Firm Dynamics and the Granular Hypothesis[*]

Vasco Carvalho[†]   Basile Grassi[‡]

March 2013

[INCOMPLETE VERSION]

[PLEASE DO NOT QUOTE NOR CIRCULATE]

Abstract

This paper quantifies the "granular" hypothesis in a firm dynamics framework. To do so, we analyze a standard firm dynamics setting (hopenhayn, 1992) with a finite number of firms, each subject to a persistent idiosyncratic productivity shocks. We show, theoretically, that the size distribution having the fattest tail among incumbents and entrants governs the output volatility rate of decay. The model, calibrated for the US economy with more than 5 million firms, generates fluctuations of aggregate TFP (respectively output) of 0.8% (respectively 2.5%). The entry rate (resp. exit rate) is procyclical (resp. a-cyclical), as in the data. Finally, the structure of the model allows to study the micro and macro impact of a shock on the biggest firm. Such a shock is contractionary at the aggregate level and expansionary at the idiosyncratic level.

1 Introduction

In modern macroeconomics, the source of aggregate fluctuations relies on fluctuations of a common component or a disturbance that affects all the agents of the economy in the same way. Even in economies with heterogeneous agents subject to idiosyncratic disturbances, the aggregate fluctuations come from a common component and the micro-level shocks average out because of a limit argument.

The micro-origins of aggregate fluctuations is left on the side. However, the question that micro-level disturbances - at the level of firms or individual technologies- account for business cycle fluctuations is worth asking. A recent literature shows that these micro-shocks can generate sizable fluctuations. Indeed, a fat-tailed size distribution of firms or a high network centrality of sectors may render local shocks into aggregate fluctuations. Yet, this literature focuses mostly on possibility results in static settings and there is little evidence to date in terms of quantitative results in dynamic settings.

The aim of this paper is to inspect the "granular" hypothesis (Gabaix 2011) in a standard quantitative firm dynamics setting. The "granular" hypothesis states that if the distribution of firms is fat tailed, then shocks to big firms do not average out and could generate sizable fluctuations. The intuition is that the small number of big firms does not allow the central limit theorem to apply and since the share of big firms in the economy is large, these deviations from the limit cannot be neglected.

In order to do this, we explore the business cycle properties of a standard firm dynamics model with no aggregate shocks (Hopenhayn 1992) and extend the Gabaix 2011 theorem to this setting. The model is calibrated so that the firm size distribution features a power-law tail as in the data. We first quantify aggregate fluctuations in this setting, we then produce impulse response functions of this economy to an idiosyncratic shock to the largest firm, and we finally inspect the cross-sectional and entry/exit properties of this model[1].

The contribution of this paper is twofold. First, we contribute to the literature on micro-origins of aggregate fluctuations by quantifying these fluctuations in a dynamic setting. Second, this framework also shows that using a continuum of firms or using the "law of large number convention" might lead to neglect sizable uncertainty.

The paper relates to two literatures: the micro-origins of aggregate fluctuations literature and the firm dynamics literature. Gabaix 2011 describes the "granular hypothesis" and shows the possibility results that we extend to our framework. Other papers studying the micro-origins of aggregate fluctuations are Acemoglu et al 2012, di Giovanni and Levchenko 2012, Carvalho 2010 and Carvalho and Gabaix 2013. This literature builds on the seminal work of Jovanovic 1987, Bak et al 1993, Scheinkman and Woodford 1994 and Horvath 1998. Some empirical evidence can be found in di

---

[1]This part is not yet completed.

Giovanni, Levchenko and Mejean 2012. This paper is also related to the firm dynamics literature: Hopenhayn 1992, Campbell 1998, Veracierto 2002, Khan and Thomas 2003, 2008, Bachman and Bayer 2009. Some papers studied aggregate fluctuations in an entry/exit framework as Lee and Mukoyama 2008, Clementi and Palazzo 2010 and Bilbiie et al. 2012 however they restrict their analysis to common aggregate shocks.

The paper is organized as follows. Section 2 derives the model. Section 3 defines the equilibrium and describes an algorithm to solve it numerically. Section 4 extends the result of Gabaix 2011 to our framework. Section 5 shows the result of the simulation of the numerical solution of this model. Finally, section 6 concludes.

2   Model Environement

We extend the Hopenyan 1992 economy to allow for a finite (but large) number of firms. There is a finite number of heterogeneous firms that differ in their productivity level, which follows a discrete Markovian process. Incumbents have access to a decreasing return to scale technology using labor as the only input. They face an operating cost at each period which leads to endogenous exit.

Although the productivity process is discrete, the number of states is large (401 in the baseline calibration) and it is a discretization of an AR(1) process using the method described in Tauchen 1986. One can think of the law of motion of the idiosyncratic (log) productivity as being:

$$\varphi_{t+1}^i = \rho \varphi_t^i + e_t^i, e_t^i \rightsquigarrow \mathcal{N}(0, \sigma_e)$$

There is a finite (but large) number of heterogeneous potential entrants, which differ in their signal about their next period productivity. To enter, potential entrants have to suffer a fixed cost.

The demand side of the economy is simplified to an exogenous labor supply which is increasing with the wage.

2.1 The incumbent's problem

As stated above, the level of idiosyncratic productivity is discrete on a grid and follows a Markov chain with a transition matrix P. The productivity space is thus described by a n_s-uple $\{\varphi_1, \ldots, \varphi_{n_s}\}$ such that $\varphi_1 < \ldots < \varphi_{n_s}$. We will say that a firm is in state (or productivity state) k when its idiosyncratic productivity is equal to φ_k. We denote $F(.|\varphi)$ the conditional distribution of the next period idiosyncratic productivity φ' given the current period idiosyncratic productivity φ.[2]

Given an aggregate state λ, and an idiosyncratic productivity level φ, the incumbent solves the following intra period problem[3]:

$$\pi^*(\lambda, \varphi) = \text{Max} \left\{ \exp(\varphi) n^\alpha - wn - c_f \right\}$$

where n is the labor input, w is the wage which depends on the current aggregate state and c_f is the operating cost that a firm should pay every period to operate. One can see that π^* is increasing in φ and decreasing in w for a given aggregate state λ. The output level is then $y(\lambda, \varphi) = \exp(\varphi)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w} \right)^{\frac{\alpha}{1-\alpha}}$. In what follows the size of a firm it will refer to its output if not otherwise specified.

The incumbent timing is the following: she draws its idiosyncratic productivity φ at the beginning of the period, pays the operating cost c_f, hires labor, produces and decides to exit or not. The next period starts by drawing a new idiosyncratic productivity. The associate Bellman equation is thus:

$$V(\lambda, \varphi) = \pi^*(\lambda, \varphi) + \beta \, \text{Max} \left\{ 0, \int_{\lambda' \in \Lambda} \int_{\varphi' \in \Phi} V(\lambda', \varphi') F(d\varphi'|\varphi) \Gamma(d\lambda'|\lambda) \right\}$$

where $\Gamma(.|\lambda)$ is the conditional distribution of λ' (the aggregate state at the next period) given λ the aggregate state at the current period and where $F(.|\varphi)$ is the conditional distribution of the next period idiosyncratic productivity φ' given the current period idiosyncratic productivity φ.

We assume that the conditional distribution $F(.|\varphi)$ is decreasing in φ which is true for an AR(1) process. Let us define:

$$\mathcal{F}(\varphi, \lambda) := \int_{\lambda' \in \Lambda} \int_{\varphi' \in \Phi} V(\lambda', \varphi') F(d\varphi'|\varphi) \Gamma(d\lambda'|\lambda)$$

For each aggregate state λ, since the instantaneous profit is increasing in the idiosyncratic productivity level, there is a unique index $s^*(\lambda)$ such that:

$$\mathcal{F}(\varphi_{s^*(\lambda)}, \lambda) \geq 0 > \mathcal{F}(\varphi_{s^*(\lambda)-1}, \lambda)$$

---

[2] Given a productivity level φ_s the distribution $F(.|\varphi_s)$ is given by the sth-row vector of the matrix P.

[3] Latter it will be shown that this aggregate state is exactly the productivity distribution.

Thus for $\varphi \geq \varphi_{s^*(\lambda)}$[4] the firm continues to operate during the next period and for $\varphi < \varphi_{s^*(\lambda)}$ the firm exits[5].

2.2 Entry

There is a constant and finite number of prospective entrants M. A share $M.G^q$ of them are of type (or signal) q, where q lies within the idiosyncratic productivity level set. The number of entrants of type q is deterministic.

If potential entrants decide to pay the cost of entry c_e, then they produce in the next period with a productivity level drawn from $F(.|q)$. The value of a successful entrant in the aggregate state λ with a type q is thus:

$$V^e(\lambda, q) = \text{Max}\left\{0, \beta \int_{\lambda' \in \Lambda} \int_{\varphi' \in \Phi} V(\lambda', \varphi') F(d\varphi'|q) \Gamma(d\lambda'|\lambda)\right\} = \text{Max}\left\{0, \beta \mathcal{F}(q, \lambda)\right\}$$

A prospective entrant will enter if and only if $V^e(\lambda, q) \geq c_e$. Since $\mathcal{F}(q, \lambda)$ is increasing in the signal q, for any aggregate state λ there is a unique index $e^*(\lambda)$ such that:

$$\mathcal{F}(\varphi_{e^*(\lambda)}, \lambda) \geq \frac{c_e}{\beta} > \mathcal{F}(\varphi_{e^*(\lambda)-1}, \lambda)$$

Thus for $q < \varphi_{e^*(\lambda)}$, we have $V^e(\lambda, q) < c_e$ and thus the q-potential entrant does not enter. Conversely, for $q \geq \varphi_{e^*(\lambda)}$, the q-potential entrant enters. One can show that as soon as $c_e \geq 0$ we have $\varphi_{e^*(\lambda)} \geq \varphi_{s^*(\lambda)}$ with equality for $c_e = 0$. To keep simple the computation, we assume $c_e = 0$ in the rest of this paper.

2.3 Law of Motion of the Productivity Distribution

The distribution of firms across the discrete state space $\{\varphi_1, \ldots, \varphi_{n_s}\}$ is a vector that we call μ_t. It is a $(n_s \times 1)$ vector equal to $(\mu_t^1, \ldots, \mu_t^{n_s})$ such that μ_t^s is equal to the number of operating firms in state s at date t.

In this section we seek to find the law of motion of the productivity distribution, i.e. what would be the next productivity distribution μ_{t+1} given the current one μ_t. The next period distribution is the sum of the evolution of incumbents and successful entrants. To establish that, we need to define two kinds of conditional distribution.

---

[4]We will use $\varphi^*(\lambda)$ and $\varphi_{s^*(\lambda)}$ indifferently.

[5]Given the state, the increasing instantaneous profit implies an increasing value function in φ and then an exit threshold. This result is similar to what is shown in Hopenyan 1992 and Clementi and Palazzo 2010.

The distribution of the date $t + 1$ incumbent conditional on the fact that they were in state s at date t is noted $f_{t+1}^{..,s}$. This $(n_s \times 1)$ vector is such that for each state k in $\{1, \ldots, n_s\}$:

$$f_{t+1}^{k,s} = \text{ the } k^{th} \text{ element of } f_{t+1}^{..,s}$$
$$:= \text{ number of incumbent in state } k \text{ at } t + 1 \text{ which were in state } s \text{ at } t$$

In the same way, let us define $g_{t+1}^{..,s}$ the distribution of successful entrants at date $t + 1$ given that they had signal s at date t. This vector is a $(n_s \times 1)$ vector such that for each state k in $\{1, \ldots, n_s\}$:

$$g_{t+1}^{k,s} = \text{ the } k^{th} \text{ element of } g_{t+1}^{k,s}$$
$$:= \text{ number of successful entrant in state } k \text{ at } t + 1 \text{ which receive a signal } s \text{ at } t$$

The period $t + 1$ productivity distribution is the sum of all this conditional distributions and thus the vector μ_{t+1} satisfies:

$$\mu_{t+1} = \sum_{s=s^*(\lambda_t)}^{n_s} f_{t+1}^{..,s} + \sum_{s=s^*(\lambda_t)}^{n_s} g_{t+1}^{..,s}$$

It is important to emphasize the fact that the $f_{t+1}^{..,s}$ and $g_{t+1}^{..,s}$ are stochastic which implies, that μ_{t+1} also is. At date $t + 1$ for $s \geq s^*(\lambda_t)$, $f_{t+1}^{..,s}$ follows a multinomial distribution with parameters the integer μ_t^s and the $(n_s \times 1)$ vector $P'_{s,.}$ where $P_{s,.}$ is the sth row vector of the matrix P (we note this $Multi(\mu_t^s, P'_{s,.})$). For $s < s^*(\lambda)$, $f_{t+1}^{..,s}$ is equal to zero. Similarly, at date $t + 1$ for $s > s^*(\lambda_t)$, $g_{t+1}^{..,s}$ follow a multinomial distribution with parameters the integer MG^q and the $(n_s \times 1)$ vector $P'_{q,.}$ i.e $Multi(MG^q, P'_{q,.})$.

To understand the above statement, let us assume that there are only three levels of productivity ($n_s = 3$) and 4 firms. These firms are distributed according to the top panel of figure 1. Let us assume that the firms have a probability to go up (respectively down) on the productivity ladder with a probability $1/2$ and to stay in the middle level with a probability $1/4$. If instead of 4 firms we had a continuum of firms, the next period we would have exactly $1/4$ of the firms at the first level, $1/2$ at the middle level and $1/4$ at the top level. This is not the case here, since the number of firms in each node is finite and thus the distribution of firms in the bottom panel of figure 1 is possible with a positive probability. In this particular, in this case the vector $(f_{t+1}^{1,2}, f_{t+1}^{2,2}, f_{t+1}^{3,2})'$ follows a multinomial distribution with a number of trials of 4 and an event probability vector $(1/2, 1/4, 1/2)'$.

It turns out that after an approximation of the multinomial distribution, the date $t + 1$ period productivity distribution vector given the current distribution follows
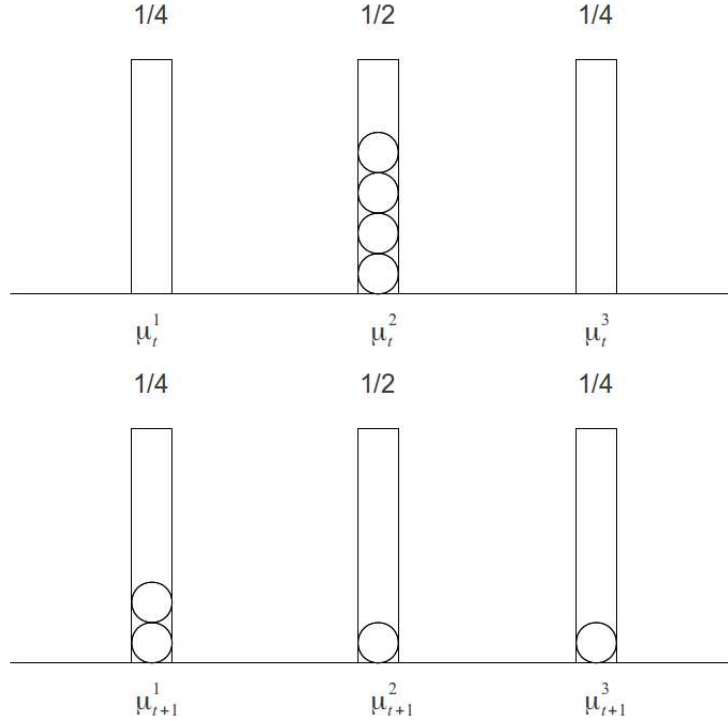
Figure 1: Why the vector $f_{t+1}^{\cdot;s}$ follows a multinomial distribution.

a multivariate Gaussian distribution with a mean and a covariance-variance matrix function of the date t productivity distribution vector.

Using the Central Limit Theorem, one can show that for a big enough n and for a probability vector p, $Multi(n,p) \approx \mathcal{N}(np, nM)$ where $M = diag(p) - p'p$. The following lemma states this formally:

Lemma 1 *Let Y be a m-random vector following a multinomial distribution with parameters $n \in \mathbb{N}$ and $p \in (0,1)^m$ i.e $Y \rightsquigarrow Multi(n,p)$. Let M be the (m,m) matrix $diag(p) - p'p$. Then:*

$$\frac{Y - np}{\sqrt{n}} \longrightarrow^{\mathcal{D}} Z$$

as n goes to infinity and where Z has a m-dimensional multivariate normal distribution with mean vector 0 and covariance matrix M. The convergence is here in distribution.

Proof: See Severini 2005 p377 Example 12.7. □

This lemma applies to the vectors $f_{t+1}^{\cdot;s}$ and $g_{t+1}^{\cdot;s}$. We define $M_s = diag(P_{s,.}) - P'_{s,.}.P_{s,.}$. Since the number μ_{t+1}^s is supposed to be large (i.e. there are many firms)[6], then $f_{t+1}^{\cdot;s}$ is approximatively distributed according $\mathcal{N}(\mu_t^s P'_{s,.}, \mu_t^s M_s)$ the multivariate normal distribution with means $\mu_t^s P'_{s,.}$ and variance-covariance matrix $\mu_t^s M_s$. The same reasoning applies for $g_{t+1}^{\cdot;s}$ which is assumed to follow a multivariate normal

---

[6]$diag(V)$, where V is a vector, is a diagonal matrix with elements of V on the diagonal.

distribution $\mathcal{N}(MG^s P'_{s,.}, MG^s M_s)$. Since the $f^{.;s}_{t+1}$ and the $g^{.;s}_{t+1}$ are independent from each other, it can easily be shown after some computation that:

$$\mu_{t+1} = (P^*_t)'(\mu_t + MG) + \varepsilon_{t+1} \tag{1}$$

where ε_{t+1} follows a multivariate normal distribution with a covariance-variance matrix $\Sigma(\mu_t)$, and P^*_t is the transition matrix P where the first $s^*(\mu_t) - 1$ rows are replaced by zeros. Let us define the mean $m(\mu_t)$ and the covariance-variance matrix $\Sigma(\mu_t)$ of the productivity distribution given the previous productivity distribution:

$$m(\mu_t) = \sum_{s=s^*(\mu_t)}^{n_s} \mu^s_t . P'_{s,.} + MG^s P'_{s,.} = (P^*_t)'(\mu_t + MG)$$

$$\Sigma(\mu_t) = \sum_{s=s^*(\mu_t)}^{n_s} (\mu^s_t + MG^s) . M_s$$

In the rest of this paper, we will show that the aggregate state is the productivity distribution. Writing the law of motion in this fashion allows to switch from a discrete to a continuous state space $\mathbb{R}^{n_s}_+$. This representation allows to show theoretical results on the aggregate fluctuations and to build an algorithm to solve this model.

2.4 Market Clearing and Aggregation

If Y_t is the aggregate output, i.e. the sum of all individual firms' output, then $Y_t = A_t(L^d_t)^\alpha$ where L^d_t is the aggregate labor demand and A_t the aggregate total factor productivity, which is equal to:

$$A_t = \left(\sum_{i=1}^{N_t} \exp(\varphi^i_t)^{\frac{1}{1-\alpha}} \right)^{1-\alpha}$$

where φ^i_t is the productivity level at date t of the ith firm among the N_t operating firms at date t. This can be rewritten by aggregating the firms which have the same productivity level:

$$A_t = \left(\sum_{s=1}^{n_s} \mu^s_t \exp(\varphi_s)^{\frac{1}{1-\alpha}} \right)^{1-\alpha} = (B'.\mu_t)^{1-\alpha}$$

where B is the $(n_s \times 1)$ vector of parameters $(\exp(\varphi_1)^{\frac{1}{1-\alpha}}, \ldots, \exp(\varphi_{n_s})^{\frac{1}{1-\alpha}})$ and where . is the matrix product.

The labor demand is $L^d(w_t) = \left(\frac{\alpha A_t}{w_t} \right)^{\frac{1}{1-\alpha}}$. The model is like a one factor model with aggregate TFP A_t. The only market we clear is the labor market. In a partial equilibrium fashion, we assume that the supply of labor at a given wage w is $L^s(w) = \underline{L}w^\gamma$ with $\gamma > 0$. The market clearing condition is then that labor supply equals

labor demand, i.e. $L^s(w_t) = L^d_t$. Solving for the wage given the date t productivity distribution μ_t yields:

$$w_t = \left(\alpha^{\frac{1}{1-\alpha}} \frac{B'.\mu_t}{\underline{L}} \right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}}$$

From this expression, one can see that the wage is fully pinned down by the distribution μ_t. Also, the distribution of productivity at $t+1$ depends only on the current distribution μ_t. The aggregate state at date t is thus $\lambda_t = \mu_t$.

3 Equilibrium

In this section, we define a deterministic stationary equilibrium which is similar to a deterministic steady state equilibrium. We also define the equilibrium of the model presented above.

3.1 Stationary Equilibrium

We define a stationary equilibrium as an equilibrium without aggregate uncertainty and thus where all variables are constant, that is to say with a deterministic aggregate state μ. The only source of uncertainty of μ is due to the fact that the $f^{\cdot,s}$ and $g^{\cdot,s}$ are random vectors. In a stationary equilibrium, we will assume that this variables are equal to their means $\mu^s P'_{s,\cdot}$ and $MG^q P'_{q,\cdot}$ respectively. This equilibrium is as if instead of considering a finite number of firms, we considered a continuum of firms. In the latter case, the $f^{\cdot,s}$ and $g^{\cdot,q}$ are not stochastic and are equal to their mean.

Let us define the matrix P^* as the matrix P where the first $s^* - 1$ rows are replaced by zeros, and the vector G^* as the vector G where the first $s^* - 1$ rows are replaced by zeros. The law of motion of μ implies $\mu = P^{*'}.\mu + MP'G^*$. Solving for this vectorial equation yields:

$$\mu = M(I - P^{*'})^{-1}P^{*'}G$$

We assume that this stationary distribution is fat tailed, like is the case in the data as shown by Gabaix 2011. In the rest of the paper, we calibrate this distribution to be fat tailed. From this, all other variables follow: the wage w, the aggregate output Y, etc...

3.2 Definition of equilibrium

[TO BE COMPLETED]

3.3 Numerical Solution Algorithm

This section describes the algorithm used to solve numerically for the equilibrium defined in section 3.2.

The state variable of this model is only the distribution of productivity $\mu \in \mathbb{R}_+^{n_s}$. Since n_s should be large, following the evolution of the distribution μ across time is not computationally feasible. To solve this model we use an algorithm similar to Krusell and Smith 1998, where we follow the evolution of the factor that matters for all the aggregate variables, namely T_t defined as:

$$T_t = \sum_{s=1}^{n_s} \mu_t^s \exp(\varphi_s)^{\frac{1}{1-\alpha}} = B'.\mu_t$$

where B is the $(n_s \times 1)$ vector $(\exp(\varphi_1)^{\frac{1}{1-\alpha}}, \ldots, \exp(\varphi_{n_s})^{\frac{1}{1-\alpha}})$.

The true evolution of T_t is:

$$T_{t+1} = B'.m(\mu_t) + B'.\varepsilon_{t+1}$$

or

$$T_{t+1} = B'.m(\mu_t) + \sqrt{B'.\Sigma(\mu_t).B}\,\epsilon_{t+1}$$

where ϵ_{t+1} is drawn from a standard univariate normal distribution. The process followed by $\log(T_t)$ is at the first order:

$$\log(T_{t+1}) = \log(B'.m(\mu_t)) + \frac{\sqrt{B'.\Sigma(\mu_t).B}}{B'.m(\mu_t)}.\epsilon_{t+1}$$

Assuming the following approximation

$$\log(B'.m(\mu_t)) = \alpha_0 + \alpha_1 \log(T_t) + w_t$$
$$\frac{\sqrt{B'.\Sigma(\mu_t).B}}{B'.m(\mu_t)} = \beta_0 + \beta_1 \log(T_t) + v_t$$

leads to the following approximate law of motion

$$\log(T_{t+1}) = \alpha_0 + \alpha_1 \log(T_t) + \beta_0.\epsilon_{t+1} + \beta_1.\epsilon_{t+1} \log(T_t) + u_t \qquad (2)$$

where u_t, v_t and w_t are error terms.

This approximate law of motion is used to compute expectations as in Krusell and Smith 1998. The coefficient are updated using estimation of this equation for a simulated series. We iterate until convergence. The algorithm is formally described bellow:

1. Guess some parameters $\alpha_0^0, \alpha_1^0, \alpha_2^0, \beta_0^0$ and β_1^0.

2. Solve jointly for the value function of an individual firm and the exit rule for all (φ, T) using the approximation law of motion 2 to compute the expectation.

3. Simulate a series of $\{T_t, \epsilon_t\}_{t=0...T}$ as follows:

   (a) Given a μ_0, compute $s^*(T_0)$ from the solution of step 2

   (b) Draw a multivariate Gaussian vector and use it to compute a μ_1, T_1 and ϵ_1, and using the law of motion of the productivity distribution described in equation (1).

   (c) Iterate from step 3a

4. Using the above simulated series, estimate the approximating rule 2.

5. Iterate from step 2 until convergence.

4   Aggregate Fluctuations

Result of Gabaix 2011 applies because μ_t is not too far from the stationary distribution μ^* and in a Gaussian way which will not affect the tail. One can expect that μ_t will also be "fat tailed" and thus, since the factor that matter is

$$T_t = \sum_{i=1}^{N_t} \exp\left(\frac{1}{1-\alpha}\varphi_t^i\right) \tag{3}$$

the rate of decay at which T_t converges to its mean will be lower than $\sqrt{N_t}$. Theorem 1 formulates this idea assuming that there is no entry and exit.

Theorem 1 *Let us assume that there is no entry and exit. Let ξ be the tail parameter of firm size distribution and assume that $\xi/\rho < 2$ and $\xi > 1$, then[7]*

$$\sigma\left(\frac{\Delta Y_t}{Y_t}\right) = \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\sigma\left(\frac{\Delta T_t}{T_t}\right)$$

$$\sigma\left(\frac{\Delta T_t}{T_t}\right) \rightsquigarrow \frac{1}{N_t^{1-\rho/\xi}}\frac{\sigma u^{1/2}}{\bar{Z}_t}$$

where $\bar{Z}_t$ is a time-dependent constant, u is a random variable with finite variance, σ is the standard deviation of $\exp(e_i^t/(1-\alpha))$ and N_t is the number of firms in period t.

Proof See appendix A.1. □

This result extends to the case of entry and exit:

---

[7]ξ is matched to be 1.03 in the calibration described below. In the calibration, ρ turns out to be 0.9796 and thus $\xi/2\rho$ is around 0.5257.

Theorem 2 *Let ξ be the tail parameter of firm size distribution and $\zeta' = \zeta(1 - \alpha)$ be the tail parameter of potential entrant size distribution. Assuming that $\xi/\rho < 2$ and $\zeta'/\rho < 2$ then*

$$\sigma\left(\frac{\Delta T_t}{T_t}\right) \sim \frac{\sigma}{N_t^{1-\rho/\xi}} \frac{\left(\frac{N_t^l}{N_t}\right)^{\rho/\xi} u^{1/2}}{\bar{I}_t} \qquad \text{if } \zeta' > \xi$$

$$\sigma\left(\frac{\Delta T_t}{T_t}\right) \sim \frac{\sigma}{N_t^{1-\rho/\zeta'}} \frac{\left(\frac{N_t^E}{N_t}\right)^{\rho/\zeta'} w^{1/2}}{\bar{I}_t} \qquad \text{if } \zeta' < \xi$$

where $\bar{I}_t$ is a time dependent constant proportional to the incumbent's average size at t, u and w are random variables with finite variance, σ is the standard deviation of $\exp(e_i^t/(1-\alpha))$ and N_t, N_t^l, N_t^E are the number of incumbents, successful incumbents and successful entrants in period t respectively.

Proof See appendix A.2. $\square$

It is worth emphasizing that in the case of entry and exit, the distribution that matters is the fattest distribution between potential entrant size distribution and incumbent size distribution. In the following calibration the relevant case is the one in which $\zeta' < \xi$ and thus the rate of decay of aggregate fluctuations is driven by potential entrant size distribution. The latter has a fatter tail than firm size distribution ($\zeta' = 0,345$ and $\xi = 1.03$).

5   Simulations

5.1   Calibration

Calibrated parameters are summarized in table 1. The parameter α governs the return to scale and is fixed at 0.8. The value is chosen to be on the lower end of recent estimates, such as Basu and Fernald 1997 and Lee 2005. The annual gross interest rate implied by the discount rate β is 4%, which is in line with most macroeconomic studies. The model is calibrated at the annual frequency. The distribution of potential entrants G is such that the distribution of $\exp(\phi)$ for entrants is a Pareto ditribution with a tail parameter ζ. The entry cost c_e is fixed equal to zero as assumed above.

We follow Tauchen 1986 to compute the matrix of transition P with $n_s = 401$ and with $[-13 * \sigma_e, 26 * \sigma_e]$ as the set for φ, a standard deviation of σ_e and an auto-covariance coefficient of ρ. Thus the process described by P is as if the (log) productivity of a firm followed:

$$\varphi_{t+1}^i = \rho\varphi_t^i + e_t^i, e_t^i \rightsquigarrow \mathcal{N}(0, \sigma_e)$$

Parameters	Value	Description
ρ	0.9796	Autocorrelation of firm level shocks
σ_e	0.1596	Std of idio. shocks
n_s	401	Number of productivity level
γ	5	Labor elasticity
α	0.8	Production function
c_f	0.0041	Operating cost
c_e	0	Entry cost
β	0.95	Discount rate
M	10^7	Number of potential entrants
$\underline{L}$	100	Parameter of the labor supply function
G	Pareto(ζ)	Distribution of $\exp(\varphi)$
ζ	1.7263	Tail parameter of entrant distr. G

Table 1: Calibration Parameters

The number of potential entrants M is set to have a number of incumbent firms $N_t = \sum_{s=1}^{n_s} \mu_t^s$ equal to 1.6×10^7. $\underline{L}$ is chosen to have a wage equal to 7. The value of the labor supply elasticity γ is fixed at 5 as in Clementi and Palazzo 2010.

After the choice of the above deep parameters, we are left with 4 parameters to calibrate. These parameters are chosen to match the entry rate, the size of entrants and exiters relative to survivors and the tail estimate of firm size distribution in the US. These targets and the corresponding references are summarized in table 2.

Statistic	Model	Data	Reference
Entry rate	0.015	0.062	Lee and Mukoyama 2008
Entrants' relative size	0.58	0.60	Lee and Mukoyama 2008
Exiters' relative size	0.00	0.49	Lee and Mukoyama 2008
Tail index of firm size dist.	1.03	1.03	Gabaix 2011

Table 2: Calibration Targets

5.2 Numerical solution

This section describes the solution given by running the algorithm presented in section 3.3 for the calibration in table 1. The approximate law of motion for state T_t turns out to be

$$\log(T_{t+1}) = 5.2507 + 0.7882 \log(T_t) - 0.5764\epsilon_{t+1} + 0.0242\epsilon_{t+1} \times \log(T_t) + u_t \quad (4)$$

The R^2 of the last step of the algorithm for this approximate law of motion is 0.99. However, it has been shown in Den Haan 2010 that this is not enough to assert the

quality of the approximate law of motion. figure 2 reproduces both the simulated path of $\log T_t$ and the one using the approximate law of motion (4).Despite some minor differences, one can see that the two paths coincide.
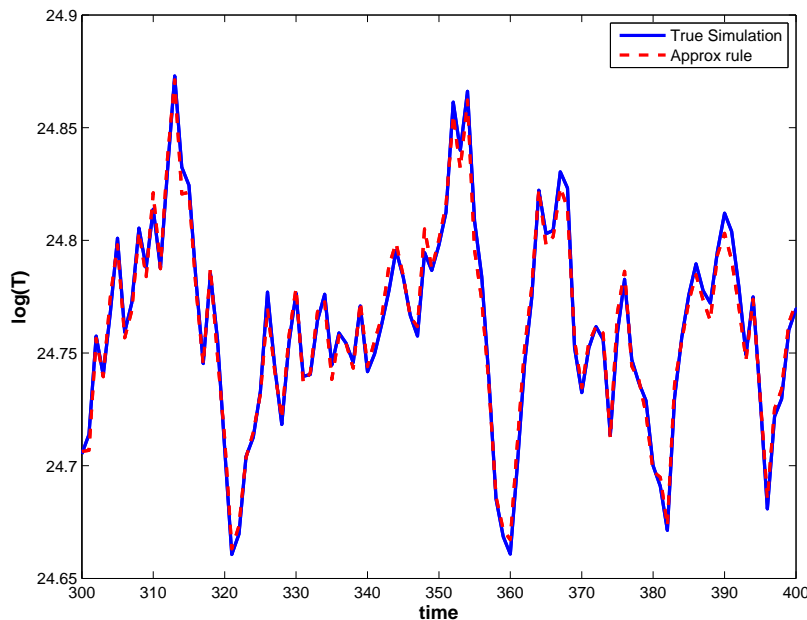

Figure 2: Simulated paths of the true and approximate evolution of T_t

After solving the model using the algorithm described in section 3.3, we compute the business cycle statistics. We simulate time series for output, hours and aggregate TFP using the law of motion (1) of the productivity distribution. These statistics are presented in table 3. First, the standard deviation of output is 2.5% which is in line with a real business cycle framework. It is slightly higher because the elasticity of labor supply in the baseline calibration is chosen to be high. This allows to match the ratio of hour volatility over output volatility (0.8), close to the value in the data. To assess the performance of the model in producing aggregate fluctuations without any aggregate shock, a better statistic is the volatility of aggregate TFP. For the baseline calibration, the standard deviation of aggregate productivity is 0.8%, which is non-negligible.

	Model		
	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(y)}$	$\rho(x,y)$
Output	2.5	1.0	1.0
Hours	2.1	0.8	1.0
Agg. Productivity	0.8	0.3	1.0

Table 3: Business Cycle Statistics

NOTE: These statistics are computed for the baseline calibration (cf. Table 1) for an economy simulated during 20,000 periods

14

The numerical solution is such that the exit/entry threshold s_t^* is constant on the relevant portion of the state space. This implies that the number of entrants does not fluctuate since the potential entrant distribution is exogenous. As shown in table 4, the exit rate fluctuates with a standard deviation of 0.2% which represents only 8.3% of output volatility. Even if the exit threshold is constant the wage, which is the only variable cost that a firm faces, moves. In this framework only small firms choose to exit and thus the wage affects their choice of exiting.

However, the number of exiters is a-cyclical as indicated by the non-significant positive correlation of exit rate and output. The intuition is as follows. In this framework booms are generally good shocks to big firms. During booms, the wage is high since labor demand is high, but small firms are not necessarily more productive. Since small firms have the same productivity but higher costs, their profitability is reduced and they exit more. However, it is also true that in some booms small firms might be more productive and thus less subject to exit. In turn, the number of exiters might go both ways during booms, depending on which firms are affected by good or bad shocks. The number of incumbents is negatively correlated with output, and thus the entry rate is pro-cyclical. At the end of the day, the entry rate is pro-cyclical and the exit rate is a-cyclical as in the data (see for example Lee and Mukoyama 2008).

x	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(Y)}$	$\rho(x,Y)$	$\rho(\frac{x}{N_t},Y)$
# Incumbents (N_t)	0.02	0.009	-0.031	na
			(<0.001)	na
# Entrants	0	0	0	0.031
			(0.000)	(<0.001)
# Exiters	0.20	0.083	0.006	0.010
			(0.381)	(0.181)

Table 4: Cyclicality of entry and exit

NOTE: The standard deviation are computed over a simulated path of 20,000 periods. The p-value are indicated in parenthesis.

5.3 Rate of Decay of Volatility

The natural question that arises is how much these fluctuations depend on the number of active firms? Table 5 presents a response. To compute this table, we increase the equilibrium number of active firms by rasing the number of potential entrants M. However, to be able to compare results we also increase the constant $\underline{L}$ such that the wage is the same across all rows.

As the number of incumbents increases, the standard deviation of both output and TFP decreases. For 100,000 incumbents the volatility of TFP is 7.7% whereas for

M	L	N_t	$\sigma(A)$	$\sigma(Y)$
10^5	1	1.6×10^5	7.67	23.90
10^6	10	1.6×10^6	2.65	8.04
10^7	100	1.6×10^7	0.86	2.57
10^8	1000	1.6×10^8	0.28	0.85

Table 5: Aggregate fluctuations and the number of firms.
NOTE: The standard deviation are computed over a simulated path of 20,000 periods and are indicated in percentage.

100 millions active firms this number drops to 0.3%. In the US economy there are about 5 millions firms, so the TFP volatility implied by this model without aggregate shocks will be between 0.9% and 2.7%.

5.4   Shock to the Biggest Firms: Impulse Response

In this section, we study the aggregate and idiosyncratic impact of a one standard deviation negative shock to the biggest firm. The left panel of figure 3 shows the long-term productivity distribution (red dashed line) along with the one where the biggest firm suffers a one standard deviation negative shock (blue line). Since the difference between these two distributions is not very large, we plot on the right panel the difference[8]. In terms of the productivity distribution, this negative shock on the biggest firm means that the mass on the highest level is moved towards the left.

Figure 4 displays the impulse response function of the aggregate variables. The impact is small: output decreases by about 0.14%, the aggregate TFP by 0.05% and hours by 0.12% compared to their long run value. The reason for that drop of aggregate variables is simple: the biggest and most productive firm suffers a drop of productivity and thus reduces its output and hours. Since this firm was big the effect on the aggregate is sizable. Note that by reducing the hours, the shock induces a drop in wage that benefits all the other firms. The aggregate variables return to their long-run values as the entry-exit process makes the productivity distribution converge back to its stationary value.

To understand what is the effect of this negative shock on all the other firms, we plot in figure 5 the response of the output of the second biggest firm[9]. The output of this firm benefits from the negative shock suffered by the biggest firm because the wage drops. The cost of the second biggest firm is reduced, and thus it can hire

---

[8]Note that the right panel x-axis scale is different.

[9]This is the response of the second biggest firm output as if it kept its productivity level constant. However, the plot will be exactly the same if we plot the output of the second biggest compared to the case where wage was set constant.
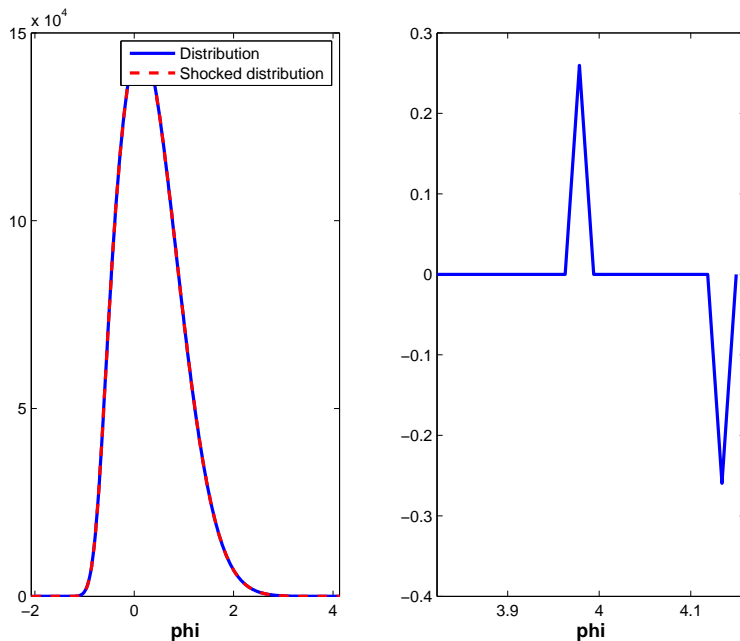
Figure 3: Firm initial distribution after the destruction of the biggest firm.
NOTE: Left panel: initial distribution (blue), long term distribution (red dash). Right panel: shock on the distribution.
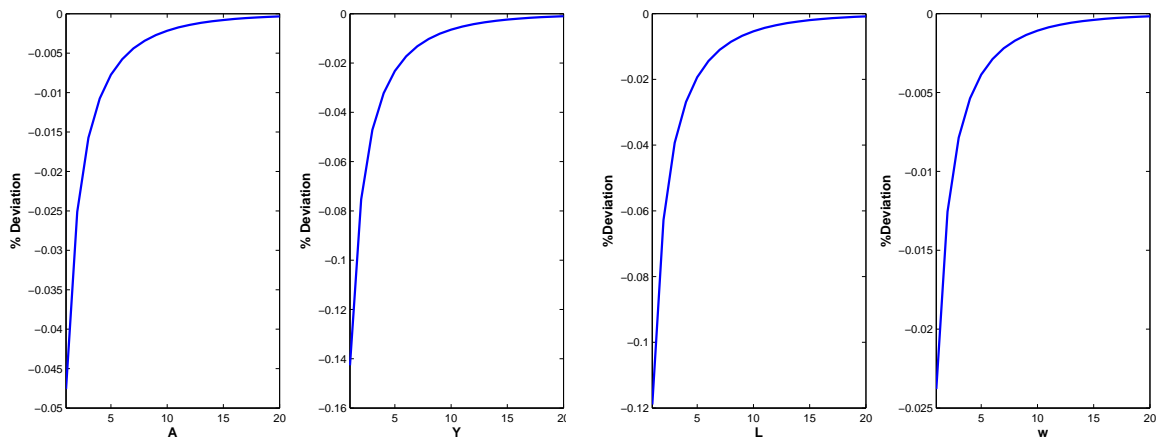

Figure 4: Aggregate response to a negative shock to the biggest firm.

more and produce more. The competition on input between firms creates a negative externality. In this case, the second biggest firm increases its output by 0.1%.

Finally, we study the impact of this negative shock on the biggest firm on the cross-sectional moments. Figure 6 displays the response of the cross-sectional standard deviation of output, output growth and TFP shocks on the left, center and right panel respectively. The cross-sectional standard deviation of output drops after this negative shock because the highest idiosyncratic output drops. This drop is about 1.4%, a sizable number. On the contrary the responses of the cross-sectional standard deviation of TFP shocks and output growth are much smaller but positive. The intuition is that the shock induces a drop in wage but does not affect the productivity
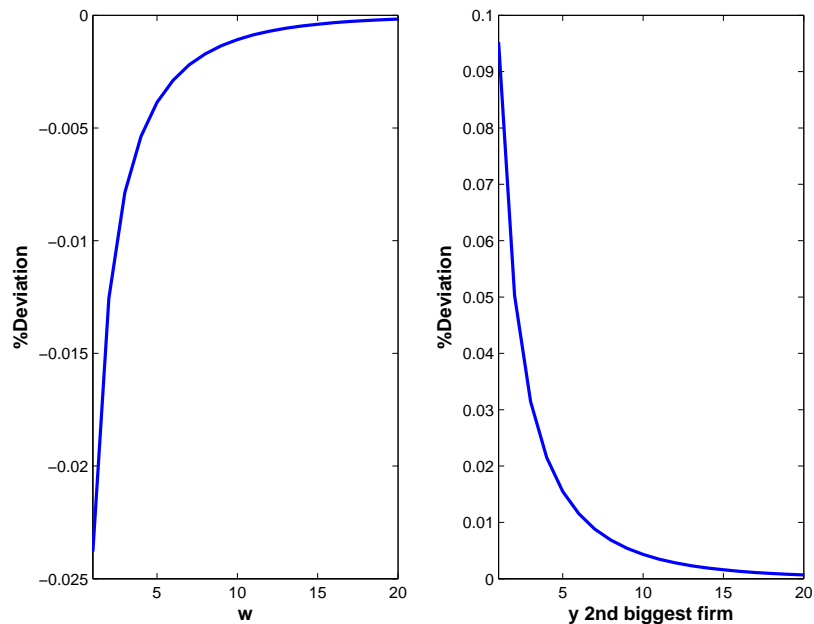
Figure 5: Idiosyncratic response to a negative shock on the biggest firm.

of small firms that are subject to exit. These small firms are thus more profitable and can suffer larger negative TFP shocks without choosing to exit. The same reasoning applies to the cross-sectional standard deviation of output growth. For this negative shocks, cross-sectional standard deviation of output is pro-cyclical whereas the same moment for TFP shocks and output growth is counter-cyclical.
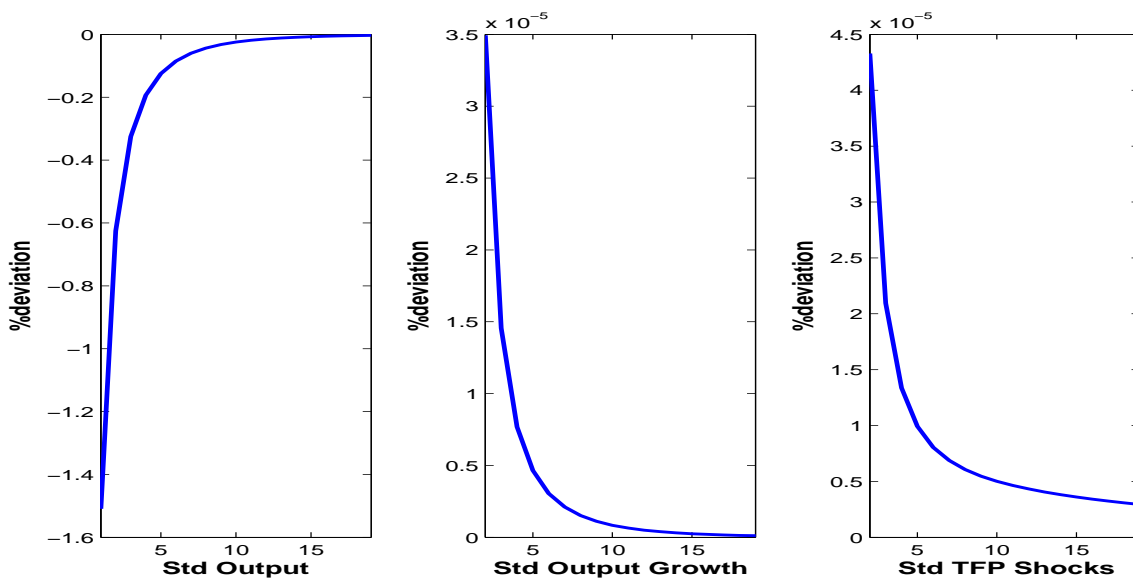

Figure 6: Cross-section moments response to a negative shock on the biggest firm.

6   Conclusion

[TO BE COMPLETED]

References

[1] Daron Acemoglu, Vasco M. Carvalho, Asuman Ozdaglar, and Alireza Tahbaz-Salehi. The network origins of aggregate fluctuations. *Econometrica*, 80(5):1977–2016, 09 2012.

[2] Ruediger Bachmann and Christian Bayer. Firm-specific productivity risk over the business cycle: Facts and aggregate implications. CESifo Working Paper Series 2844, CESifo Group Munich, 2009.

[3] Per Bak, Kan Chen, Jose Scheinkman, and Michael Woodford. Aggregate fluctuations from independent sectoral shocks: self-organized criticality in a model of production and inventory dynamics. *Ricerche Economiche*, 47(1):3–30, March 1993.

[4] Susanto Basu and John G Fernald. Returns to scale in u.s. production: Estimates and implications. *Journal of Political Economy*, 105(2):249–83, April 1997.

[5] Florin O. Bilbiie, Fabio Ghironi, and Marc J. Melitz. Endogenous entry, product variety, and business cycles. *Journal of Political Economy*, 120(2):304 – 345, 2012.

[6] Jeffrey Campbell. Entry, exit, embodied technology, and business cycles. *Review of Economic Dynamics*, 1(2):371–408, April 1998.

[7] Vasco M Carvalho. Aggregate fluctuations and the network structure of inter-sectoral trade. mimeo, 2010.

[8] Vasco M. Carvalho and Xavier Gabaix. The great diversification and its undoing. *American Economic Review*, Forthcoming, 2013.

[9] Gian Luca Clementi and Dino Palazzo. Entry, exit, firm dynamics, and aggregate fluctuations. Working Paper Series 27 10, The Rimini Centre for Economic Analysis, January 2010.

[10] Wouter J. Den Haan. Assessing the accuracy of the aggregate law of motion in models with heterogeneous agents. *Journal of Economic Dynamics and Control*, 34(1):79–99, January 2010.

[11] Julian di Giovanni and Andrei A. Levchenko. Country size, international trade, and aggregate fluctuations in granular economies. *Journal of Political Economy*, 120(6):1083 – 1132, 2012.

[12] Julian di Giovanni, Andrei A. Levchenko, and Isabelle Mejean. Firms, destinations, and aggregate fluctuations. CEPR Discussion Papers 9168, C.E.P.R. Discussion Papers, October 2012.

[13] Xavier Gabaix. The granular origins of aggregate fluctuations. *Econometrica*, 79(3):733–772, 05 2011.

[14] Hugo A Hopenhayn. Entry, exit, and firm dynamics in long run equilibrium. *Econometrica*, 60(5):1127–50, September 1992.

[15] Michael Horvath. Cyclicality and sectoral linkages: Aggregate fluctuations from independent sectoral shocks. *Review of Economic Dynamics*, 1(4):781–808, October 1998.

[16] Boyan Jovanovic. Micro shocks and aggregate risk. *The Quarterly Journal of Economics*, 102(2):395–409, May 1987.

[17] Aubhik Khan and Julia K. Thomas. Nonconvex factor adjustments in equilibrium business cycle models: do nonlinearities matter? *Journal of Monetary Economics*, 50(2):331–360, March 2003.

[18] Aubhik Khan and Julia K. Thomas. Idiosyncratic shocks and the role of nonconvexities in plant and aggregate investment dynamics. *Econometrica*, 76(2):395–436, 03 2008.

[19] Per Krusell and Anthony A. Smith. Income and wealth heterogeneity in the macroeconomy. *Journal of Political Economy*, 106(5):867–896, October 1998.

[20] Yoonsoo Lee. The importance of reallocations in cyclical productivity and returns to scale: evidence from plant-level data. Working Paper 0509, Federal Reserve Bank of Cleveland, 2005.

[21] Yoonsoo Lee and Toshihiko Mukoyama. Entry, exit and plant-level dynamics over the business cycle. Working Paper 0718, Federal Reserve Bank of Cleveland, 2008.

[22] Jose A Scheinkman and Michael Woodford. Self-organized criticality and economic fluctuations. *American Economic Review*, 84(2):417–21, May 1994.

[23] Thomas A. Severini. *Elements of Distribution Theory*. Cambridge University Press, 2005.

[24] George Tauchen. Finite state markov-chain approximations to univariate and vector autoregressions. *Economics Letters*, 20(2):177–181, 1986.

[25] Marcelo L. Veracierto. Plant-level irreversible investment and equilibrium business cycles. *American Economic Review*, 92(1):181–197, March 2002.

A Proof

A.1 Proof of Theorem 1

Theorem 3 *Let us assume that there is no entry and exit. Let ξ be the tail parameter of firm size distribution and assume that $\xi/\rho < 2$ and $\xi > 1$, then[10]*

$$\sigma(\frac{\Delta Y_t}{Y_t}) = \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\sigma(\frac{\Delta T_t}{T_t})$$

$$\sigma(\frac{\Delta T_t}{T_t}) \rightsquigarrow \frac{1}{N_t^{1-\rho/\xi}}\frac{\sigma u^{1/2}}{\bar{Z}_t}$$

where $\bar{Z}_t$ is a time-dependent constant, u is a random variable with finite variance, σ is the standard deviation of $\exp(e_i^t/(1-\alpha))$ and N_t is the number of firms in period t.

Proof: This proof follows closely Gabaix 2011.

Let us first compute the aggregate output Y_t as a function of only T_t:

$$Y_t = \sum_{i=1}^{N_t} y_t^i = \sum_{i=1}^{N_t} \exp(\varphi_t^i)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}}T_t$$

Recall that:

$$w_t = \left(\alpha^{\frac{1}{1-\alpha}}T_t\right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}}$$

Substituting this expression of the wage in the latter equation and taking the growth rate yields:

$$Y_t = \alpha^{\frac{\alpha\gamma}{\gamma(1-\alpha)+1}}(T_t)^{1-\frac{\alpha}{\gamma(1-\alpha)+1}}$$

$$\frac{\Delta Y_t}{Y_t} = \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\frac{\Delta T_t}{T_t}$$

Since we have assumed that there is no entry and exit, an incumbent firm indexed by i in period t is still incumbent in period $t+1$. We still index it by i.

$$\frac{\Delta T_t}{T_t} = \frac{\sum_{i=1}^{N_t}\exp(\frac{\varphi_{t+1}^i}{1-\alpha}) - \exp(\frac{\varphi_t^i}{1-\alpha})}{\sum_{i=1}^{N_t}\exp(\frac{\varphi_t^i}{1-\alpha})}$$

but

$$\exp(\frac{\varphi_{t+1}^i}{1-\alpha}) - \exp(\frac{\varphi_t^i}{1-\alpha}) = \exp(\frac{\varphi_t^i}{1-\alpha})^\rho\exp(\frac{e_t^i}{1-\alpha}) - \exp(\frac{\varphi_t^i}{1-\alpha})$$

---

[10]ξ is matched to be 1.03 in the calibration described below. In the calibration, ρ turns out to be 0.9796 and thus $\xi/2\rho$ is around 0.5257.

where e_t^i is drawn from a normal distribution with mean zero and variance σ_e. Furthermore the e_t^i are iid across firms and time. Let us define $\sigma = \sqrt{\mathbb{V}ar\left(\exp(\frac{e_t^i}{1-\alpha})\right)}$ and $Z_t^i = \exp(\frac{\varphi_t^i}{1-\alpha})$.

$$\mathbb{V}ar\left(\Delta\exp(\frac{\varphi_t^i}{1-\alpha})\right) = (Z_t^i)^{2\rho}\sigma^2$$

Let us drop the time subscript. Using the fact that productivity is independent across firms, we have $\sigma\left(\frac{\Delta T_t}{T_t}\right) = \sigma h$ with $h = \frac{N^{-1}\left(\sum_{i=1}^{N}(Z_i)^{2\rho}\right)^{1/2}}{N^{-1}\sum_{i=1}^{N}(Z_i)}$.

It is clear that Z_i is drawn from the same distribution as firm size and is iid across firms, thus the distribution of Z_i has a tail distributed as a power law with a tail parameter ξ which is assumed to be greater than one. It follows from the law of large numbers that $N^{-1}\sum_{i=1}^{N}(Z_i) \longrightarrow \mathbb{E}(Z_i) := \bar{Z}$ almost surely. Thus we can say that:

$$h \rightsquigarrow \frac{N^{-1}\left(\sum_{i=1}^{N}(Z_i)^{2\rho}\right)^{1/2}}{\bar{Z}}$$

Since $Z_i^{2\rho}$ has a power law distributed tail with parameter $\xi/\rho < 2$, using the Lévy theorem in the appendix of Gabaix 2011, we have

$$N^{-2\rho/\xi}\left(\sum_{i=1}^{N}(Z_i)^{2\rho}\right) \longrightarrow^{\mathcal{D}} u$$

where u is a standard Lévy distribution with parameter $2\rho/\xi$ and the convergence is in distribution. It follows that

$$N^{1-\rho/\xi}h \rightsquigarrow \frac{u^{1/2}}{\bar{Z}}$$

from which follow the results. $\square$

A.2   Proof of Theorem 2

Theorem 4 *Let ξ be the tail parameter of firm size distribution and $\zeta' = \zeta(1-\alpha)$ be the tail parameter of potential entrant size distribution. Assuming that $\xi/\rho < 2$ and $\zeta'/\rho < 2$ then*

$$\sigma(\frac{\Delta T_t}{T_t}) \sim \frac{\sigma}{N_t^{1-\rho/\xi}}\frac{\left(\frac{N_t^l}{N_t}\right)^{\rho/\xi}u^{1/2}}{\bar{I}_t} \qquad\qquad \text{if } \zeta' > \xi$$

$$\sigma(\frac{\Delta T_t}{T_t}) \sim \frac{\sigma}{N_t^{1-\rho/\zeta'}}\frac{\left(\frac{N_t^E}{N_t}\right)^{\rho/\zeta'}w^{1/2}}{\bar{I}_t} \qquad\qquad \text{if } \zeta' < \xi$$

where $\bar{I}_t$ is a time dependent constant proportional to the incumbent's average size at t, u and w are random variables with finite variance, σ is the standard deviation of $\exp(e_i^t/(1-\alpha))$ and N_t, N_t^l, N_t^E are the number of incumbents, successful incumbents and successful entrants in period t respectively.

Proof:

Note that

$$T_{t+1} = \sum_{l \text{ successful incumbent at } t} \exp(\frac{\varphi_{t+1}^l}{1-\alpha}) + \sum_{e \text{ successful entrant at } t} \exp(\frac{\varphi_{t+1}^e}{1-\alpha})$$

and

$$T_t = \sum_{l \text{ successful incumbent at } t} \exp(\frac{\varphi_t^l}{1-\alpha}) + \sum_{x \text{ exiters at } t} \exp(\frac{\varphi_t^x}{1-\alpha})$$

Let us define $Z_t^l = \exp(\frac{\varphi_t^l}{1-\alpha})$ for l-successful incumbent at t, $E_{t+1}^e = \exp(\frac{\varphi_{t+1}^e}{1-\alpha})$ for e successful entrant at t and $X_t^x = \exp(\frac{\varphi_t^x}{1-\alpha})$ for x exiters at t.

The growth rate of T_t is:

$$\frac{\Delta T_t}{T_t} = \frac{1}{T_t}\left(\sum_l \Delta Z_t^l + \sum_e E_{t+1}^e - \sum_x X_t^x\right)$$

Note that

$$\mathbb{V}ar\left(\Delta Z_t^l\right) = (Z_t^l)^{2\rho}\sigma^2$$

and

$$\mathbb{V}ar\left(E_{t+1}^e\right) = (E_t^e)^{2\rho}\sigma^2$$

where $E_t^e = \exp(\frac{q_t^e}{1-\alpha})$ with q_t^e the signal at t of the successful entrant e.

This leads to

$$\mathbb{V}ar\frac{\Delta T_t}{T_t} = \frac{\sigma^2}{(T_t)^2}\left(\sum_l (Z_t^l)^{2\rho} + \sum_e (E_t^e)^{2\rho}\right)$$

since the variance conditional on date t of X_t^x is equal to zero.

Denoting N_t^l, N_t^E and N_t^X the number of successful incumbents, successful entrants and exiters at date t respectively. According to the law of large number, we have:

$$(N^l)^{-1}\sum_l Z_t^l \to \mathbb{E}Z_t^l := \bar{Z}_t$$

$$(N^X)^{-1}\sum_l X_t^x \to \mathbb{E}X_t^x := \bar{X}_t$$

It is straightforward that

$$N^{-1}T_t \sim \frac{N_t^l}{N_t}\bar{Z}_t + \frac{N_t^X}{N_t}\bar{X}_t := \bar{I}_t$$

the average of incumbent size at date t (both successful and exiters).

The distribution of the random variable Z_t^l as a power law tail with parameters ξ, the tail parameter of firm size distribution (since only small firms exit). The distribution

23

of the random variable Z_t^l as a power law tail with parameters $\zeta' = \zeta(1-\alpha)$ the tail parameter of entrant size distribution (since only big entrants are successful).

Since $\xi/\rho < 2$ and $\zeta'/\rho < 2$ and using the Lévy theorem of the appendix of Gabaix 2011, we have

$$(N_t^l)^{-2\rho/\xi} \sum_l (Z_t^l)^{2\rho} \to^d u$$

$$(N_t^X)^{-2\rho/\zeta'} \sum_x (X_t^x)^{2\rho} \to^d w$$

where u and w are standard Lévy distribution with parameters $\xi/2\rho$ and $\zeta'/2\rho$ respectively.

Computing the two above results yields

$$\mathbb{V}ar \frac{\Delta T_t}{T_t} \sim N^{-2} \frac{\sum_l (Z_t^l)^{2\rho} + \sum_x (X_t^x)^{2\rho}}{(\bar{I}_t)^2}$$

Note the numerator of the right hand side is equivalent to

$$N^{-2+2\rho/\xi} \left(\left(\frac{N_t^l}{N_t}\right)^{2\rho/\xi} u \right) \text{ if } \zeta' > \xi$$

since $N^{2\rho(1/\zeta - 1/\xi)} \to 0$ in this case.

Similarly:

$$N^{-2+2\rho/\zeta'} \left(\left(\frac{N_t^E}{N_t}\right)^{2\rho/\zeta'} u \right) \text{ if } \zeta' < \xi$$

This gives the results:

$$\sigma(\frac{\Delta T_t}{T_t}) \sim \frac{\sigma}{N_t^{1-\rho/\xi}} \frac{\left(\frac{N_t^l}{N_t}\right)^{\rho/\xi} u^{1/2}}{\bar{I}_t} \qquad\qquad \text{if } \zeta' > \xi$$

$$\sigma(\frac{\Delta T_t}{T_t}) \sim \frac{\sigma}{N_t^{1-\rho/\zeta'}} \frac{\left(\frac{N_t^E}{N_t}\right)^{\rho/\zeta'} w^{1/2}}{\bar{I}_t} \qquad\qquad \text{if } \zeta' < \xi$$

$\square$