an alternative $z \in \mathcal{X}$ such that either

(a) $z \succ y$, $x \mathrel{R} z$ and $y \not R\, z$, or

(b) $x \succ z$, $z \mathrel{R} y$ and $z \not R\, x$.

The 'missed opportunity' in (1) is that $x \mathrel{R} y$ (the hoped-for outcome when offering $\{x, y\}$ for a vote) could potentially have been obtained 'for free' by offering votes on $\{x, z\}$ and $\{z, y\}$, via a 'favourable imposition of transitivity'. The 'risk' in (2)(a) is that if the vote on $\{x, y\}$ were to go badly (so that $y \mathrel{R} x$), then $y \mathrel{R} z$ would follow—an 'unfavourable imposition of transitivity'.

Definition 11. A strategy *never misses an opportunity* (*never takes a risk*) iff it does not miss an opportunity (take a risk) on the path.

Theorem 3. A strategy is regret-free iff it never misses an opportunity or takes a risk.

We give a joint proof of Theorems 2 and 3 in appendix D. The argument is illustrated in Figure 3.

The 'only if' part of Theorem 3 asserts that opportunity-missing and risk-taking are errors, in the sense that committing one of them will lead to a non-W-unimprovable ranking under some W. This is intuitive, but requires some work to show. To appreciate why, suppose that σ offers a pair $x \succ y$ under W, and that doing so either misses an opportunity or takes a risk. It is then easy to find another tournament W' such that the outcome R of σ

may be reached by offering $\{1, 3\}$ and then $\{2, 3\}$. But a strategy that does this cannot be regret-free because it has outcome R' also under the general will $1 \mathrel{W'} 2 \mathrel{W'} 3 \mathrel{W'} 1$, and R' is not a W'-unimprovable ranking (since the more aligned ranking $\succ$ is W'-feasible).

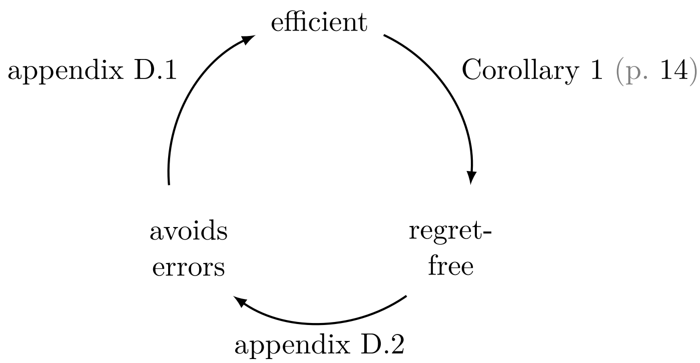


Figure 3 – Proof of Theorems 2 and 3.

under W' ranks y above x, whereas some other W'-feasible ranking R' ranks $x \; R' \; y$. But that is not enough: to ensure that R' is more aligned with $\succ$ than R, it must be that *every* pair $\{z, w\}$ ranked 'right' by R ($z \succ w$ and $z \; R \; w$) is also ranked 'right' by R' ($z \; R' \; w$). We construct W' and R' with these properties in appendix D.2.

The 'if' part of Theorem 3 asserts that these are the *only* errors. Thus to achieve regret-freeness, it suffices to avoid missing opportunities and taking risks, separately after each history. The proof in appendix D.1 begins with an arbitrary non-efficient strategy σ, meaning one whose outcome R under some tournament W ranks some pair $x, y \in \mathcal{X}$ as $y \; R \; x$ even though $x \succ y$ and $x \; W \; y$. The pair $\{x, y\}$ cannot have been voted on (else x would have prevailed), so $y \; R \; x$ must have been determined by an imposition of transitivity. We show that avoiding the two errors suffices to preclude such 'unfavourable' impositions of transitivity, so that σ must have committed the one or the other.

Theorem 3 tells us that non-error pairs are attractive, but it does not promise that they exist. In particular, there could conceivably be histories at which the chair has committed no errors, but is now forced to do so because no unranked pairs remain that can be offered without missing an opportunity or taking a risk. The following proposition rules out this scenario:

Proposition 2. After any history at which the chair has not missed an opportunity or taken a risk, there is an unranked pair of alternatives which can be offered without missing an opportunity or taking a risk.

The proof is in appendix E.

Remark 7. Proposition 2 shows that the characterisation in Theorem 3 of regret-free behaviour is tight: for any W and sequence of pairs that is error-free under W, some regret-free strategy offers this sequence under W.[32]

Theorem 3, augmented by Proposition 2, allows us to give the following simple 'myopic' advice to the chair. After each history, inspect the current proto-ranking to identify an unranked pair of alternatives that would not miss an opportunity or take a risk; that such a pair exists is guaranteed by Proposition 2. Offer any such pair for a vote. By Theorem 3, the outcome will be W-unimprovable whatever the general will W.

6 A characterisation of insertion sort

What, if anything, makes the insertion-sort strategy special? In this section, we show that insertion sort is characterised (up to outcome-equivalence) by a lexicographic property: among all strategies, it optimises the position of the chair's favourite alternative; among such strategies, it optimises the position of the chair's second-favourite alternative; and so on (Theorem 4, §6.2). We further show that recursively applying the *amendment procedure* is outcome-equivalent to insertion sort (Proposition 3, §6.3).

6.1 Outcome-equivalence

We call *outcome-equivalent* those strategies that have the same outcome under every general will:

Definition 12. Two strategies are *outcome-equivalent* iff for every tournament W, they have the same outcome under W.

Say that two strategies σ, σ' *offer the same votes* iff for any tournament W, the pairs offered by σ under W are exactly those offered by σ' (not necessarily in the same order). Strategies that offer the same votes are clearly outcome-equivalent, and the converse is in fact also true.[33]

[32]In particular, a strategy can be constructed which offers this sequence under W and commits no errors under other general wills. Such a strategy is regret-free by Theorem 3.

[33]Suppose that σ (σ') offers (does not offer) some pair $\{x, y\}$ under some tournament W. Let W' agree with W for every pair but $\{x, y\}$. Then σ' has the same outcome under W as under W', whereas σ does not—so σ and σ' are not outcome-equivalent.

6.2 Lexicographic characterisation of insertion sort

Let $R^\sigma(W)$ denote the outcome of a strategy σ under a tournament W. Given an alternative $x \in \mathcal{X}$, write

$$N_x^\sigma(W) := |\{y \in \mathcal{X} : x \succ y \text{ and } x \; R^\sigma(W) \; y\}|$$

for the number of $\succ$-worse alternatives y that end up ranked below x when σ is used under W. Let $\mathcal{W}$ be the set of all tournaments on $\mathcal{X}$.

The chair wishes, for each alternative $x \in \mathcal{X}$, to get each $\succ$-worse alternative y ranked below x. We call one strategy *better for x* than another iff the former tends to rank more such ys below x:

Definition 13. Given an alternative $x \in \mathcal{X}$ and strategies σ, σ', say that σ is *better for x* than σ' iff for each $k \in \{1, \ldots, n-1\}$, we have

$$|\{W \in \mathcal{W} : N_x^\sigma(W) \geq k\}| \geq |\{W \in \mathcal{W} : N_x^{\sigma'}(W) \geq k\}|.$$

In other words, N_x^σ first-order stochastically dominates $N_x^{\sigma'}$.

Definition 14. Given an alternative $x \in \mathcal{X}$ and a set Σ of strategies, a strategy $\sigma \in \Sigma$ is *best for x among Σ* iff it is better for x than any $\sigma' \in \Sigma$.

For each alternative $x \in \mathcal{X}$, the chair prefers that every $\succ$-worse y be ranked below x. Regret-free strategies differ in the priority they give to different (x, y) pairs. The *lexicographic* priority focusses firstly on pairs in which x is the chair's favourite alternative, then on those in which x is her second-favourite alternative, and so on:

Definition 15. Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$. A strategy is *lexicographic* iff among all strategies, it is best for 1; among such strategies, it is best for 2; among such strategies, it is best for 3; and so on.

Theorem 4. A strategy is outcome-equivalent to insertion sort iff it is lexicographic.

The proof is in appendix F.

6.3 The (recursive-)amendment algorithm

In this section, we re-interpret the insertion-sort strategy in terms more familiar in the agenda-setting literature.

Amendment algorithm. Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$. First, pit $n-1$ against n. Then pit $n-2$ against the winner. Then pit $n-3$ against the previous round's winner. And so on. Call the winner of the final round the *final winner*.

The amendment algorithm is designed to choose a single alternative. In particular, the final winner is ranked top, since any other alternative either lost a vote to it, or lost a vote to an alternative that lost to it, etc. Because it approximates procedure in the legislatures of many English-speaking and Scandinavian countries, the amendment procedure has received a great deal of attention in the literature (see the references in §1.1).

The natural way to extend the amendment algorithm to obtain a ranking (rather than a single choice) is the following recursion:

Recursive-amendment algorithm. Label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$. First, run the amendment algorithm on $\mathcal{X}$, and write y_1 for the final winner. Next, run the amendment algorithm on $\mathcal{X} \setminus \{y_1\}$, writing y_2 for the final winner.[34] Then run the amendment algorithm on $\mathcal{X} \setminus \{y_1, y_2\}$ to obtain a final winner y_3. And so on. The resulting ranking is $y_1 \; R \cdots R \; y_{n-1} \; R \; y_n$, where y_n denotes the unique element of $\mathcal{X} \setminus \{y_1, \ldots, y_{n-1}\}$.

Remark 8. Viewed as a sorting algorithm, recursive amendment is exactly *selection sort*—see e.g. Knuth (1998, §5.2.3).

With three alternatives, recursive amendment coincides with insertion sort (Figure 2 on p. 15 describes both). But with four or more alternatives, they differ, as we show by example in supplemental appendix L.

Call any strategy that coincides with the recursive-amendment algorithm on the path an *amendment strategy*. Amendment strategies offer the same votes as insertion sort:

Proposition 3. Any amendment strategy is outcome-equivalent to the insertion-sort strategy.

The proof is in appendix F.

7 Strategic voting

We have assumed history-invariant voting: a voter either votes for x at every history at which $\{x, y\}$ is offered, or else votes for y at every such history. In

[34] When the amendment algorithm demands a vote on a pair $\{x, y\}$ that is already ranked as (wlog) $x \; R \; y$, treat this as if the vote occurred and had outcome $x \; W \; y$.

this section, we investigate the reasonableness of this assumption when voters are strategic. We provide a rationale for a particular type of history-invariant behaviour: *sincere* voting.

In particular, we show that sincere voting is (uniquely) *dominant* in the following sense: any other strategy is not obviously better (is obviously worse) against any (some) profile of strategies of the chair and other voters. Here, 'obviously better' means precisely that the outcome is more aligned with the voter's preference over alternatives.

7.1 Environment

Let each voter $i \in \{1, \dots, I\}$ have a preference $\succ_i$ over the alternatives $\mathcal{X}$. Like the chair, a voter is assumed to weakly prefer rankings that are more aligned with her preference over alternatives.

A *strategy* σ_i of a voter specifies, after each history and for every offered pair $\{x, y\}$, whether x or y should be voted for. Recall that a history records only which pairs were offered and which alternative won in each pair, not who voted how. The definition of a strategy therefore rules out complex path-dependence. We shall return to this point (Remark 9 below).

A strategy is *history-invariant* iff for each pair $x, y \in \mathcal{X}$, either the strategy votes for x whenever $\{x, y\}$ is offered, or else it always votes for y. The leading example of such a strategy is the *sincere* strategy, which instructs a voter to always vote for whichever alternative she likes better.

For a strategy σ of the chair and strategies $\sigma_1, \dots, \sigma_I$ of the voters, write $R(\sigma, \sigma_1, \dots, \sigma_I)$ for the outcome (the ranking that results).

Definition 16. Let σ_i, σ_i' be strategies of voter i, and σ, σ_{-i} strategies of the chair and the other voters. σ_i' is *obviously better than* σ_i against σ, σ_{-i} iff $R(\sigma, \sigma_i', \sigma_{-i})$ is distinct from, and more aligned with $\succ_i$ than, $R(\sigma, \sigma_i, \sigma_{-i})$.

When one strategy is obviously better than another, it yields a better outcome no matter what voter i's exact preference over rankings, given only the weak assumption that voter i weakly prefers rankings more aligned with her preference $\succ_i$ over alternatives. By contrast, comparing strategies that are not related by 'obviously better than' involves trade-offs.

7.2 Sincere voting is dominant

Definition 17. A strategy σ_i of a voter is *dominant* iff for any alternative strategy σ_i',

($\nexists$) there exist no strategies σ, σ_{-i} of the chair and other voters against which σ_i' is obviously better than σ_i, and

($\exists$) there exist strategies σ, σ_{-i} of the chair and other voters against which σ_i is obviously better than σ_i'.

Dominance is strong. (Albeit not as strong as conventional dominance, since 'obviously better' is only a partial ordering.) Observe that there can be at most one dominant strategy. In fact, there is exactly one:

Proposition 4. For each voter, the sincere strategy is (uniquely) dominant.

In words, deviating from sincere voting results in a no better (a worse) outcome against any (some) strategies of the chair and the other voters, in the sense of 'more aligned with $\succ_i$ than'. The proof is in appendix G.

Remark 9. Proposition 4 remains true, with the same proof, if the definition of dominance is strengthened to allow the alternative strategy σ_i' to be an 'extended strategy' that can condition on who voted how in previous periods.

Appendices

A Standard definitions

This appendix collects the definitions of order-theoretic concepts used in this paper. Let $\mathcal{A}$ be a non-empty set, and Q a binary relation on it. Recall that Q is formally a subset of $\mathcal{A} \times \mathcal{A}$, and that '$a \; Q \; b$' is shorthand for '$(a, b) \in Q$'. For $a, b \in \mathcal{A}$, we write $a \not{Q} \; b$ iff it is not the case that $a \; Q \; b$. For $a, b \in \mathcal{A}$ such that $a \; Q \; b$, we denote by $[a, b]_Q$ the *Q-order interval*

$$[a, b]_Q \coloneqq \{a, b\} \cup \{c \in \mathcal{A} : a \; Q \; c \; Q \; b\}.$$

Q is *reflexive* (*irreflexive*) iff $a \; Q \; a$ ($a \not{Q} \; a$) for every $a \in \mathcal{A}$, *total* iff $a \; Q \; b$ or $b \; Q \; a$ for any distinct $a, b \in \mathcal{A}$, *complete* iff it is reflexive and total, *asymmetric* iff $a \; Q \; b$ implies $b \not{Q} \; a$ for $a, b \in \mathcal{A}$, and *transitive* iff $a \; Q \; b \; Q \; c$ implies $a \; Q \; c$ for $a, b, c \in \mathcal{A}$.

The *transitive closure* of Q, denoted $\operatorname{tr} Q$, is the smallest (in the sense of set inclusion) transitive relation that contains Q.[35] The *strict part* of Q is the binary relation $\operatorname{str} Q$ such that $a \operatorname{str} Q \; b$ iff $a \; Q \; b$ and $b \not{Q} \; a$. For two binary relations Q and Q' on $\mathcal{A}$, Q' is an *extension* of Q iff both $Q \subseteq Q'$ and $\operatorname{str} Q \subseteq \operatorname{str} Q'$.

[35]Every relation possesses a transitive closure because the maximal relation $\mathcal{A} \times \mathcal{A}$ is transitive and the intersection of transitive relations is transitive.

B Additional material for §2

This appendix complements the exposition of the environment in §2. We formally define histories, strategies, outcomes and paths (§B.1), show that all and only tournaments can be general wills (§B.2), and provide characterisations of W-feasibility (§B.3) and of 'more aligned with than' (§B.4).

B.1 Formal definitions (§2.1–§2.2)

In this appendix, we formally define histories, strategies, outcomes and paths. A history is a sequence $((x_t, y_t))_{t=1}^T$, where $\{x_t, y_t\}$ is the pair offered in period t, and x_t is the winner (i.e. $x_t \; W \; y_t$):

Definition 18. There is exactly one *history of length* 0 (the 'empty history'). A *history of length* $T \in \mathbf{N}$ is a sequence $((x_t, y_t))_{t=1}^T$ of ordered pairs of alternatives satisfying $x_t \neq y_t$ and $x_t \not R_{t-1} \; y_t \not R_{t-1} \; x_t$ for each $t \in \{1, \dots, T\}$, where $R_0 := \varnothing$ and $R_t := \operatorname{tr}(R_{t-1} \cup \{(x_t, y_t)\})$ for each $t \in \{1, \dots, T\}$. (Here 'tr' denotes the transitive closure.) The history is *terminal* iff R_T is total.

Let $\mathcal{H}_T$ be the set of all non-terminal histories of length $T \geq 0$, and write $\mathcal{H} := \bigcup_{T=0}^{\infty} \mathcal{H}_T$ for all non-terminal histories.[36] A strategy offers, after each non-terminal history, a pair of alternatives that are unranked at that history:

Definition 19. A *strategy of the chair* is a map $\sigma : \mathcal{H} \to 2^{\mathcal{X}}$ such that for each non-terminal history $h \in \mathcal{H}$, we have $\sigma(h) = \{x, y\}$ for some alternatives $x, y \in \mathcal{X}$ such that $(h, (x, y))$ and $(h, (y, x))$ are histories.

A strategy σ and a tournament W generate a terminal history $((x_t, y_t))_{t=1}^T$ as follows: for each $t \in \{1, \dots, T\}$, (x_t, y_t) is given by

$$\{x_t, y_t\} := \sigma\left(((x_s, y_s))_{s=1}^{t-1}\right) \quad \text{and} \quad x_t \; W \; y_t.$$

This history is associated with a sequence of proto-rankings $(R_t)_{t=0}^T$, as outlined in Definition 18 above.[37]

Definition 20. The *outcome* of a strategy σ under a tournament W is the ranking R_T associated with the terminal history they generate.

A strategy and a tournament also generate a sequence of non-terminal histories—namely, all truncations of their generated terminal history. If a non-terminal history is generated by σ and some tournament, we say that it belongs to the *path* of σ.

[36] Histories can only be so long: $\mathcal{H}_T$ is empty for (and only for) $T \geq |\mathcal{X}| \times (|\mathcal{X}| - 1)/2$.

[37] Namely, $R_0 = \varnothing$ and $R_t = \operatorname{tr}(R_{t-1} \cup \{(x_t, y_t)\}) = \operatorname{tr}\left(\bigcup_{s=1}^{t} \{(x_s, y_s)\}\right)$ for $t \geq 1$.

B.2 Which binary relations are general wills? (§2.2)

In this appendix, we prove that all and only tournaments are legitimate general wills. A *voting behaviour* V_i specifies for each pair $x, y \in \mathcal{X}$ whether voter i will vote for x ($x \ V_i \ y$) or for y ($y \ V_i \ x$). Formally:

Definition 21. A *voting behaviour* is a tournament. A *voting profile* is a collection $(V_i)_{i=1}^I$ of voting behaviours.

The general will of a voting profile $(V_i)_{i=1}^I$ is the tournament W such that $x \ W \ y$ iff $x \ V_i \ y$ for a majority of voters $i \in \{1, \dots, I\}$. The following shows that all (and only) tournaments are the general will of some voting profile $(V_i)_{i=1}^I$, even if we insist that each voter's behaviour V_i be transitive (meaning that it can be rationalised as sincere):

Fact 1. For a binary relation W on $\mathcal{X}$, the following are equivalent:

(1) W is a tournament.

(2) For every $I \in \mathbf{N}$ odd, W is the general will of some profile $(V_i)_{i=1}^I$ of voting behaviours.

(3) For some $I \in \mathbf{N}$ odd, W is the general will of some profile $(V_i)_{i=1}^I$ of *transitive* voting behaviours.

Proof. It is immediate that (2) and (3) (separately) imply (1). To see that (1) implies (2), simply observe that a tournament W is the general will of the voting profile $(V_i)_{i=1}^I = (W)_{i=1}^I$ for any $I \in \mathbf{N}$. The fact that (1) implies (3) follows from McGarvey's (1953) theorem. ■

B.3 A characterisation of W-feasibility (§2.3)

This appendix contains a characterisation of feasibility used in our proofs.

Definition 22. Let R be a ranking. Alternatives $x, y \in \mathcal{X}$ with $x \ R \ y$ are *R-adjacent* iff there is no $z \in \mathcal{X}$ such that $x \ R \ z \ R \ y$.

Observation 1. Let W be a tournament, and R a ranking. The following are equivalent:

(1) R is W-feasible.

(2) For any R-adjacent $x, y \in \mathcal{X}$ with $x \ R \ y$, we have $x \ W \ y$.

Condition (2) admits a graph-theoretic interpretation. Think of W as a directed graph with vertices $\mathcal{X}$ and a directed edge from x to y iff $x \; W \; y$ (as in Example 1 on p. 10), and think of a ranking R as a sequence of alternatives: the highest-ranked, the second-highest-ranked, and so on.[38] Condition (2) requires precisely that R be a directed path in W.

Proof. (2) implies (1): Let R satisfy (2). Then R is the outcome under W of any strategy that offers a vote on each R-adjacent pair of alternatives.

(1) implies (2): Let R be W-feasible, and let $x, y \in \mathcal{X}$ be R-adjacent with $x \; R \; y$. By W-feasibility, there is a strategy σ whose outcome under W is R. Along its induced history, it is determined that $x \; R \; y$. Since x, y are R-adjacent, this cannot be via an imposition of transitivity. So it must occur in a vote on $\{x, y\}$, in which x wins—thus $x \; W \; y$. ■

B.4 A characterisation of 'more aligned with than' (§2.4)

In this appendix, we provide a characterisation of 'more aligned with than', and use it to prove the claims made in §2.4 (p. 10) about the applications.

Observation 2. For rankings $\succ$, R and R', the following are equivalent:[39]

(1) R is more aligned with $\succ$ than R'.

(2) For every non-empty set $X \subseteq \mathcal{X}$, the R-highest alternative in X is identical to or $\succ$-better than the R'-highest alternative in X.

Proof. (1) implies (2): We prove the contra-positive. Suppose that R, R' do not satisfy (2), so that there is a non-empty $X \subseteq \mathcal{X}$ whose R-highest element x is (strictly) $\succ$-worse than its R'-highest element x'. Then R' ranks x, x' 'right' ($x' \succ x$ and $x' \; R' \; x$) whereas R ranks them 'wrong' ($x \; R \; x'$), so R is not more aligned with $\succ$ than R'.

(2) implies (1): We prove the contra-positive. Suppose that R is not more aligned with $\succ$ than R', so that there are alternatives $x, x' \in \mathcal{X}$ with $x' \succ x$, $x' \; R' \; x$ and $x \; R \; x'$. Then the R-highest alternative in $X \coloneqq \{x, x'\}$ is (strictly) $\succ$-worse than the R'-highest. ■

Hiring (continued)**.** We claimed that a more aligned ranking is precisely one that hires a weakly $\succ$-better candidate at every realisation of uncertainty. This follows immediately from Observation 2.

[38] Formally: identify R with the sequence $(x_k)_{k=1}^{|\mathcal{X}|}$ such that $x_1 \; R \; x_2 \; R \cdots R \; x_{|\mathcal{X}|}$.

[39] This is an instance of the Milgrom–Shannon (1994) comparative statics theorem: viewing $(\mathcal{X}, \succ)$ as an ordered set of actions and R, R' as (strict) preferences, (1) says that R single-crossing dominates R', and (2) says that R chooses higher actions than R' does.

Party lists (continued)**.** Enrich the model so that only a random subset $X \subseteq \mathcal{X}$ of candidates is available.[40] We claim that R is more aligned with $\succ$ than R' iff for every realisation (K, X) and every $k \leq K$, the k^{th} $\succ$-best candidate hired by R is weakly $\succ$-better than the k^{th} $\succ$-best hired by R'.

To prove the 'only if' part, let R be more aligned with $\succ$ than R', and fix an $X \subseteq \mathcal{X}$ and a K. Assume without loss of generality that $K \leq |X|$. Label the candidates $\{x_1, \ldots, x_K\}$ hired by R under X so that $x_1 \; R \cdots R \; x_K$, and similarly write $x'_1 \; R' \cdots R' \; x'_K$ for those hired by R'. We must show that $x_k \succeq x'_k$ for every $k \in \{1, \ldots, K\}$. We proceed by strong induction on k. The base case $k = 1$ is immediate from Observation 2.

For the induction step, suppose for some $k \in \{2, \ldots, K\}$ that $x_\ell \succeq x'_\ell$ for every $\ell < k$. Define $Y \coloneqq X \setminus \{x_1, \ldots, x_{k-1}, x'_1, \ldots, x'_{k-1}\}$. We have $x_\ell \succeq x'_\ell \succ x'_k$ for every $\ell < k$ by the induction hypothesis. It follows that $x'_k \in Y$, and hence that x'_k is the R'-highest alternative in Y. If $x_k \in Y$, then x_k is the R-highest alternative in Y, whence $x_k \succeq x'_k$ by Observation 2. If instead $x_k \notin Y$, then $x_k = x'_\ell \succ x'_k$ for some $\ell < k$.

For the 'if' part, we prove the contra-positive. Suppose that R is not more aligned with $\succ$ than R'. Then by Observation 2, there is a subset $X' \subseteq \mathcal{X}$ such that R hires a strictly $\succ$-worse candidate than R' at the realisation $(K, X) = (1, X')$ of uncertainty.

C An extension lemma

This appendix presents an extension lemma that is used in the proofs of Theorems 2 and 3 in appendix D and of Proposition 4 in appendix G. Recall from appendix A the definitions of 'extension' and of order intervals $[x, y]_R$.

Extension lemma. Let R be a proto-ranking, and let $A \subseteq \mathcal{X}$ be such that $[x, y]_R \subseteq A$ for all $x, y \in A$ with $x \; R \; y$. Then the binary relation $R \cup A^2$ admits a complete and transitive extension.

The extension lemma is used directly in the proof of Theorems 2 and 3 (in particular, in the proof of Lemma 4 in appendix D.2). Before proving it, we deduce the corollary used in the proof in appendix G of Proposition 4.

Extension corollary. Let R be a proto-ranking, and let $x, y \in \mathcal{X}$ be distinct alternatives such that $x \not\!R \; y \not\!R \; x$. Then there is a ranking $R' \supseteq R$ such that x, y are R'-adjacent with $x \; R' \; y$.

That is, any proto-ranking that does not rank $x, y \in \mathcal{X}$ can be extended to a ranking that ranks x above y, with nothing in-between.

[40]The grand set $X = \mathcal{X}$ can occur with high probability, if desired.

Proof. Let R and $x, y \in \mathcal{X}$ satisfy the hypothesis. Then $R \cup \{x, y\}^2$ admits a complete and transitive extension Q by the extension lemma. Note that $x \; Q \; y \; Q \; x$. It follows that by appropriately breaking indifferences in Q, we may obtain a ranking $R' \supseteq R \cup \{(x, y)\}$ such that x, y are R'-adjacent with $x \; R' \; y$. ■

To prove the extension lemma, we use a theorem due to Suzumura (1976).

Definition 23. A binary relation Q on a set $\mathcal{A}$ is *Suzumura-consistent* iff for $a_1, \ldots, a_K \in \mathcal{A}$, $a_1 \; Q \; a_2 \; Q \cdots Q \; a_{K-1} \; Q \; a_K$ implies that either $a_1 \; Q \; a_K$ or $a_1 \not{Q} \; a_K \not{Q} \; a_1$.

Suzumura's extension theorem. A binary relation admits a complete and transitive extension iff it is Suzumura-consistent.

See Bossert and Suzumura (2010, p. 45) for a proof.

Proof of the extension lemma. Let R and A satisfy the hypothesis; we seek a complete and transitive extension of the relation $Q := R \cup A^2$. By Suzumura's extension theorem, it suffices to show that for any finite sequence of alternatives $(w_k)_{k=1}^K$ such that $w_1 \; Q \cdots Q \; w_K$, we have either $w_1 \; Q \; w_K$ or $w_1 \not{Q} \; w_K \not{Q} \; w_1$. There are two cases.

Case 1: $w_k \; R \; w_{k+1}$ for every $k < K$. Then $w_1 \; R \; w_K$ since R is transitive (being a proto-ranking), so $w_1 \; Q \; w_K$.

Case 2: $\{w_k, w_{k+1}\} \subseteq A$ for some $k < K$. Let k' (k'') be the smallest (largest) such $k < K$, so that $w_1 \; R \cdots R \; w_{k'}$ if $k' > 1$ and $w_{k''+1} \; R \cdots R \; w_K$ if $k'' < K - 1$. Assume toward a contradiction that $w_K \; Q \; w_1$ and $w_1 \not{Q} \; w_K$. Since $\{w_1, w_K\} \nsubseteq A$ (as otherwise $w_1 \; Q \; w_K \; Q \; w_1$), it must be that $w_K \; R \; w_1$. It follows by transitivity of R that $w_{k''+1} \; R \; w_K \; R \; w_1 \; R \; w_{k'}$, and thus $\{w_K, w_1\} \subseteq [w_{k''+1}, w_{k'}]_R$. Note that $[w_{k''+1}, w_{k'}]_R \subseteq A$ since $w_{k''+1}, w_{k'} \in A$ by construction. Therefore $\{w_K, w_1\} \subseteq A$, which implies that $w_K \; Q \; w_1 \; Q \; w_K$—a contradiction. ■

D Proof of Theorems 2 and 3 (§5, pp. 17 and 19)

We prove Theorems 2 and 3 jointly, in the manner depicted in Figure 3 (p. 20). We already showed that efficiency implies regret-freeness (Corollary 1, p. 14). We shall establish the other two parts in §D.1 and §D.2.

D.1 Proof that error-avoiding strategies are efficient

The proof relies on two intermediate results, Lemma 2 and Corollary 3 below. We first require an abstract fact about the transitive closure operation:

Observation 3. Consider a proto-ranking R and unranked alternatives $x, y \in \mathcal{X}$ (i.e. $x \not R\ y \not R\ x$). Let R' be the transitive closure of $R \cup \{(z, w)\}$, and suppose that $y\ R'\ x$. Then (a) either $y\ R\ z$ or $y = z$, and (b) either $w\ R\ x$ or $w = x$.

Proof. Since R' is the transitive closure of $R \cup \{(z, w)\}$ and $y\ R'\ x$, there must be a sequence $(z_k)_{k=1}^K$ of alternatives with $z_1 = y$, $z_K = x$ and

$$(z_k, z_{k+1}) \in R \cup \{(z, w)\} \quad \text{for every } k < K.$$

Since $y \not R\ x$ and R is transitive, we must have $(z_k, z_{k+1}) = (z, w)$ for some $k < K$. The result follows since R is transitive. ■

Definition 24. Let R be a proto-ranking. An ordered pair of alternatives $(x, y) \in \mathcal{X}$ is a *missed opportunity in R* iff there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$, $y\ R\ x$ and $y \not R\ z \not R\ x$.

Lemma 2. Consider a proto-ranking R that contains no missed opportunities. Let $x \succ y$ be alternatives with $y \not R\ x$. Suppose that offering $\{z, w\}$ does not miss an opportunity or take a risk at R, and that doing this leads to a proto-ranking R' such that $y\ R'\ x$. Then $\{z, w\} = \{x, y\}$.

That is: if no errors are made, then a misfortune $y\ R'\ x$ can occur only as the result of a vote on $\{x, y\}$, not via an imposition of transitivity. This captures the idea that avoiding errors is enough to preclude unfavourable impositions of transitivity.

Proof. Let R, x, y, z, w and R' satisfy the hypothesis of the lemma, and assume wlog that $z \succ w$. We must show that $z = x$ and $w = y$.

Claim. $w\ R'\ z$.

Proof of the claim. Suppose toward a contradiction that $w \not R'\ z$. We will show that R contains a missed opportunity.

Since R' is induced from R by offering $\{z, w\}$, and $w \not R'\ z$, it must be that R' is the transitive closure of $R \cup \{(z, w)\}$. Since $y \not R\ x$ and $y\ R'\ x$, it follows by Observation 3 that (a) either $y\ R\ z$ or $y = z$, and (b) either $w\ R\ x$ or $w = x$. Now consider two cases.

Case 1: $z \succ x$ or $z = x$. We will show that $z \succ x \succ y$, $y\ R\ z$, and $y \not R\ x \not R\ z$, so that (z, y) is a missed opportunity in R. Both $x \succ y$ and $y \not R\ x$ hold by hypothesis. For $x \not R\ z$, suppose to the contrary that $x\ R\ z$; then since $w\ R\ x$ or $w = x$ by property (b), we have $w\ R\ z$ by transitivity of R, a contradiction.

To obtain $y\ R\ z$, observe that $z \succ y$ since by hypothesis $z \succ x$ or $z = x$, and we know that $x \succ y$ and that $\succ$ is transitive. Thus $z \neq y$, whence $y\ R\ z$ follows by property (a). To see that $z \succ x$, simply note that $z = x$ is impossible because $y\ R\ z$ and $y \not R\ x$.

Case 2: $x \succ z$. We will show that $x \succ z \succ w$, $w\ R\ x$, and $w \not R\ z \not R\ x$, so that (x, w) is a missed opportunity in R. Both $x \succ z \succ w$ and $w \not R\ z$ hold by hypothesis. For $z \not R\ x$, suppose to the contrary that $z\ R\ x$; then since $y\ R\ z$ or $y = z$ by property (a), we have $y\ R\ x$ by transitivity of R, a contradiction.

To obtain $w\ R\ x$, observe that $x \succ w$ since by hypothesis $x \succ z$, and we know that $z \succ w$ and that $\succ$ is transitive. Thus $w \neq x$, whence $w\ R\ z$ follows by property (b). □

In light of the claim, R' must be the transitive closure of $R \cup \{(w, z)\}$. Since $y \not R\ x$ and $y\ R'\ x$, applying Observation 3 yields that (a) either $y\ R\ w$ or $y = w$, and (b) either $z\ R\ x$ or $z = x$.

We claim that

$$z \neq x \succ w \tag{1}$$

is impossible. Suppose toward a contradiction that (1) holds; we will show that offering $\{z, w\}$ takes a risk at R, i.e. that $x \succ w$, $z\ R\ x$ and $w \not R\ x$. We have $x \succ w$ by (1), and $z\ R\ x$ by (1) and property (a). To see that $w \not R\ x$, suppose to the contrary that $w\ R\ x$; then since $y\ R\ w$ or $y = w$ by property (a), it follows by transitivity of R that $y\ R\ x$, a contradiction.

Now suppose toward a contradiction that $\{z, w\} \neq \{x, y\}$. We claim that

$$z \succ y \neq w. \tag{2}$$

If $z = x$, then $z \succ y$ is immediate, and $w \neq y$ follows since $\{z, w\} \neq \{x, y\}$ by hypothesis. Suppose instead that $z \neq x$. Since (1) cannot hold, it must be that either $w \succ x$ or $w = x$. Since $x \succ y$, it follows by transitivity of $\succ$ that $w \succ y$, so that $w \neq y$. Furthermore, since $z \succ w$, transitivity of $\succ$ yields $z \succ y$. So (2) holds.

It remains to derive a contradiction from $\{z, w\} \neq \{x, y\}$, using the fact that (2) must hold. We shall show that $z \succ y$, $y\ R\ w$ and $y \not R\ z$, so that offering $\{z, w\}$ takes a risk at R. We obtain $z \succ y$ from (2), and $y\ R\ w$ from (2) and property (a). And it must be that $y \not R\ z$ because $y\ R\ z$ together with property (b) and transitivity of R imply the contradiction $y\ R\ x$. ■

Corollary 3. Suppose that R contains no missed opportunities, and that offering $\{z, w\}$ (where $z \not R\ w \not R\ z$) does not miss an opportunity or take a risk at R. Then the proto-ranking R' induced by offering $\{z, w\}$ contains no missed opportunities.

Proof. Let R, z, w and R' be as in the hypothesis of the lemma, and suppose toward a contradiction that there is a missed opportunity (x, y) in R'.

We claim that $y \not R\, x$ and $y\ R'\ x$, so that Lemma 2 is applicable. It must be that $y \not R\, x$, for otherwise (x, y) would be a missed opportunity in R. That $y\ R'\ x$ is immediate from the fact that (x, y) is a missed opportunity in R'.

It follows by Lemma 2 that $\{z, w\} = \{x, y\}$. But since (x, y) is a missed opportunity in R', there is an alternative $z' \in \mathcal{X}$ such that $x \succ z' \succ y$ and $y \not R'\, z' \not R'\, x$, and thus $y \not R\, z' \not R\, x$ since $R \subseteq R'$. Thus offering $\{z, w\} = \{x, y\}$ misses an opportunity at R—a contradiction. ■

Armed with Lemma 2 and Corollary 3, we are ready to prove that error-avoiding strategies are efficient.

Proposition. A strategy that never misses an opportunity or takes a risk is efficient.

Proof. Take a strategy σ that is not efficient, and suppose that it never misses an opportunity or takes a risk; we shall derive a contradiction. Since σ is not efficient, there exists a tournament W such that the outcome R of σ under W fails to be W-efficient, which is to say that $x \succ y$, $x\ W\ y$ and $y\ R\ x$ for some alternatives $x, y \in \mathcal{X}$.

Write $\varnothing = R_0 \subseteq R_1 \subseteq \cdots \subseteq R_{T'} = R$ for the sequence of proto-rankings associated with the terminal history generated by σ and W. Let $T \leq T'$ be the first period in which x, y are ranked ($y \not R_{T-1}\, x \not R_{T-1}\, y$ and $y\ R_T\ x$). Since $x\ W\ y$ and $y\ R_T\ x$, it cannot be that $\{x, y\}$ is voted on in period T.

Because $R_0 = \varnothing$ contains no missed opportunities and σ never misses an opportunity or takes a risk, Corollary 3 promises that R_{T-1} contains no missed opportunities. Thus by Lemma 2, σ must offer $\{x, y\}$ in period T—a contradiction. ■

D.2 Proof that regret-free strategies avoid errors

The proof relies on two lemmata. For the first, recall from appendix A the notation $[x, y]_R$ for R-order intervals.

Lemma 3. Given a pair of alternatives $x, y \in \mathcal{X}$, let R' be a ranking such that $x\ R'\ y$, and let W be a tournament that agrees with R' on every pair $\{z, w\} \nsubseteq [x, y]_{R'}$. Then the outcome under W of any strategy agrees with R' on every pair $\{z, w\} \nsubseteq [x, y]_{R'}$.

Proof. Let x, y, R' and W satisfy the hypothesis, and let R be the outcome under W of some strategy of the chair.

Claim. Let $\{z, w\} \nsubseteq [x, y]_{R'}$ be such that (a) $z\ R'\ w$ and (b) $x\ R'\ z$ or $w\ R'\ y$. Then $z\ R\ w$.

Proof of the claim. Assume that $x\ R'\ z$; we omit the similar argument for the case $w\ R'\ y$. Suppose toward a contradiction that $w\ R\ z$. Label the alternatives $[w, z]_R \equiv \{x_1, \ldots, x_K\}$ so that

$$w = x_1\ R \cdots R\ x_K = z.$$

Since R is W-feasible, we have $x_1\ W\ \cdots\ W\ x_K$ by Observation 1 (appendix B.3, p. 27).

It must be that $x\ R'\ w$, since otherwise $z\ R'\ w$ and the transitivity of R' would produce the contradiction $z\ R'\ x$. We thus have $x\ R'\ w$ and $w \notin [x, y]_{R'}$, which means that $y\ R'\ w$.

Suppose that $y\ R'\ x_k$ for every $k < K$. Then $\{x_k, x_{k+1}\} \nsubseteq [x, y]_{R'}$ for every $k < K$. Since R' agrees with W on pairs $\{z', w'\} \nsubseteq [x, y]_{R'}$, it follows that $x_k\ R'\ x_{k+1}$ for every $k < K$, whence $w\ R'\ z$ by transitivity of R'—a contradiction.

Suppose instead that $x_{k'}\ R'\ y$ for some $k' < K$, and let k be the smallest such k'. Since $y\ R'\ w = x_1$, it must be that $k > 1$. By definition of k, we have $x_k\ R'\ y\ R'\ x_{k-1}$. On the one hand, the transitivity of R' demands that $x_k\ R'\ x_{k-1}$. On the other hand, since $\{x_{k-1}, x_k\} \nsubseteq [x, y]_{R'}$ (because $x_{k-1}\ \not R'\ y$ by definition of k) and $x_{k-1}\ W\ x_k$, we must have $x_{k-1}\ R'\ x_k$—a contradiction. $\square$

Now fix a pair $\{z, w\} \nsubseteq [x, y]_{R'}$ such that $z\ R'\ w$; we must show that $z\ R\ w$. If either $x\ R'\ z$ or $w\ R'\ y$, then $z\ R\ w$ follows from the claim.

Suppose instead that $z\ R'\ x$ and $y\ R'\ w$. Observe that $\{z, x\} \nsubseteq [x, y]_{R'}$, (a) $z\ R'\ x$ and (b) $x\ R'\ y$. We may therefore apply the claim to $\{z, x\}$ to obtain $z\ R\ x$. Similarly applying the claim to $\{x, w\}$ yields $x\ R\ w$, whence $z\ R\ w$ follows by transitivity of R. ■

Lemma 4. Let R be a proto-ranking, and let $x, y, z \in \mathcal{X}$ be such that

$$\{x, y, z\}^2 \cap R \subseteq \{(x, z)\}.$$

Then there exists a ranking $R' \supseteq R \cup \{(x, z), (z, y)\}$ such that $[x, y]_{R'} = [x, z]_R \cup \{y\}$.

To interpret the conclusion, observe that the properties of R' are equivalent to the following: (a) $x\ R'\ z$ and $[x, z]_{R'} = [x, z]_R$, and (b) $z\ R'\ y$ and

$[z,y]_{R'} = \{z,y\}$.[41] In words, Lemma 4 runs as follows. Suppose that a proto-ranking R ranks x above z, and has nothing else to say about $\{x,y,z\}$.[42] Call the (possibly empty) set of alternatives ranked below x and above z (i.e. $[x,z]_R \setminus \{x,z\}$) the 'in-between set'. The lemma asserts that R may be extended to a ranking R' that (i) adds nothing to the in-between set ($[x,z]_{R'} = [x,z]_R$) and that (ii) ranks y immediately below z ($z \; R' \; y$ and $[z,y]_{R'} = \{z,y\}$).

The proof of Lemma 4 relies on the extension lemma from appendix C.

Proof. Let a proto-ranking R and alternatives $x,y,z \in \mathcal{X}$ satisfy the hypothesis. Define $A \coloneqq [x,z]_R \cup \{y\}$.

Claim. For any $x', y' \in A$ such that $x' \; R \; y'$, we have $[x',y']_R \subseteq A$.

Proof of the claim. Fix alternatives $x', y' \in A$ with $x' \; R \; y'$. By definition of A, it suffices to show that $\{x', y'\} \not\ni y$. So suppose toward a contradiction that $x' = y$; the other case is similar. We have $y' \neq y$ since $x' \; R \; y'$ and R is irreflexive (being a proto-ranking). Since $y' \in A$, it follows that $y' \in [x,z]_R$. By $x' \; R \; y'$ and the transitivity of R, we obtain $y = x' \; R \; z$. But $y \not\!R \; z$ by hypothesis—a contradiction. $\square$

By the claim, the extension lemma in appendix C (p. 29) is applicable, so there exists a complete and transitive extension Q of the binary relation $R \cup A^2$. Since $z' \; Q \; w' \; Q \; z'$ for any $z', w' \in A$, we have in particular that $w \; Q \; y \; Q \; w$ for any $w \in [x,z]_R$. We may therefore obtain the desired ranking R' by appropriately breaking indifferences in Q. $\blacksquare$

With Lemmata 3 and 4 in hand, we are ready to prove that regret-free strategies avoid errors.

Proposition. A regret-free strategy never misses an opportunity or takes a risk.

[41] The formal proof of equivalence is as follows. If a ranking R' has properties (a) and (b), then $x \; R' \; z \; R' \; y$ by (a) and (b), and

$$\begin{aligned}
[x,y]_{R'} &= [x,z]_{R'} \cup [z,y]_{R'} \\
&= [x,z]_{R'} \cup \{y\} && \text{since } [z,y]_{R'} = \{y\} \text{ by (b)} \\
&= [x,z]_R \cup \{y\} && \text{since } [x,z]_{R'} = [x,z]_R \text{ by (a).}
\end{aligned}$$

If a ranking R' has the properties in the lemma, then $x \; R' \; z \; R' \; y$, so

$$[x,z]_{R'} \cup [z,y]_{R'} = [x,y]_{R'} = [x,z]_R \cup \{y\}.$$

Hence $[x,z]_{R'} = [x,z]_R$ (whence (a)) and $[z,y]_{R'} = \{z,y\}$ (whence (b)).

[42] For simplicity, neglect the case $\{x,y,z\}^2 \cap R = \varnothing$.

Proof. We shall prove the contra-positive. To that end, fix a strategy σ and a tournament W such that σ misses an opportunity or takes a risk under W. We shall construct a tournament W' such that the outcome R of σ under W' fails to be W'-unimprovable. In particular, we shall find a W'-feasible ranking $R' \neq R$ that is more aligned with $\succ$ than R.

Let T be the first period in which σ either misses an opportunity or takes a risk under W. Write R_{T-1} for the associated end-of-period-$(T-1)$ proto-ranking, and let $\{x, y\}$ be the pair offered in period T.

We shall consider three cases, based on the behaviour of σ under W in period T. By hypothesis, $\{x, y\}$ either misses an opportunity or takes a risk at R_{T-1}. If $\{x, y\}$ misses an opportunity, there is an alternative $z \in \mathcal{X}$ such that $x \succ z \succ y$ and $y \not R_{T-1}\ z \not R_{T-1}\ x$. It cannot be that $x\ R_{T-1}\ z\ R_{T-1}\ y$, as this would imply that $x\ R_{T-1}\ y$, contradicting the fact that $\{x, y\}$ is offered in period T. Thus one of the following must hold:

(a) $x \not R_{T-1}\ z \not R_{T-1}\ y$,

(b) $x\ R_{T-1}\ z \not R_{T-1}\ y$, or

(c) $x \not R_{T-1}\ z\ R_{T-1}\ y$.

If instead $\{x, y\}$ takes a risk, then there is a $z \in \mathcal{X}$ such that either

(d) $z \succ y$, $x\ R_{T-1}\ z$, and $y \not R_{T-1}\ z$, or

(e) $x \succ z$, $z\ R_{T-1}\ y$, and $z \not R_{T-1}\ x$.

This yields three cases, as follows. Case 1 is (a). Case 2 encompasses both (b) and (d) under the (slightly more general) hypothesis that 'there exists a $z \in \mathcal{X}$ such that $z \succ y$, $z \not R_{T-1}\ y \not R_{T-1}\ z$, and $x\ R_{T-1}\ z$'. Finally, case 3 encompasses (c) and (e) under the hypothesis that 'there exists a $z \in \mathcal{X}$ such that $x \succ z$, $z \not R_{T-1}\ x \not R_{T-1}\ z$ and $z\ R_{T-1}\ y$. Since cases 2 and 3 are analogous, we omit the proof for the latter.

Case 1: $\exists z \in \mathcal{X}$ such that $x \succ z \succ y$ and $\{x, z, y\}^2 \cap R_{T-1} = \varnothing$. By Lemma 4, there is a ranking R' such that

$$R' \supseteq R_{T-1} \cup \{(x, z), (z, y)\} \quad \text{and} \quad [x, y]_{R'} = \{x, y, z\}.$$

Define W' to equal R', except that $y\ W'\ x$. Clearly W' is a tournament, and R' is W'-feasible by Observation 1 (appendix B.3, p. 27) since x, y are not R'-adjacent. Denote by R the outcome of σ under W'. It remains to show that $R \neq R'$, and that R' is more aligned with $\succ$ than R.

For the former, since $x\ R'\ y$, it suffices to show that $y\ R\ x$. To this end, observe that that $R_{T-1} \subseteq W'$. Thus the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W, which means in particular that $\{x, y\}$ is offered in period T. Since $y\ W'\ x$, it follows that $y\ R\ x$, as desired.

To show that R' is more aligned with $\succ$ than R, observe that W' agrees with R' on every pair $\{w, w'\} \nsubseteq \{x, y, z\} = [x, y]_{R'}$. It follows by Lemma 3 that R and R' agree on every pair $\{w, w'\} \nsubseteq \{x, y, z\}$. Since $x \succ z \succ y$ and $x\ R'\ z\ R'\ y$, it follows that R' is more aligned with $\succ$ than R.

Case 2: $\exists z \in \mathcal{X}$ such that $z \succ y$, $z\ \not{R}_{T-1}\ y\ \not{R}_{T-1}\ z$ and $x\ R_{T-1}\ z$. We shall begin with an auxiliary ranking R'', then use it to construct our tournament W' and ranking R'. By Lemma 4, there is a ranking

$$R'' \supseteq R_{T-1} \cup \{(x, z), (z, y)\}$$

such that

$$[x, y]_{R''} = [x, z]_{R_{T-1}} \cup \{y\}. \tag{3}$$

Define

$$X \coloneqq \{w \in [x, z]_{R_{T-1}} \setminus \{x\} : w \succ y\},$$

and let W' be such that

(i) $w\ W'\ y$ for every $w \in X$,

(ii) $y\ W'\ w$ for every $w \in [x, z]_{R_{T-1}} \setminus X$, and

(iii) W' agrees with R'' on every other pair.

Denote by R the outcome of σ under the tournament W'.

Observe that (i) $y\ \not{R}_{T-1}\ w$ for every $w \in X$ (since otherwise $y\ R_{T-1}\ w\ R_{T-1}\ z$, contradicting the case-2 hypothesis), (ii) $w\ \not{R}_{T-1}\ y$ for every $w \in [x, z]_{R_{T-1}} \setminus X$ (otherwise either $x = w\ R_{T-1}\ y$ or $x\ R_{T-1}\ w\ R_{T-1}\ y$, whence $x\ R_{T-1}\ y$), and (iii) $R_{T-1} \subseteq R''$. Thus $R_{T-1} \subseteq W'$ by definition of the latter. It follows that the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W, which means in particular that $\{x, y\}$ is offered in period T. Since $y\ W'\ x$, we thus have $y\ R\ x$.

Since $X \subseteq [x, y]_{R_{T-1}} \subseteq [x, y]_{R''}$ (by definition of X and (3)), W' agrees with R'' on every pair $\{w, w'\} \nsubseteq [x, y]_{R''}$. It follows by Lemma 3 that R agrees with R'' on every pair $\{w, w'\} \nsubseteq [x, y]_{R''}$. Therefore y, x are R-adjacent, so that

$$[y, z]_R = \{y\} \cup [x, z]_R = \{y\} \cup [x, z]_{R_{T-1}} = [x, y]_{R''}. \tag{4}$$

It follows that $X \cup \{x\} \subseteq [y, z]_R$.

Define $X' \coloneqq X \cup \{x\}$, and label its elements $X' \equiv \{a_1, \ldots, a_K\}$ so that $a_1 \; R \; \cdots \; R \; a_K$. Similarly label $[y,z]_R \setminus X' \equiv \{b_1, \ldots, b_L\}$ so that $b_1 \; R \cdots R \; b_L$.[43] Let R' be the ranking that

(I) agrees with R on any pair $\{w, w'\} \not\subseteq [y,z]_R$, and

(II) ranks the elements of $[y,z]_R$ as $a_1 \; R' \cdots R' \; a_K \; R' \; b_1 \; R' \cdots R' \; b_L$.

We have now constructed a tournament W' and a ranking R'. Recall that R is the outcome of σ under W'. It remains to show that

(dist) R' is distinct from R,

(alig) R' is more aligned with $\succ$ than R, and

(feas) R' is W'-feasible.

For (dist), observe that since $x \in X'$ and $y \in [y,z]_R \setminus X'$, we have $x = a_k \; R' \; b_\ell = y$ for some k and ℓ.[44] Since $y \; R \; x$, it follows that $R' \neq R$.

For (alig), fix a pair $w, w' \in \mathcal{X}$ with $w \; R \; w'$ and $w' \; R' \; w$; we must show that $w' \succ w$. By definition of R', it must be that $w' \in X' = X \cup \{x\}$ and that $w \in [y,z]_R \setminus X'$. Thus $w' \succ y$ (by $x \succ y$ and the definition of X) and either $y = w$ or $y \succ w$, whence $w' \succ w$ by transitivity of $\succ$.

It remains to establish (feas). To this end (recalling Observation 1 in appendix B.3, p. 27), fix an R'-adjacent pair $w, w' \in \mathcal{X}$ with $w' \; R' \; w$; we must show that $w' \; W' \; w$. There are two cases.

Sub-case (2)(a): $\{w, w'\} \not\subseteq [y,z]_R$. Then $w' \; R \; w$ by part (I) of the definition of R'. Since R agrees with R'' on any pair $\{z', z''\} \not\subseteq [y,z]_R$, it follows that $w' \; R'' \; w$. It therefore suffices to show that W' agrees with R'' on $\{w, w'\}$. Recalling the definition (i)–(iii) of W',

– If $\{w, w'\} \not\ni y$, then W' agrees with R'' on $\{w, w'\}$ by (iii).

– If $w' = y \in [y,z]_R$, then $w \notin [y,z]_R = [x,z]_{R_{T-1}} \supseteq X$ by (4), so neither (i) nor (ii) applies to the pair $\{w, y\} = \{w, w'\}$. Thus by (iii), W' agrees with R'' on $\{w, w'\}$.

– If $w = y \in [y,z]_R$, then $w' \notin [y,z]_R \supseteq [x,z]_{R_{T-1}} \supseteq X$ by (4), so neither (i) nor (ii) applies to the pair $\{y, w'\} = \{w, w'\}$. Thus by (iii), W' agrees with R'' on $\{w, w'\}$.

[43] $[y,z]_R \setminus X'$ is non-empty since y belongs to it.

[44] In fact, $k = \ell = 1$ since y is R-highest in $[y,z]_R$ and (recall) y, x are R-adjacent.

Sub-case (2)(b): $\{w, w'\} \subseteq [y, z]_R$. Recall the definition (i)–(iii) of W'. Recall also part (II) of the definition of R'. Observe that $y = b_1$ since y is R-highest in $[y, z]_R$, and $y \notin X' = X \cup \{x\}$. Furthermore, $x = a_1$ since y, x are R-adjacent (recall (4)) and $x \in X$. Finally, remark that $K \geq 2$ since $z \in X' = X \cup \{x\}$ and $z \neq x$.

Suppose first that $w' = y = b_1$. Then since w', w are R'-adjacent with $w' \; R' \; w$, we have $w = b_2 \notin X' \supseteq X$. Thus part (i) does not apply to the pair $\{w, y\} = \{w, w'\}$. Thus part (ii) applies, yielding $w' = y \; W' \; w$.

Suppose instead that $w = y = b_1$. Then since w', w are R'-adjacent with $w' \; R' \; w$, we have $w' = a_K \in X' = X \cup \{x\}$. Since $x = a_1$ and $K \geq 2$, we have $w' \neq x$. Thus $w' \in X$, so that (i) applies to the pair $\{y, w'\} = \{w, w'\}$, yielding $w' \; W' \; y = w$.

Finally, suppose that $\{w, w'\} \not\ni y$. Then W' and R'' agree on the pair $\{w, w'\}$ by (iii), so it suffices to show that $w' \; R'' \; w$.

Since $b_1 = y \notin \{w, w'\}$, we have either $\{w, w'\} \subseteq X'$ or $\{w, w'\} \subseteq [y, z]_R \setminus X'$. Thus R and R' agree on $\{w, w'\}$ by part (I) of the definition of R', so that $w' \; R \; w$.

Now label $[w', w]_R \equiv \{z_1, \ldots, z_J\}$ so that $z_1 \; R \; \cdots \; R \; z_J$. Since R is W'-feasible, we have $z_1 \; W' \; \cdots \; W' \; z_J$ by Observation 1 (appendix B.3, p. 27). By the hypotheses $w' \in [y, z]_R$ and $w' \neq y$, we must have $y \; R \; w'$ and thus $y \notin [w', w]_R$. This together with the fact that $z_j \; W' \; z_{j+1}$ for each $j < J$ implies, via part (iii) of the definition of W', that $z_j \; R'' \; z_{j+1}$ for each $j < J$. It follows by transitivity of R'' that $w' = z_1 \; R'' \; z_J = w$, as desired. ■

E Proofs of Propositions 1 and 2 (§5, pp. 18 and 20)

In this appendix, we establish tightness for the characterisations of regret-freeness in Theorems 2 and 3. We begin in §E.1 with a lemma, then use it to deduce Proposition 2 (§E.2) and Proposition 1 (§E.3).

E.1 A lemma

Definition 25. Given a proto-ranking R and alternatives $x \succ y$ and $z \neq w$, say that $\{z, w\}$ *makes* $\{x, y\}$ *an error at* R iff both $x \not R \; y \not R \; x$ and $z \not R \; w \not R \; z$, and one of the following holds:

- $x \succ z \succ y$, $y \not R \; z \not R \; x$, and $w \in \{x, y\}$.
- $z \succ y$, $x \; R \; z$ and $w = y$.
- $x \succ z$, $z \; R \; y$ and $w = x$.

If $\{z, w\}$ makes $\{x, y\}$ an error, then offering $\{x, y\}$ either misses an opportunity or takes a risk at R, and the chair 'should' offer $\{z, w\}$ instead.[45]

Recall from appendix D.1 (p. 31) the definition of a missed opportunity.

Lemma 5. Let R be a proto-ranking, and let $A \subseteq \mathcal{X}^2$ be a non-empty set of pairs of distinct alternatives. Suppose that for any pair $\{x, y\} \in A$, there is a pair $\{z, w\} \in A$ that makes $\{x, y\}$ an error at R. Then R contains a missed opportunity.

Proof. Let R and A satisfy the hypothesis. Then there is a pair $\{z, w\} \in A$ and another pair $\{z', w'\} \in A$ that makes $\{z, w\}$ an error at R. Assume (wlog) that $z \succ w$ and $z' \succ w'$. Since $\{z, w\} \neq \{z', w'\}$, we must have either $z \neq z'$ or $w \neq w'$. Assume that $z \neq z'$; the case $w \neq w'$ is similar.

First claim. There exists a sequence $(x_t)_{t=1}^T$ in $\mathcal{X}$ with $T \geq 2$ and $x_1 \neq x_2$ such that for every $t \leq T$, writing $x_{T+1} \coloneqq x_1$,

(i) if $x_t \succ x_{t+1}$ then $x_{t+1} \not R\, x_t$, and

(ii) if $x_{t+1} \succ x_t$ then $x_t \mathrel{R} x_{t+1}$.

Proof of the first claim. Define $\{x_1, y_1\} \coloneqq \{z, w\}$ and $\{x_2, y_2\} \coloneqq \{z', w'\}$. By the hypothesis of the lemma, there is a pair $\{x_3, y_3\} \in A$ with (wlog) $x_3 \succ y_3$ that makes $\{x_2, y_2\}$ an error at R, a $\{x_4, y_4\} \in A$ with $x_4 \succ y_4$ that makes $\{x_3, y_3\}$ an error at R, and so on. Since A is finite, $\{x_1, y_1\}$ makes $\{x_T, y_T\}$ an error for some $T \in \mathbf{N}$. We have $T \geq 2$ and $x_1 \neq x_2$ by construction, and (i)–(ii) must hold because $\{x_{t+1}, y_{t+1}\}$ makes $\{x_t, y_t\}$ an error at R. □

Let $(x_t)_{t=1}^T$ be a minimal sequence satisfying the conditions of the first claim (one with no strict subsequence that satisfies the conditions).

Second claim. $x_t \neq x_s$ for all distinct $t, s \in \{1, \dots, T\}$.

Proof of the second claim. Suppose toward a contradiction that $x_t = x_{t+1}$; then the sequence $x_1, \dots, x_{t-1}, x_{t+1}, \dots, x_T$ satisfies the conditions of the first claim, contradicting the minimality of $(x_t)_{t=1}^T$. Assume for the remainder that $x_t \neq x_{t+1}$ for every $t \in \{1, \dots, T\}$.

Suppose toward a contradiction that $x_t = x_s$, where $t + 1 < s$. Then the sequence $x_{t+1}, \dots, x_s$ satisfies the conditions of the first claim, which is absurd since $(x_t)_{t=1}^T$ is minimal. □

[45] This is heuristic, as offering $\{z, w\}$ might itself miss an opportunity or take a risk at R.

In light of the second claim, we may re-label the sequence $(x_t)_{t=1}^T$ so that $x_1 \succ x_t$ for every $t \in \{2, \ldots, T\}$. Let $t' \leq T$ be the least $t \geq 2$ such that $x_T \succ \cdots \succ x_t$. (So $t' = T$ exactly if $x_t \succ x_T$ for every $t < T$.) We shall show that $\{x_1, x_{t'}\}$ is a missed opportunity in R; in particular, that $t' \geq 3$ and

(a) $x_1 \succ x_{t'-1} \succ x_{t'}$,

(b) $x_{t'} \; R \; x_1$, and

(c) $x_{t'} \not R \; x_{t'-1} \not R \; x_1$.

For (b), if $t' = T$ then $x_{t'} = x_T \; R \; x_1$ by property (ii), and if not then $x_{t'} \; R \cdots R \; x_T \; R \; x_1$ by property (ii), whence $x_{t'} \; R \; x_1$ by transitivity of R. The second half $x_{t'-1} \succ x_{t'}$ of (a) holds by definition of t'. The first half $x_{t'} \not R \; x_{t'-1}$ of (c) then follows by property (i). Since $x_{t'} \; R \; x_1$, it follows that $t' - 1 \neq 1$, which is to say that $t' \geq 3$. The first half $x_1 \succ x_{t'-1}$ of (a) then holds by construction. Finally, the second half $x_{t'-1} \not R \; x_1$ of (c) must hold since otherwise the sequence $(x_t)_{t=1}^{t'-1}$ would satisfy the conditions of the first claim, in contradiction with the minimality of $(x_t)_{t=1}^T$. ■

E.2 Proof of Proposition 2 (p. 20)

At a history at which the chair has committed no errors, the proto-ranking clearly contains no missed opportunities. The following therefore implies Proposition 2.

Proposition 2*. Let R be a non-total proto-ranking containing no missed opportunities. Then there exist distinct $x, y \in \mathcal{X}$ such that $x \not R \; y \not R \; x$ and offering a vote on $\{x, y\}$ neither misses an opportunity nor takes a risk at R.

Proof. Let R be a non-total proto-ranking, and suppose that for any distinct $x, y \in \mathcal{X}$ with $x \not R \; y \not R \; x$, offering a vote on $\{x, y\}$ either misses an opportunity or takes a risk at R. We shall show that R contains a missed opportunity.

Let A be the set of all pairs $\{x, y\} \subseteq \mathcal{X}$ with $x \neq y$ and $x \not R \; y \not R \; x$. The set A is non-empty since R is not total. By hypothesis, for any $\{x, y\} \in A$, offering $\{x, y\}$ either misses an opportunity or takes a risk at R, implying that some $\{z, w\} \in A$ makes $\{x, y\}$ an error at R. It follows by Lemma 5 (§E.1, p. 40) that R contains a missed opportunity. ■

E.3 Proof of Proposition 1 (p. 18)

Lemma 6. Fix a tournament W, let R be a W-efficient ranking, and let $R' \subseteq R$ be a non-total proto-ranking containing no missed opportunities.

Then there exist distinct $x, y \in \mathcal{X}$ such that $x \not R' y \not R' x$, W and R agree on $\{x, y\}$, and offering $\{x, y\}$ does not miss an opportunity or take a risk at R'.

Proof of Proposition 1. Fix a tournament W and a W-efficient ranking R. By Proposition 2 (p. 20; already proved), it suffices to find a terminal history $((x_t, y_t))_{t=1}^T$, with associated proto-rankings $(R_t)_{t=0}^T$,[46] such that

- for every $t \in \{1, \dots, T\}$, $x_t \; W \; y_t$ and $x_t \; R \; y_t$, and
- for every $t \in \{2, \dots, T\}$, offering $\{x_t, y_t\}$ does not miss an opportunity or take a risk at R_{t-1}.

Such a terminal history is obtained by repeatedly applying Lemma 6. ■

Proof of Lemma 6. We shall prove the contra-positive. Fix a tournament W and a W-efficient ranking R, and let $R' \subseteq R$ be a non-total proto-ranking. Suppose that for any distinct $x, y \in \mathcal{X}$ with $x \not R' y \not R' x$ such that W and R agree on $\{x, y\}$, offering $\{x, y\}$ misses an opportunity or takes a risk at R'. We will show that R' contains a missed opportunity.

Let A be the set of all pairs $\{x, y\} \in \mathcal{X}^2$ such that $x \succ y$, $x \not R' y \not R' x$, and there is no $z \in \mathcal{X}$ such that $x \; R \; z \; R \; y$. The set A is non-empty since it includes any R-adjacent pair $\{x, y\}$ with $x \not R' y \not R' x$, and there must be such a pair since R' is non-total and $R' \subseteq R$. By Lemma 5 (§E.1, p. 40), it suffices to show that for any pair $\{x, y\} \in A$, there is a pair $\{z, w\} \in A$ that makes $\{x, y\}$ an error at R'.

So fix a pair $\{x, y\} \in A$. We claim that W and R must agree on $\{x, y\}$. If x, y are R-adjacent, then this holds by Observation 1 (appendix B.3, p. 27) since R is W-feasible. If x, y are not R-adjacent, then since no $z \in \mathcal{X}$ satisfies $x \; R \; z \; R \; y$, it must be that $y \; R \; x$. Since $x \succ y$ and R is W-efficient, it follows that $y \; W \; x$, so that W and R agree on $\{x, y\}$.

It follows from the (contra-positive) hypothesis that offering $\{x, y\}$ either misses an opportunity or takes a risk at R'. Consider each in turn.

Case 1: $\{x, y\}$ misses an opportunity. In this case there is a $z \in \mathcal{X}$ satisfying $x \succ z \succ y$ and $y \not R' z \not R' x$. Since $\{x, y\} \in A$, we must have either $z \; R \; x$ or $y \; R \; z$. Assume that $z \; R \; x$; the case $y \; R \; z$ is analogous. Since $R' \subseteq R$, we have $x \not R' z \not R x$. Thus the pair $\{x, z\}$ lives in A and makes $\{x, y\}$ an error at R'.

Case 2: $\{x, y\}$ takes a risk. Assume that there is a $z \in \mathcal{X}$ such that $z \succ y$, $x \; R' \; z$ and $y \; R' \; z$; the case in which $x \succ z$, $z \; R' \; y$ and $z \not R' x$ is similar. Then $\{y, z\}$ makes $\{x, y\}$ an error at R'. To see that $\{y, z\}$ belongs to A, observe

[46] Recall that $R_0 = \varnothing$ and $R_t = \text{tr}\left(\bigcup_{s=1}^t \{(x_s, y_s)\}\right)$ for each $t \in \{1, \dots, T\}$.

that (i) $x \mathrel{R} z$ since $x \mathrel{R'} z$ and $R' \subseteq R$, that (ii) $y \mathrel{R} z$ by (i) and $\{x, y\} \in A$, and finally that (iii) $z \not\mathrel{R'} y$ since otherwise $x \mathrel{R'} z$ and the transitivity of R' would imply the falsehood $x \mathrel{R'} y$. ■

F Proofs of Theorem 4 and Proposition 3 (§6, pp. 22–23)

In this appendix, we prove our results concerning the outcome-equivalents of insertion sort. We prove Theorem 4 and Proposition 3 in §F.1 using two lemmata, then prove these lemmata in §F.2–§F.4. Throughout, we label the alternatives $\mathcal{X} \equiv \{1, \ldots, n\}$ so that $1 \succ \cdots \succ n$.

F.1 Proofs using lemmata

In this section, we state a definition and two lemmata, and use these to prove Theorem 4 and Proposition 3.

Definition 26. Let Σ_0 be the set of all strategies. For every integer $k \in \{1, \ldots, n-2\}$, let Σ_k be the set of all strategies σ with the following property: for any tournament W and alternative $j \leq k$, labelling the alternatives $\{j+1, \ldots, n\} \equiv \{x_{j+1}, \ldots, x_n\}$ as

$$x_{j+1} \mathrel{R^\sigma(W)} \cdots \mathrel{R^\sigma(W)} x_n,$$

the first vote involving j that σ offers under W is on $\{j, x_{j+1}\}$; if j loses, then a second vote involving j is offered, namely on $\{j, x_{j+2}\}$; if j loses again, then a third vote involving j is offered, namely on $\{j, x_{j+3}\}$; and so on.

The definition of Σ_{n-2} describes a natural generalisation of insertion sort: for each alternative j, given how the $\succ$-worse alternatives $\{x_{j+1}, \ldots, x_n\}$ are ultimately ranked, the same votes involving j are offered, in the same order, though not necessarily in adjacent periods.[47] Each Σ_k for $k < n-2$ is defined by the same property restricted to those alternatives j that are $\succ$-better than or equal to k, so that $\Sigma_0 \supseteq \Sigma_1 \supseteq \cdots \supseteq \Sigma_{n-2}$.

As the definition suggests, the strategies in Σ_{n-2} are those that offer the same votes as insertion sort:

Lemma 7. A strategy is outcome-equivalent to insertion sort iff it belongs to Σ_{n-2}.

[47] Note a subtlety in the definition: although we label $\{x_{j+1}, \ldots, x_n\}$ according to the outcome $R^\sigma(W)$ of σ under W, a strategy in Σ_k need not (as insertion sort would) have totally ranked $\{j+1, \ldots, n\}$ before offering votes involving j.

Proof of Proposition 3. By inspection, any amendment strategy belongs to Σ_{n-2}, so is outcome-equivalent to insertion sort by Lemma 7. ■

The prove Theorem 4, a further lemma is required:

Lemma 8. Given $k \in \{1, \dots, n-2\}$, a strategy $\sigma \in \Sigma_{k-1}$ is best for k among Σ_{k-1} iff it belongs to Σ_k.

Proof of Theorem 4. Let $B_0 = \Sigma_0$ be the set of all strategies, and for each $k \in \{1, \dots, n-2\}$, let B_k be the set of strategies in B_{k-1} that are best for k among B_{k-1}.[48] A lexicographic strategy is precisely one that lives in B_{n-2}.

By Lemma 7, a strategy is outcome-equivalent to insertion sort iff it lives in Σ_{n-2}; so what must be shown is that $\Sigma_{n-2} = B_{n-2}$. We shall prove the stronger claim that $\Sigma_k = B_k$ for each $k \in \{0, \dots, n-2\}$ by (weak) induction on k. The base case $k = 0$ holds by definition of Σ_0 and B_0. For the induction step, suppose that $\Sigma_{k-1} = B_{k-1}$; then $\Sigma_k = B_k$ by Lemma 8. ■

The remainder of this appendix is devoted to proving Lemmata 7 and 8. We begin in §F.2 with two preliminary results, then prove Lemma 8 in §F.3 and Lemma 7 in §F.4.

F.2 Preliminary results

The following lemma is used in the proof of Lemma 7.

Lemma 9. Given a $k \in \{1, \dots, n-2\}$, consider a strategy $\sigma \in \Sigma_k$, a tournament W, an alternative $j \leq k$ and some $m \in \{1, \dots, n-j\}$. Suppose that under W, σ offers at least m votes involving j, and that j loses the first $m-1$ of these. Let ℓ be the m^{th} alternative pitted against j, and let R be the proto-ranking associated with the history after which the vote on $\{j, \ell\}$ occurs. Then $\ell \; R \; i$ for any $i \neq \ell$ such that $i > j$ and j did not lose against i prior to the vote on $\{j, \ell\}$.

Proof. Suppose toward a contradiction that $\ell \not R \; i$ for some $i \neq \ell$ with $i > j$ such that j did not lose to i prior to the vote on $\{j, \ell\}$. Then there exists a ranking W' such that $R \subseteq W'$ and $i \; W' \; \ell$. (Note that a ranking is precisely a transitive tournament.) Being a ranking, W' is the only W'-feasible ranking (by Observation 1 in appendix B.3, p. 27), so that $R^\sigma(W') = W'$, and in particular $i \; R^\sigma(W') \; \ell$.

Let T be the period in which σ offers $\{j, \ell\}$ (i.e. the m^{th} vote involving j) under W. Since $R \subseteq W'$, the history of length $T-1$ generated by σ and

[48] Let $B_k \coloneqq \varnothing$ if B_{k-1} is empty. (It is in fact non-empty, but we haven't proved it yet.)

W' is the same as that generated by σ and W. So in particular, under W', σ offers $\{j, \ell\}$ in period T, σ does not offer $\{j, i\}$ in an earlier period, and j does not win a vote in any earlier period. Since $\sigma \in \Sigma_k$, it follows that $\ell \; R^\sigma(W') \; i$, a contradiction. ■

The proof of Lemma 8 relies on the following.

Lemma ♮. Given $k \in \{2, \dots, n-2\}$, if $N_k^\sigma(W) > 0$ for some $\sigma \in \Sigma_{k-1}$ and tournament W, then prior to winning its first vote, k is pitted only against $\succ$-worse alternatives.

Proof. We prove the contra-positive. Let $k \in \{2, \dots, n-2\}$, $\sigma \in \Sigma_{k-1}$ and a tournament W be such that σ pits k against some $j < k$ under W and k wins no vote before the one against j; we must show that $N_k^\sigma(W) = 0$. Let R be the proto-ranking associated with the history after which the vote on $\{j, k\}$ occurs. It suffices to show that $\ell \; R \; k$ for all $\ell > k$.

Claim. j wins no vote prior to the one against k.

Proof of the claim. Suppose toward a contradiction that the first alternative against which j wins is $\ell \neq k$. Then $j \; R \; \ell$. Since $\sigma \in \Sigma_{k-1}$, the vote on $\{j, \ell\}$ is the m^{th} involving j, for some $m < n - j$. It follows by Lemma 9 (above) that $\ell \; R \; k$, which together with $j \; R \; \ell$ and the transitivity of R yields $j \; R \; k$. On the other hand, since $\{j, k\}$ is offered after a history with proto-ranking R, we must have $j \not R \; k \not R \; j$. Contradiction! □

Fix an $\ell > k$; we shall show that $\ell \; R \; k$. Since $\sigma \in \Sigma_{k-1}$, the vote on $\{j, k\}$ is the m^{th} involving j, for some $m < n - j$. It must be that σ offers $\{j, \ell\}$ prior to $\{j, k\}$, since otherwise Lemma 9 would yield $k \; R \; \ell$, contradicting the hypothesis that k wins no votes prior to the one against j. Since j wins no vote prior to the one against k and $m < n - j$, Lemma 9 yields $\ell \; R \; k$, as desired. ■

F.3 Proof of Lemma 8 (§F.1, p. 44)

We shall use the probabilistic notation $\Pr(E|F) := |E \cap F|/|F|$ for the fraction of tournaments in $F \subseteq \mathcal{W}$ that belong to $E \cap F \subseteq F$, and similarly $\Pr(E) := \Pr(E|\mathcal{W})$. This corresponds the thought experiment in which the general will W is drawn uniformly at random from $\mathcal{W}$.

We must establish that membership of Σ_k is necessary and sufficient for being best for k among Σ_{k-1}. We prove sufficiency and necessity in turn, making use of Lemma ♮ from the previous section.

Proof of sufficiency. Fix k, a strategy $\sigma \in \Sigma_{k-1}$ and an $m \in \{1, \dots, n-k\}$. We shall derive an upper bound for $\Pr(N_k^\sigma \geq m)$, then show that it is attained if $\sigma \in \Sigma_k$.

For each $\ell \in \{1, \dots, n-k\}$, let $F_\ell \subseteq \mathcal{W}$ be the set of tournaments under which σ offers at least ℓ votes involving alternative k, with alternative k losing the first $\ell - 1$ of these and winning the ℓ^{th}.

Claim. $\Pr(F_\ell) \leq 1/2^\ell$ for each $\ell \in \{1, \dots, n-k\}$, with equality if $\sigma \in \Sigma_k$.

Proof of the claim. A tournament W lies in F_ℓ iff under σ and W,

– alternative k loses its first vote (probability $1/2$),

– a second vote involving alternative k occurs (probability ≤ 1, with equality if $\sigma \in \Sigma_k$) and k loses (probability $1/2$),

…

– an $(\ell - 1)^{\text{th}}$ vote involving alternative k occurs (probability ≤ 1, with equality if $\sigma \in \Sigma_k$) and k again loses (probability $1/2$), and

– an ℓ^{th} vote involving alternative k occurs (probability ≤ 1, with equality if $\sigma \in \Sigma_k$) and k wins (probability $1/2$).

Thus

$$\Pr(F_\ell) \leq \frac{1}{2} \times \left(1 \times \frac{1}{2}\right)^{\ell-1} = \frac{1}{2^\ell}, \quad \text{with equality if } \sigma \in \Sigma_k. \qquad \square$$

By Lemma ♮ (§F.2, p. 45), if $N_k^\sigma(W) > 0$, then prior to winning its first vote, k was only pitted against $\succ$-worse alternatives. There are $n-k$ of these: $\{k+1, \dots, n\}$. Thus if $N_k^\sigma(W) \geq m$ holds, then alternative k cannot have lost strictly more than $n-m-k$ votes and must have won at least one—so in particular, $W \in F_1 \cup \cdots \cup F_{n-k-m+1}$. Thus

$$\begin{aligned}
\Pr(N_1^\sigma \geq m) &= \sum_{\ell=1}^{n-k-m+1} \Pr(F_\ell) \Pr(N_1^\sigma \geq m | F_\ell) \\
&\leq \sum_{\ell=1}^{n-k-m+1} \Pr(F_\ell) && (\sharp) \\
&\leq \sum_{\ell=1}^{n-k-m+1} \frac{1}{2^\ell}, && (\flat)
\end{aligned}$$

where $(\flat)$ holds by the claim.

Now suppose that $\sigma \in \Sigma_k$; we shall show that ($\sharp$) and ($\flat$) hold with equality, so that σ attains the bound. For ($\flat$), this follows from the claim. For ($\sharp$), fix an $\ell \in \{1, \ldots, n-k-m+1\}$ and a $W \in F_\ell$; we must show that $N_k^\sigma(W) \geq m$. Label $\{k+1, \ldots, n\} \equiv \{x_{k+1}, \ldots, x_n\}$ so that

$$x_{k+1} \; R^\sigma(W) \ldots R^\sigma(W) \; x_n.$$

Since $W \in F_\ell$, we have by definition of Σ_k that $k \; R^\sigma(W) \; x_{\ell+1}$. Thus $k \; R^\sigma(W) \; x_{\ell'}$ for each $\ell' \in \{\ell+1, \ldots, n\}$, so that $N_k^\sigma(W) \geq n - \ell \geq m$. ■

Proof of necessity. Take a strategy σ in $\Sigma_{k-1} \setminus \Sigma_k$. Since σ belongs to Σ_{k-1}, it satisfies the inequalities ($\sharp$) and ($\flat$) in the sufficiency argument. Suppose that one or the other holds strictly for some $m \in \{1, \ldots, n-k\}$, so that σ fails to attain the bound in the sufficiency proof. Since any $\sigma' \in \Sigma_k$ attains the bound for every m by the (just-proved) sufficiency part, it follows that σ is not best for k among Σ_{k-1}. It therefore suffices to find an $m \in \{1, \ldots, n-k\}$ such that either ($\sharp$) or ($\flat$) holds strictly.

Since $\sigma \in \Sigma_{k-1} \setminus \Sigma_k$, there is a tournament W such that, labelling the alternatives $\{k+1, \ldots, n\} \equiv \{x_{k+1}, \ldots, x_n\}$ so that

$$x_{k+1} \; R^\sigma(W) \ldots R^\sigma(W) \; x_n,$$

one of the following holds under W:

(a) σ pits k against some $j \neq x_{k+1}$ prior to pitting it against x_{k+1}.

(b) For some $\ell \in \{k+1, \ldots, n-k\}$, σ offers at least ℓ votes involving k, the first $\ell - 1$ of which are against $x_{k+1}, \ldots, x_\ell$ and are all lost by k, and the ℓ^{th} of which is against some $j \neq x_{\ell+1}$.

(c) For some $\ell \in \{k+1, \ldots, n-k\}$, σ offers exactly $\ell - 1$ votes involving k, against $x_{k+1}, \ldots, x_\ell$, each of which is lost by k.

Case (a). We shall exhibit a tournament $W' \in F_1$ such that $N_k^\sigma(W') < n - k$, so that ($\sharp$) holds strictly for $m = n - k$. This is trivial if $N_k^\sigma(W) = 0$ (let $W' := W$), so suppose that $N_k^\sigma(W) > 0$. Let $j \neq x_{k+1}$ be the first alternative pitted against k by σ under W, and let R be the proto-ranking associated with the history after which this occurs. Clearly $j \not R \; k \not R \; x_{k+1}$. Since $\sigma \in \Sigma_{k-1}$ and $N_k^\sigma(W) > 0$, Lemma $\natural$ (§F.2, p. 45) implies that $j > k$. So $x_{k+1} \; R^\sigma(W) \; j$ (since $j \in \{k+1, \ldots, n\} = \{x_{k+1}, \ldots, n\}$ and $j \neq x_{k+1}$), and thus $j \not R \; x_{k+1}$.

It follows that there is a ranking W' such that $R \subseteq W'$ and $x_{k+1}\ W'\ k\ W'\ j$.[49] (Note that a ranking is precisely a transitive tournament.) Let T be the period in which σ offers $\{j,k\}$ under W. Since $R \subseteq W'$, the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W, and thus $\{j,k\}$ is the first pair involving k that σ offers under W'. Thus $W' \in F_1$ since $k\ W'\ j$. Being a ranking, W' is the only W'-feasible ranking (by Observation 1 in appendix B.3, p. 27), so $R^\sigma(W') = W'$. Thus $N_k^\sigma(W') < n-k$ since $x_{k+1}\ W'\ k$.

Case (b). If $j > k$, then case-(a) argument yields a $W' \in F_\ell$ such that $N_k^\sigma(W') < n-k-\ell+1$, so that (♯) holds strictly for $m = n-k-\ell+1$. Suppose instead that $j < k$. Let W' be the tournament that agrees with W on every pair, except that $k\ W'\ j$. Clearly $W' \in F_\ell$. By (the contra-positive of) Lemma ♮, we have $N_k^\sigma(W') = 0$. Thus $\Pr(N_k^\sigma \geq 1 | F_\ell) < 1$, so that (♯) holds strictly for $m = 1$.

Case (c). Let E be the set of tournaments under which σ offers at least $\ell - 1$ votes involving k, the first $\ell - 1$ of which k loses. Then $\Pr(E) > 0$, and the probability that σ offers at least ℓ votes conditional on E is strictly less than 1. The argument for the claim in the sufficiency proof therefore yields $\Pr(F_\ell) < 1/2^\ell$, so that (♭) holds strictly for e.g. $m = 1$. ■

F.4 Proof of Lemma 7 (§F.1, p. 43)

We must show that membership of Σ_{n-2} is necessary and sufficient for outcome-equivalence to insertion sort. For the sufficiency part, we shall make use of Lemma 9 in §F.3.

Proof of sufficiency. Let $\sigma \in \Sigma_{n-2}$, fix a tournament W, and let R (R') be the outcome of σ (of insertion sort) under W. We will show for each $k \in \{n-2, \dots, 1\}$ that R agrees with R' on $\{k, \dots, n\}$, using induction on k. For the base case $k = n-2$, let $\{j, \ell\}$ be the first pair offered by σ, where $j < \ell$; it suffices to show that $j = n-1$. Suppose to the contrary; then by Lemma 9 (§F.3, p. 44), prior to the first vote, ℓ is already ranked above every $i > j$ such that $i \neq \ell$, which is absurd. The induction step is immediate from the fact that $\sigma \in \Sigma_{n-2}$. ■

The proof of necessity refers to the argument for Lemma 8 in §F.3 above.

[49] To see why, observe that the transitive closure R' of $R \cup \{(x_{k+1}, k)\}$ is a proto-ranking since $k \not R\ x_{k+1}$. Since $j \neq x_{k+1}$ and $j \not R\ x_{k+1}$, we have $j \not R'\ k$ by Observation 3 (appendix D.1, p. 31), and thus the transitive closure R'' of $R' \cup \{(k, j)\}$ is also a proto-ranking. Let W' be any ranking that contains R''.

Proof of necessity. Let $\sigma \notin \Sigma_{n-2}$; we must show that it is not outcome-equivalent to insertion sort. Note that there is a (unique) $k \in \{1, \ldots, n-2\}$ such that $\sigma \in \Sigma_{k-1} \setminus \Sigma_k$. Recall the bound in the proof of the sufficiency part of Lemma 8 (§F.3 above). Since insertion sort belongs to Σ_k, it attains the bound for every $m \in \{1, \ldots, n-k\}$ by the Lemma 8 sufficiency argument, and thus all of its outcome-equivalents do. By the Lemma 8 necessity argument, σ fails to attain the bound for some $m \in \{1, \ldots, n-k\}$, so is not among the outcome-equivalents of insertion sort. ■

G Proof of Proposition 4 (§7, p. 25)

Fix a voter i, and let $\sigma_i^\star$ be her sincere strategy. We must establish properties ($\nexists$) and ($\exists$) in the definition of dominance. For the latter, we shall make use of the extension corollary in appendix C (p. 29).

Property ($\nexists$)*:* Fix strategies σ, σ_{-i} of the chair and other voters and a non-sincere strategy σ_i' of voter i, and suppose that $R' \coloneqq R(\sigma, \sigma_i', \sigma_{-i})$ is distinct from $R \coloneqq R(\sigma, \sigma_i^\star, \sigma_{-i})$; we must show that R' is not more aligned with $\succ_i$ than R. Let T be the first period in which the proto-rankings R_T and R_T' differ, and let $\{x, y\}$ be the pair voted on in this period, where (wlog) $x \; R_T \; y$ and $y \; R_T' \; x$. The two strategy profiles generate the same length-$(T-1)$ history h (by definition of T), and thus the same period-T votes $\sigma_j(h)$ by the other voters $j \neq i$. So voter i is pivotal after history h, and since $\sigma_i^\star$ is sincere it must be that $x \succ_i y$. Thus $R' \supseteq R_T'$ is not more aligned with $\succ_i$ than $R \supseteq R_T$.

Property ($\exists$)*:* Take any non-sincere strategy σ_i'. Choose strategies σ', σ_{-i}' such that σ_i' votes non-sincerely along the terminal history induced by the strategy profile $(\sigma', \sigma_i', \sigma_{-i}')$, and let T be the first period in which this occurs. Then the proto-ranking in period $T-1$ is the same under the strategy profiles $(\sigma', \sigma_i', \sigma_{-i}')$ and $(\sigma', \sigma_i^\star, \sigma_{-i}')$; call it R_{T-1}. Write $\{x, y\}$ for the pair of alternatives that are voted on in period T, where (wlog) $x \succ_i y$.

By the extension corollary in appendix C (p. 29), there exists a ranking $R \supseteq R_{T-1}$ with $x \; R \; y$ and x, y R-adjacent. Let R' be exactly R, except with the positions of x and y reversed. Clearly R is more aligned with $\succ_i$ than R', and the two are distinct.

It thus suffices to find strategies σ and σ_{-i} such that $R(\sigma, \sigma_i^\star, \sigma_{-i}) = R$ and $R(\sigma, \sigma_i', \sigma_{-i}) = R'$. For the chair, let $\sigma \coloneqq \sigma'$. As for σ_{-i}, let half of the other voters $j \in I \setminus \{i\}$ vote according R (i.e. vote for x over y iff $x \; R \; y$), and the rest vote according to R'. ■

Supplemental appendices

H A characterisation of our protocol

In this appendix, we show that among all possible rules of interaction between the chair and committee that lead to a ranking, the 'transitive' protocol studied in this paper (described in §2.1) is the only one (up to restriction) that denies the chair arbitrary power and that allows votes only on pairs. This protocol is thus the natural one, given the restriction to pairwise votes. Non-binary votes raise issues that are beyond the scope of this paper.[50]

A *ballot* is a set of two or more alternatives. An *election* is (B, V), where B is a ballot and V is a map $\{1, \ldots, I\} \to B$ specifying what alternative each voter votes for. An *electoral history* is a finite sequence of elections with distinct ballots. For two (distinct) electoral histories h, h', we write $h \sqsubseteq (\sqsubset)\ h'$ iff h is a truncation of h'.

A *protocol* specifies for each (permitted) electoral history either (1) a set of ballots that the chair is permitted to offer or (2) a ranking. Formally:

Definition 27. A *protocol* is $(\mathcal{H}, \rho)$, where

(1) $\mathcal{H}$ is a non-empty set of electoral histories such that

- if h' belongs to $\mathcal{H}$, then so does any $h \sqsubseteq h'$, and
- if $h = ((B_1, V_1), \ldots, (B_t, V_t))$ belongs to $\mathcal{H}$, then so does $h' = ((B_1, V_1), \ldots, (B_t, V_t'))$ for any $V_t' : \{1, \ldots, n\} \to B_t$.

Call $h \in \mathcal{H}$ *terminal* (in $\mathcal{H}$) iff there is no $h' \sqsupset h$ in $\mathcal{H}$.

(2) ρ is a map that assigns a ranking to each terminal $h \in \mathcal{H}$.

Call an electoral history *binary* iff each ballot has exactly two elements. A *binary protocol* $(\mathcal{H}, \rho)$ is one whose $\mathcal{H}$ consists of binary electoral histories. For any binary electoral history $h = ((\{x_s, y_s\}, V_s))_{s=1}^t$, where wlog $|\{i : V_s(i) = x_s\}| > I/2$ for each $s \in \{1, \ldots, t\}$, let R^h denote the transitive closure of $\bigcup_{s=1}^t \{(x_s, y_s)\}$.[51] The *transitive protocol* is the binary protocol that permits the chair to offer a ballot $\{x, y\}$ after binary electoral history

[50] Unlike in the binary case, there is no 'most natural' non-binary protocol. In particular, reasonable protocols can differ in what they deem the committee to have 'decided' in a vote on three or more alternatives in which none won an outright majority.

[51] If h is the empty electoral history, then $R^h = \varnothing$.

h exactly if the pair x, y is unranked by R^h, and assigns the ranking R^h to each terminal h.[52]

To deny the chair arbitrary power, we focus on protocols that rank x above y whenever x won an outright majority and y was also on the ballot:

Definition 28. A protocol $(\mathcal{H}, \rho)$ satisfies *committee sovereignty* iff for any terminal $h = ((B_t, V_t))_{t=1}^T \in \mathcal{H}$ such that $|\{i : V_t(i) = x\}| > I/2$ and $y \in B_t \setminus \{x\}$ for some $t \in \{1, \ldots, T\}$, we have $x \; \rho(h) \; y$.

For binary protocols, committee sovereignty is equivalent to imposing transitivity after every vote:

Observation 4. A binary protocol $(\mathcal{H}, \rho)$ satisfies committee sovereignty iff $\rho(h) \supseteq R^h$ for every terminal $h \in \mathcal{H}$.

That is, any pair linked by a chain of majorities ($x \; R^h \; y$) must be ranked accordingly ($x \; \rho(h) \; y$), and so cannot be offered for a vote.[53]

Proof. Let $(\mathcal{H}, \rho)$ be binary and satisfy committee sovereignty, and take a terminal $h = ((\{x_t, y_t\}, V_t))_{t=1}^T \in \mathcal{H}$, where wlog $x_t \; R^h \; y_t$ for each $t \in \{1, \ldots, T\}$. Then $\rho(h) \supseteq \bigcup_{t=1}^T \{(x_t, y_t)\}$ by committee sovereignty, whence $\rho(h) \supseteq R^h$ because $\rho(h)$ is transitive and R^h is by definition the smallest transitive relation containing $\bigcup_{t=1}^T \{(x_t, y_t)\}$.

For the converse, let $(\mathcal{H}, \rho)$ be binary with $\rho(h) \supseteq R^h$ for every terminal $h \in \mathcal{H}$. Take any terminal $h = ((\{x_t, y_t\}, V_t))_{t=1}^T \in \mathcal{H}$ and suppose that $|\{i : V_t(i) = x_t\}| > I/2$; we must show that $x_t \; \rho(h) \; y_t$. Since $x_t \; R^h \; y_t$, this follows immediately from $\rho(h) \supseteq R^h$. ■

More is needed to deny the chair excessive power: she must also be required to offer enough ballots to give the committee a fair say. To formalise this, write $x \; S^h \; y$ for an electoral history $h = ((B_t, V_t))_{t=1}^T$ iff

$$x, y \in B_t \quad \text{and} \quad |\{i : V_t(i) = x\}| \geq |\{i : V_t(i) = y\}|$$

for some $t \in \{1, \ldots, T\}$, and say that h *gives the committee a say on* x, y iff $\{z_1, z_L\} = \{x, y\}$ for some sequence $z_1 \; S^h \; z_2 \; S^h \cdots S^h \; z_L$ of alternatives.

[52] Explicitly it is $(\mathcal{H}^\star, \rho^\star)$, where $\mathcal{H}^\star$ consists of all binary electoral histories h' such that

$$h \sqsubset ((\{x_1, y_1\}, V_1), \ldots, (\{x_t, y_t\}, V_t)) \sqsubseteq h' \quad \text{implies} \quad x_t \not R^h \; y_t \not R^h \; x_t,$$

(so that $h \in \mathcal{H}$ is terminal iff R^h is a ranking,) and $\rho^\star(h) \coloneqq R^h$ for each terminal $h \in \mathcal{H}^\star$.

[53] Formally: if $x \; R^h \; y$, then no terminal $h' \sqsupseteq h$ can feature the ballot $\{x, y\}$ (except in h). For otherwise there would be a terminal h' in which y beats x in a vote, so that $x \; R^{h'} \; y \; R^{h'} \; x$, which is impossible since $\rho(h') \supseteq R^{h'}$ and $\rho(h')$ is a ranking.

Definition 29. A protocol $(\mathcal{H}, \rho)$ satisfies *democratic legitimacy* iff every terminal $h \in \mathcal{H}$ gives the committee a say on each pair of alternatives.

Write $\tau(\mathcal{H})$ for the terminal elements of $\mathcal{H}$. A protocol $(\mathcal{H}, \rho)$ is a *restriction* of $(\mathcal{H}', \rho')$ iff $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}')$ and $\rho = \rho'|_{\tau(\mathcal{H})}$.[54] To wit, anything the chair can do under $(\mathcal{H}, \rho)$, she can also do under $(\mathcal{H}', \rho')$.

Proposition 5. A protocol is binary and satisfies committee sovereignty and democratic legitimacy iff it is a restriction of the transitive protocol.

Thus any binary protocol that does not give the chair arbitrary power must be the transitive protocol, possibly with limitations on what unranked pairs the chair may offer at some histories. Neglecting such limitations as ad hoc, we arrive at the transitive protocol.

Proof. Any restriction of the transitive protocol $(\mathcal{H}^\star, \rho^\star)$ satisfies the three properties since $(\mathcal{H}^\star, \rho^\star)$ does and the properties are preserved under restriction. For the converse, let $(\mathcal{H}, \rho)$ satisfy the three properties; we must show that $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}^\star)$ and $\rho = \rho^\star|_{\tau(\mathcal{H})}$.

To establish $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}^\star)$, we show separately that $\mathcal{H} \subseteq \mathcal{H}^\star$ and that any $h \in \tau(\mathcal{H}) \subseteq \mathcal{H}^\star$ is terminal in $\mathcal{H}^\star$. For the former, fix a pair of electoral histories $h \sqsubset h' = ((\{x_s, y_s\}, V_s))_{s=1}^t \in \mathcal{H}$. We must show that the pair x_t, y_t is unranked by R^h, so suppose toward a contradiction that $x_t \; R^h \; y_t$. Then we must have $x_t \; \rho(h'') \; y_t$ for any terminal $h'' \in \mathcal{H}$ such that $h'' \sqsupseteq h$ since $\rho(h'') \supseteq R^{h''} \supseteq R^h$ by Observation 4. In particular, this must hold for any terminal $h'' \in \mathcal{H}$ with first $t-1$ elections $(\{x_1, y_1\}, V_1), \ldots, (\{x_{t-1}, y_{t-1}\}, V_{t-1})$ and t^{th} election $(\{x_t, y_t\}, V_t')$, where V_t' satisfies $|\{i : V_t'(i) = y_t\}| > I/2$. But for such an h'', committee sovereignty of $(\mathcal{H}, \rho)$ clearly demands that $y_t \; \rho(h'') \; x_t$—a contradiction.

For the latter, let $h \in \tau(\mathcal{H}) \subseteq \mathcal{H}^\star$; we must show that h is terminal in $\mathcal{H}^\star$, meaning precisely that R^h is total. Since h is binary, a pair x, y is ranked by R^h iff h gives the committee a say on x, y. And h gives the committee a say on every pair since $(\mathcal{H}, \rho)$ satisfies democratic legitimacy.

To show that $\rho = \rho^\star|_{\tau(\mathcal{H})}$, fix an $h \in \tau(\mathcal{H})$. Then $\rho(h) \supseteq R^h = \rho^\star(h)$ by Observation 4 and the definition of $\rho^\star$, and the containment must be an equality since $\rho(h)$ and $\rho^\star(h)$ are both rankings. ■

I How many W-feasible rankings are W-unimprovable?

This appendix contains two results. In §I.1, we show that for a given tournament W, every W-feasible ranking is W-unimprovable iff W is transitive. In

[54] $\tau(\mathcal{H}) \subseteq \tau(\mathcal{H}')$ holds exactly if $\mathcal{H} \subseteq \mathcal{H}'$ *and* any $h \in \tau(\mathcal{H})$ is terminal in $\mathcal{H}'$.

§I.2, we show that on average across tournaments W, only a small fraction of W-feasible rankings are W-unimprovable if there are many alternatives.

I.1 When is agenda-setting valuable?

Given the general will W, the value of agenda-setting lies in being able to reach a W-unimprovable ranking rather than some (necessarily W-feasible) ranking that is not W-unimprovable. This motivates the following definition:

Definition 30. Given her preference $\succ$ (a ranking), the chair *benefits from agenda-setting* under a tournament W iff there exists a W-feasible ranking that is not W-unimprovable.

Proposition 6. For a tournament W, the following are equivalent:

(1) W is not a ranking (i.e. is not transitive).

(2) For some $\succ$, the chair benefits from agenda-setting under W.

(3) For every $\succ$, the chair benefits from agenda-setting under W.

In words, agenda-setting is valuable precisely because it allows the chair to exploit Condorcet cycles: the chair benefits whenever there is a cycle in W, and otherwise does not benefit.

The proof relies on concepts and results from §3–§4.

Proof. (3) immediately implies (2). To see that (2) implies (1), consider the contra-positive: if W is a ranking, then it is clearly the only W-feasible ranking, so the chair does not benefit from agenda-setting for any $\succ$.

To show that (1) implies (3), fix any ranking $\succ$ and any tournament W that is not a ranking; it suffices to exhibit distinct W-feasible rankings R and R' such that R is more aligned with $\succ$ than R'. Call a ranking W-*anti-efficient* iff it is W-efficient when the chair's preference is reversed: for any pair $x, y \in \mathcal{X}$ of alternatives with $x \prec y$ and $x \; W \; y$, we have $x \; R \; y$.

By Theorem 1 (p. 16), there exist W-feasible rankings R and R' which are, respectively, W-efficient and W-anti-efficient. We claim that R is more aligned with $\succ$ than R'. To show this, take $x, y \in \mathcal{X}$ with $x \succ y$. If $x \; W \; y$, then $x \; R \; y$ since R is W-efficient. If instead $y \; W \; x$, then $y \; R' \; x$ since R' is W-anti-efficient. In sum, either $x \; R' \; y$ fails or $x \; R \; y$ holds, which is to say that $x \; R' \; y$ implies $x \; R \; y$.

It remains only to show that R and R' are distinct. Since W is not a ranking, there must be $x, y, z \in \mathcal{X}$ such that $x \; W \; y \; W \; z \; W \; x$. Suppose wlog that $x \succ z$. There are three cases. If $x \succ y \succ z$, then $x \; R \; y \; R \; z$ and

$z\ R'\ x\ R'\ y$, which are distinct. If $y \succ x \succ z$, then $y\ R\ z\ R\ x$ and $z\ R'\ x\ R'\ y$, which are distinct. If $x \succ z \succ y$, then $x\ R\ y\ R\ z$ and $y\ R'\ z\ R'\ x$, which are distinct. ■

I.2 Most W-feasible rankings are not W-unimprovable

The following asserts that when there are enough alternatives, only a small fraction of a typical W's W-feasible rankings are W-unimprovable.

Proposition 7. For each $n \in \mathbf{N}$, let R^n (W^n) denote a uniform random draw from the set of all rankings (tournaments) on $\mathcal{X}_n \coloneqq \{1, \ldots, n\}$, with R^n and W^n independent. Then

$$\Pr\left(R^n \text{ is } W^n\text{-unimprovable} | R^n \text{ is } W^n\text{-feasible}\right) \to 0 \quad \text{as } n \to \infty.$$

Proof. Fix any $n \geq 5$, and define $K_n \coloneqq \lfloor (n-1)/4 \rfloor$. Further fix a ranking R and a tournament W on $\mathcal{X}_n$, and label the alternatives $\mathcal{X}_n = \{x_1, \ldots, x_n\}$ so that $x_1\ R\ \ldots\ R\ x_n$. Given $k \in \{1, \ldots, K_n\}$, say that R *admits a local W-improvement at* $(x_{4k-2}, x_{4k-1}, x_{4k})$ iff both

- $x_{4k-3}\ W\ x_{4k}\ W\ x_{4k-2}$ and $x_{4k-1}\ W\ x_{4k+1}$, and
- $x_{4k} \succ x_{4k-1}$ and $x_{4k} \succ x_{4k-2}$.

If R admits a local W-improvement at $(x_{4k-2}, x_{4k-1}, x_{4k})$, then it fails to be W-unimprovable since the ranking

$$x_1\ R'\ \cdots\ R'\ x_{4k-3}\ R'\ x_{4k}\ R'\ x_{4k-2}\ R'\ x_{4k-1}\ R'\ x_{4k+1}\ R'\ \cdots\ R'\ x_n$$

is then W-feasible (by Observation 1 in appendix B.3 (p. 27)) and more aligned with $\succ$.

For each $n \geq 5$, let $(X_k^n)_{k=1}^n$ be random variables such that

$$\{X_1^n, \ldots, X_n^n\} = \mathcal{X}_n \quad \text{and} \quad X_1^n\ R^n\ \ldots\ R^n\ X_n^n \quad \text{a.s.}$$

The events '$X_{4k}^n \succ X_{4k-1}^n$ and $X_{4k}^n \succ X_{4k-2}^n$' are independent across $k \in \{1, \ldots, K_n\}$ and each have probability 1/4. It follows by Observation 1 that conditional on R^n being W^n-feasible, the events

$$\text{`}R^n \text{ admits a local } W^n\text{-improvement at } \left(X_{4k-2}^n, X_{4k-1}^n, X_{4k}^n\right)\text{'}$$

are independent across $k \in \{1, \ldots, K_n\}$ and have probability $(1/2)^5$. Thus

$$\Pr\left(R^n \text{ is } W^n\text{-unimprovable} | R^n \text{ is } W^n\text{-feasible}\right) \leq \left(1 - (1/2)^5\right)^{K_n},$$

which vanishes as $n \to \infty$ since $K_n = \lfloor (n-1)/4 \rfloor$ diverges. ■

J Indecisive votes

In this appendix, we allow the vote on a pair of alternatives to be indecisive, in which case the chair may choose how they are ranked. (This occurs e.g. when the chair is a voting member of the committee.) To that end, we re-interpret $x \; W \; y$ to mean that the chair is *permitted* to rank x above y, and allow for the possibility that both $x \; W \; y$ and $y \; W \; x$. A vote on $\{x, y\}$ with $x \; W \; y$ is *indecisive* if also $y \; W \; x$, and *decisive* otherwise.

The general will W must still be total and irreflexive, but not necessarily asymmetric. By appeal to an argument similar to that for Fact 1 (appendix B.2, p. 27), *any* total and irreflexive relation W should be considered.

A history still records what pairs were offered and how each pair was ranked, and a *strategy* now specifies not only what pair to offer after each history, but also how to rank them if the vote is indecisive. Note that a history does not record whether a vote was decisive or not, and thus that we rule out strategies that condition on this information. We show in supplemental appendix K how this restriction may be dropped.

Regret-free and efficient strategies are defined as before, with 'for any tournament W' replaced by 'for any total and irreflexive W'. By Lemma 1 (p. 13), efficiency still implies regret-freeness.

When the chair offers $\{x, y\}$ with $x \succ y$ and the vote is indecisive, we say that she *ranks in her interest* iff she ranks x above y, and *against her interest* otherwise. Augment the definition of the insertion sort in §4 so that the chair ranks in her interest whenever a vote is indecisive. Theorem 1 (§4, p. 16) remains true, with the same proof: insertion sort is efficient, and thus regret-free.

The characterisations of regret-free strategies (Theorems 2 and 3 in §5, pp. 17 and 19) extend as follows:

Theorem (2+3)′. For a strategy σ, the following are equivalent:

(a) σ is regret-free.

(b) σ is efficient.

(c) σ never misses an opportunity, takes a risk, or ranks against the chair's interest.

Proof. We establish the implications depicted in Figure 3 (p. 20). The proof that (c) implies (b) given in appendix D.1 applies essentially unchanged. That (b) implies (a) follows from Lemma 1 (p. 13).

To show that (a) implies (c), observe that the proof in appendix D.2 establishes that regret-free strategies never miss an opportunity or take a risk. It therefore remains only to show that a regret-free strategy must not rank against the chair's interest.

We prove the contra-positive. Let σ be a strategy that ranks against the chair's interest under some tournament W; we shall find a tournament W' such that the outcome R of σ under W' fails to be W'-unimprovable. In particular, we shall exhibit a W'-feasible ranking $R' \neq R$ that is more aligned with $\succ$ than R.

Let T be the first period in which σ ranks against the chair's interest under W. Write R_{T-1} for the associated end-of-period-$(T-1)$ proto-ranking, and let $\{x, y\}$ be the pair offered in period T. By hypothesis, $x \; W \; y \; W \; x$, and the chair chooses to rank y above x.

By the extension corollary in appendix C (p. 29), there exists a ranking $R' \supseteq R_{T-1} \cup \{(x, y)\}$ such that x, y are R'-adjacent. Define a tournament W' by $W' := R' \cup \{(y, x)\}$, and denote by R the outcome of σ under W'. Clearly R' is W'-feasible. It remains to show that $R \neq R'$ and that R' is more aligned with $\succ$ than R.

For the former, since $x \; R' \; y$, it suffices to show that $y \; R \; x$. To this end, observe that that $R_{T-1} \subseteq R' \subseteq W'$. Thus the history of length $T-1$ generated by σ and W' is the same as that generated by σ and W, which means in particular that $\{x, y\}$ is offered in period T, and that y is ranked above x if the vote is indecisive. Under W', the vote is indeed indecisive ($x \; W' \; y \; W' \; x$), and thus $y \; R \; x$ as desired.

To show that R' is more aligned with $\succ$ than R, observe that W' agrees with R' on every pair $\{z, w\} \nsubseteq \{x, y\} = [x, y]_{R'}$. It follows by Lemma 3 in appendix D.2 (p. 33) that R and R' agree on every pair $\{z, w\} \neq \{x, y\}$. Since $x \succ y$ and $x \; R' \; y$, it follows that R' is more aligned with $\succ$ than R. ■

All of the remaining results also extend: the characterisations of regret-freeness are tight (Propositions 1 and 2 in §5, pp. 18 and 20), the outcome-equivalents of insertion sort are (include) the lexicographic (amendment) strategies (Theorem 4 and Proposition 3 in §6, pp. 22 and 23), and sincere voting is dominant (Proposition 4 in §7, p. 25).

K Strategies with extended history-dependence

By definition, a strategy does not condition on who voted how in the past. To relax this restriction, let an *extended history* be a sequence of pairs offered and votes cast by each member on each pair, and let an *extended strategy* assign to each extended history an unranked pair of alternatives.

Recall from appendix B.2 (p. 27) the definition of voting profiles. The *outcome* of an extended strategy σ under a voting profile $(V_i)_{i=1}^I$ is the final ranking that results. A strategy is *regret-free* (*efficient*) iff its outcome under every voting profile $(V_i)_{i=1}^I$ is W-unimprovable (W-efficient), where W denotes the general will of $(V_i)_{i=1}^I$.

Insertion sort is clearly an extended strategy, so is efficient by Theorem 1 (§4, p. 16). Our characterisation of regret-freeness (Theorems 2 and 3 in §5, pp. 17 and 19) remains valid:

Theorem (2+3)″. For an extended strategy σ, the following are equivalent:

(a) σ is regret-free.

(b) σ is efficient.

(c) σ never misses an opportunity or takes a risk.

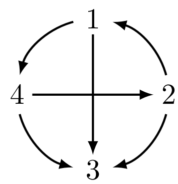
Proof. We prove the implications depicted in Figure 3 (p. 20). That (c) implies (b) follows from the argument in appendix D.1, which applies unchanged to extended strategies. That (b) implies (a) follows from Lemma 1 (p. 13).

To show that (a) implies (c), we prove the contra-positive by augmenting the argument in appendix D.2. Take an extended strategy σ that misses an opportunity or takes a risk under some voting profile $(V_i)_{i=1}^I$, and let t be the first period in which this occurs. Let W be the general will of $(V_i)_{i=1}^I$. Construct an alternative general will W' exactly as in the proof in appendix D.2. Construct in addition a voting profile $(V_i')_{i=1}^I$ whose general will is W', and such that the extended history up to time t under σ and $(V_i')_{i=1}^I$ is the same as under σ and $(V_i)_{i=1}^I$. The argument in appendix D.2 ensures that the outcome of σ under $(V_i')_{i=1}^I$ fails to be W'-unimprovable. Thus σ fails to be regret-free. ■

L Insertion sort and recursive amendment are distinct

Proposition 3 (p. 23) asserts that recursive amendment is outcome-equivalent to insertion sort. The following example shows that the two are nonetheless distinct: they offer the same votes, but not always in the same order.

Example 3. Consider alternatives $\mathcal{X} = \{1, 2, 3, 4\}$, where the chair's preference is $1 \succ 2 \succ 3 \succ 4$ and the general will W is as follows:



Insertion sort first offers $\{3, 4\}$ ($4\ R\ 3$). It then inserts 2 into $\{3, 4\}$ by offering $\{2, 4\}$ ($4\ R\ 2$), then $\{2, 3\}$ ($4\ R\ 2\ R\ 3$). It finally inserts 1 into $\{2, 3, 4\}$ by offering $\{1, 4\}$.

Recursive amendment also first offers $\{3, 4\}$ (4 wins). It then pits the winner 4 against 2 (4 wins again), and then 4 against 1, leaving 1 the final winner. Next, the amendment algorithm is applied to $\{2, 3, 4\}$, with final winner 4.[55] Finally, the amendment algorithm is applied to $\{2, 3\}$, and the (final) winner 2 is ranked third.

M Relation to ranking methods

In this appendix, we investigate the link with the social choice literature mentioned in §1.1 (p. 4). We recast the chair's problem as a choice among ranking methods, characterise the constraint set of this problem, and compare its solutions to ranking methods in the literature.

A *ranking method* is a map that assigns to each tournament a ranking. Each strategy σ induces a ranking method, namely the one whose value at a tournament W is the outcome of σ under W. Call a ranking method ρ *feasible* iff it is induced by some strategy, and *regret-free* iff $\rho(W)$ is W-unimprovable for every W. Clearly the chair's problem in §2 can be re-formulated as a choice between ranking methods, where the constraint set consists of the feasible ranking methods and the objective is to choose a regret-free one.

For a tournament W and rankings R, R', say that R is *more aligned with W than R'* iff for any pair $x, y \in \mathcal{X}$ of alternatives with $x\ W\ y$, if $x\ R'\ y$ then also $x\ R\ y$. This is exactly the definition in the text (§2.4, p. 10), except that we allow W to be any tournament (not necessarily a ranking).

Definition 31. A ranking method ρ is *faithful* iff for every tournament W, no ranking $R \neq \rho(W)$ is more aligned with W than $\rho(W)$.

[55] Since $4\ R\ 2$ and $4\ R\ 3$ had already been determined, no votes are actually offered.

Faithfulness clearly admits a normative interpretation. It is a natural strengthening of Condorcet consistency, the requirement that $\rho(W)$ rank x highest if $x \; W \; y$ for every alternative $y \neq x$. The following shows that it also has a positive interpretation:

Observation 5. A ranking method ρ is faithful iff $\rho(W)$ is W-feasible for every tournament W.

Proof. Fix a ranking method ρ and a tournament W, and write $R \coloneqq \rho(W)$. If R is W-feasible, then any $R' \neq R$ fails to be more aligned with W since it must rank some R-adjacent pair $x \; R \; y$ as $y \; R' \; x$, where $x \; W \; y$ by Observation 1 (appendix B.3, p. 27). If R is not W-feasible, then by Observation 1 there is an R-adjacent pair $x \; R \; y$ such that $y \; W \; x$, so the ranking $R' \neq R$ that agrees with R on every pair but x, y is more aligned with W. ■

By Observation 5, any feasible ranking method must be faithful. The converse does not hold, because feasibility also imposes restrictions across tournaments. To describe these constraints, we introduce a second property:

Definition 32. A ranking method ρ is *consistent* iff whenever $\rho(W) \neq \rho(W')$ for two tournaments W and W', there are alternatives $x, y \in \mathcal{X}$ such that $x \; W \; y \; W' \; x$ and

$$x \; \rho(W'') \; y \quad \text{iff} \quad x \; W'' \; y \quad \text{for every tournament } W'' \supseteq W \cap W'.$$

This property is mathematically natural, but we do not think that it has any normative appeal. Instead, it captures constraints that the rules of the interaction impose on the chair:

Proposition 8. A ranking method is feasible iff it is faithful and consistent.

Call a ranking method ρ *efficient* iff $\rho(W)$ is W-efficient for every tournament W. (W-efficiency is defined on p. 12.) By Theorem 2 (p. 17), a feasible ranking method is regret-free iff it is efficient. Thus:

Corollary 4. A ranking method is feasible and regret-free iff it is faithful, consistent and efficient.

While faithfulness has a normative interpretation, consistency and efficiency are 'positive' in nature: the former is a constraint imposed by the rules of the game, while the latter is defined in terms of the chair's self-interested preference $\succ$. This makes feasible and regret-free ranking methods quite different from those studied in the literature, which are characterised by purely normative axioms (e.g. Rubinstein (1980) for the Copeland method). Indeed, standard ranking methods such as those of Copeland and Kemeny–Slater are neither consistent nor efficient, though the latter is faithful.

Proof of Proposition 8. For necessity, let ρ be feasible. Then ρ is faithful by Observation 5. To show that it is consistent, let σ be a strategy inducing ρ, and fix tournaments W and W' such that $\rho(W) \neq \rho(W')$. Let t be the first period in which the history generated by σ and W differs from that generated by σ and W', and let $\{x, y\}$ be the pair offered in this period. Then W and W' disagree on $\{x, y\}$. Furthermore, the pair $\{x, y\}$ is clearly offered in period t of the history generated by σ and any $W'' \supseteq W \cap W'$, so that $x\ \rho(W'')\ y$ iff $x\ W''\ y$.

For sufficiency, let ρ be faithful and consistent; we shall construct a strategy that induces ρ. For each history h, let W_h and W'_h be tournaments such that

(a) if $h = ((x_t, y_t))_{t=1}^T$, then $x_t\ W_h\ y_t$ and $x_t\ W'_h\ y_t$ for each $t \in \{1, \dots, T\}$, and

(b) W_h and W'_h disagree on any pair that is not voted on in h.

Since ρ is faithful, $\rho(W_h)$ is W_h-feasible and $\rho(W'_h)$ is W'_h-feasible by Observation 5. Thus by Observation 1 (appendix B.3, p. 27), we have $\rho(W_h) = \rho(W'_h)$ iff h is terminal. Since ρ is consistent, we may for each non-terminal history h choose a pair $\sigma(h) := \{x, y\} \subseteq \mathcal{X}$ that satisfies

(c) $x\ W_h\ y\ W'_h\ x$ and

(d) $x\ W''\ y$ iff $x\ \rho(W'')\ y$ for any tournament $W'' \supseteq W_h \cap W'_h$.

Claim. Let $h = ((x_t, y_t))_{t=1}^T \neq \varnothing$ be a history such that $\{x_1, y_1\} = \sigma(\varnothing)$ and

$$\{x_t, y_t\} = \sigma\left(((x_s, y_s))_{s=1}^{t-1}\right) \quad \text{for each } t \in \{2, \dots, T\}.$$

Then (i) for any tournament W'' with $x_t\ W''\ y_t$ for each $t \in \{1, \dots, T\}$, we have $x_t\ \rho(W'')\ y_t$ for each $t \in \{1, \dots, T\}$, and (ii) the pair $\sigma(h)$ is unranked by the transitive closure of $\bigcup_{t=1}^T \{(x_t, y_t)\}$.

Proof of the claim. For the first part, fix a $t \in \{1, \dots, T\}$ and a tournament W'' such that $x_s\ W''\ y_s$ for each $s \in \{1, \dots, T\}$. Define $h' := ((x_s, y_s))_{s=1}^{t-1}$ (meaning $h' = \varnothing$ if $t = 1$), noting that $\sigma(h') = \{x_t, y_t\}$. We have $W'' \supseteq W_{h'} \cap W'_{h'}$ since $W_{h'}$ and $W'_{h'}$ satisfy (b), whence $x_t\ \rho(W'')\ y_t$ by (d).

For the second part, we have $x_t\ \rho(W_h)\ y_t$ and $x_t\ \rho(W'_h)\ y_t$ for every $t \in \{1, \dots, T\}$ by (a) and the first part of the claim, implying that $\rho(W_h)$ and $\rho(W'_h)$ (being transitive) agree on every pair ranked by the transitive closure of $\bigcup_{t=1}^{T}\{(x_t, y_t)\}$. Since $\rho(W_h)$ and $\rho(W'_h)$ disagree on the pair $\sigma(h)$ by (c) and (d), it follows that $\sigma(h)$ is unranked by the transitive closure. □

By the second part of the claim, σ is a well-defined strategy.[56] To show that it induces ρ, fix a tournament W, and let $h = ((x_t, y_t))_{t=1}^{T}$ be the terminal history generated by σ and W; we must demonstrate that $\rho(W)$ is the transitive closure of $\bigcup_{t=1}^{T}\{(x_t, y_t)\}$. Since both are rankings, it suffices to show that $x_t\ \rho(W)\ y_t$ for every $t \in \{1, \dots, T\}$. And this follows from the claim since $x_t\ W\ y_t$ for every $t \in \{1, \dots, T\}$. ■

References

Apesteguia, J., Ballester, M. A., & Masatlioglu, Y. (2014). A foundation for strategic agenda voting. *Games and Economic Behavior*, *87*, 91–99. doi:10.1016/j.geb.2014.05.006

Arrow, K. J. (1950). A difficulty in the concept of social welfare. *Journal of Political Economy*, *58*(4), 328–346. doi:10.1086/256963

Arrow, K. J. (1951). *Social choice and individual values*. New York, NY: Wiley.

Arrow, K. J. (1963). *Social choice and individual values* (2nd). New Haven, CT: Yale University Press.

Banks, J. S. (1985). Sophisticated voting outcomes and agenda control. *Social Choice and Welfare*, *1*(4), 295–306. doi:10.1007/BF00649265

Barberà, S. & Gerber, A. (2017). Sequential voting and agenda manipulation. *Theoretical Economics*, *12*(1), 211–247. doi:10.3982/TE2118

Bell, D. E. (1982). Regret in decision making under uncertainty. *Operations Research*, *30*(5), 961–981. doi:10.1287/opre.30.5.961

Black, D. (1958). *The theory of committees and elections*. Cambridge: Cambridge University Press.

Bossert, W. & Suzumura, K. (2010). *Consistency, choice, and rationality*. Cambridge, MA: Harvard University Press.

Charon, I. & Hudry, O. (2010). An updated survey on the linear ordering problem for weighted or unweighted tournaments. *Annals of Operations Research*, *175*(1), 107–158. doi:10.1007/s10479-009-0648-7

[56] We actually defined σ only on the path. Off the path, any behaviour will do.

Copeland, A. (1951). *A reasonable social welfare function*. Notes from University of Michigan seminar on applications of mathematics to the social sciences.

Curello, G. & Sinander, L. (2019). *The lattice structure of preference comparison*. Working paper, 19 Feb 2019. arXiv: 1902.07260

Daniels, H. E. (1969). Round-robin tournament scores. *Biometrika*, *56*(2), 295–299. doi:10.2307/2334422

Farquharson, R. (1969). *Theory of voting*. New Haven, CT: Yale University Press.

Fishburn, P. C. (1983). Transitive measurable utility. *Journal of Economic Theory*, *31*(2), 293–317. doi:10.1016/0022-0531(83)90079-0

Gershkov, A., Kleiner, A., Moldovanu, B., & Shi, X. (2019). *The art of compromising: Voting with interdependent values and the flag of the Weimar Republic*. Working paper, 9 Sep 2019.

Gershkov, A., Moldovanu, B., & Shi, X. (2017). Optimal voting rules. *Review of Economic Studies*, *84*(2), 688–717. doi:10.1093/restud/rdw044

Gershkov, A., Moldovanu, B., & Shi, X. (2019a). *Monotonic norms and orthogonal issues in multidimensional voting*. Working paper, 6 Sep 2019.

Gershkov, A., Moldovanu, B., & Shi, X. (2019b). Voting on multiple issues: What to put on the ballot? *Theoretical Economics*, *14*(2), 555–596. doi:10.3982/TE3193

González-Díaz, J., Hendrick, R., & Lohmann, E. (2014). Paired comparisons analysis: An axiomatic approach to ranking methods. *Social Choice and Welfare*, *42*(1), 139–169. doi:10.1007/s00355-013-0726-2

Gordon, G. J. (1999). Regret bounds for prediction problems. In *Proceedings of the twelfth annual conference on computational learning theory* (pp. 29–40). New York, NY: Association for Computing Machinery. doi:10.1145/307400.307410

Groseclose, T. & Milyo, J. (2010). Sincere versus sophisticated voting in Congress: Theory and evidence. *Journal of Politics*, *72*(1), 60–73. doi:10.1017/s0022381609990478

Horan, S. (2020). *Agendas in legislative decision-making*. Working paper, 5 Apr 2020.

Kemeny, J. G. (1959). Mathematics without numbers. *Daedalus*, *88*(4), 577–591. JSTOR: 20026529

Kendall, M. G. (1955). Further contributions to the theory of paired comparisons. *Biometrics*, *11*(1), 43–62. doi:10.2307/3001479

Kleiner, A. & Moldovanu, B. (2017). Content-based agendas and qualified majorities in sequential voting. *American Economic Review*, *107*(6), 1477–1506. doi:10.1257/aer.20160277

Knuth, D. E. (1998). *The art of computer programming, volume 3: Sorting and searching* (2nd). Upple Saddle River, NJ: Addison–Wesley.

Landha, K. K. (1994). Coalitions in congressional voting. *Public Choice*, *78*(1), 43–63. doi:10.1007/BF01053365

Loomes, G. & Sugden, R. (1982). Regret theory: An alternative theory of rational choice under uncertainty. *The Economic Journal*, *92*(368), 805–824. doi:10.2307/2232669

McGarvey, D. C. (1953). A theorem on the construction of voting paradoxes. *Econometrica*, *21*(4), 608–610. doi:10.2307/1907926

Milgrom, P. & Shannon, C. (1994). Monotone comparative statics. *Econometrica*, *62*(1), 157–180. doi:10.2307/2951479

Miller, N. R. (1977). Graph-theoretical approaches to the theory of voting. *American Journal of Political Science*, *21*(4), 769–803. doi:10.2307/2110736

Moon, J. W. & Pullman, N. J. (1970). On generalized tournament matrices. *SIAM Review*, *12*(3), 384–399. doi:10.1137/1012081

Myerson, R. B. (1991). *Game theory*. Cambridge, MA: Harvard University Press.

Ordeshook, P. C. & Palfrey, T. R. (1988). Agendas, strategic voting, and signaling with incomplete information. *American Journal of Political Science*, *32*(2), 441–466. doi:10.2307/2111131

Oscarsson, H. (2019). *Rekordlåg personröstning*. In U. Andersson, B. Rönnerstrand, P. Öhberg, & A. Bergström (Eds.), *Storm och stiltje* (pp. 107–116). Gothenburg: SOM-institutet.

Poole, K. T. & Rosenthal, H. (1997). *Congress: A political-economic history of roll call voting*. New York, NY: Oxford University Press.

Roessler, C., Shelegia, S., & Strulovici, B. (2018). Collective commitment. *Journal of Political Economy*, *126*(1), 347–380. doi:10.1086/694294

Rousseau, J.-J. (1755a). *Discours sur l'origine et les fondements de l'inégalité parmi les hommes*. Amsterdam: Michel Rey.

Rousseau, J.-J. (1755b). *Economie ou oeconomie*. In J.-B. d'Alembert & D. Diderot (Eds.), *Encyclopédie, ou dictionnaire raisonné des sciences, des arts et des métiers* (Vol. 5, pp. 337–349). Paris: le Breton, David, Durand and Briasson.

Rousseau, J.-J. (1762). *Du contrat social; ou principes du droit politique*. Amsterdam: Michel Rey.

Rubinstein, A. (1980). Ranking the participants in a tournament. *SIAM Journal on Applied Mathematics*, *38*(1), 108–111. doi:10.1137/0138009

Savage, L. J. (1951). The theory of statistical decision. *Journal of the American Statistical Association*, *46*(253), 55–67. doi:10.2307/2280094

Slater, P. (1961). Inconsistencies in a schedule of paired comparisons. *Biometrika*, *48*(3–4), 303–312. doi:10.1093/biomet/48.3-4.303

Slutzki, G. & Volij, O. (2005). Ranking participants in generalized tournaments. *International Journal of Game Theory*, *33*(2), 255–270. doi:10.1007/s00182-005-0197-5

Suzumura, K. (1976). Remarks on the theory of collective choice. *Economica*. New Series, *43*(172), 381–390. doi:10.2307/2553273

Wei, T.-H. (1952). *Algebraic foundations of ranking theory* (doctoral thesis, University of Cambridge).

Wilkerson, J. D. (1999). "Killer" amendments in Congress. *American Political Science Review*, *93*(3), 535–552. doi:10.2307/2585573

Young, H. P. (1986). Optimal ranking and choice from pairwise comparisons. In B. Grofman & G. Owen (Eds.), *Information pooling and group decision making* (pp. 113–122). Greenwich, CT: JAI Press.

Young, H. P. (1988). Condorcet's theory of voting. *American Political Science Review*, *82*(4), 1231–1244. doi:10.2307/1961757

Young, H. P. & Levenglick, A. (1978). A consistent extension of Condorcet's election principle. *SIAM Journal on Applied Mathematics*, *35*(2), 285–300. doi:10.1137/0135023

Zermelo, E. (1929). Die Berechnung der Turnier-Ergebnisse als ein Maximumproblem der Wahrscheinlichkeitsrechnung. *Mathematische Zeitschrift*, *29*(1), 436–460. doi:10.1007/BF01180541

Zinkevich, M. (2003). Online convex programming and generalized infinitesimal gradient ascent. In *Proceedings of the twentieth international conference on machine learning* (pp. 928–935). Washington, DC: Association for the Advancement of Artificial Intelligence.