Charity Auctions for the Happy Few*

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Abstract

Recent literature has shown that all-pay auctions raise more money for charity than winner-pay auctions. We demonstrate that the first and second-price winner-pay auctions outperform first-price all-pay auctions when bidders are sufficiently asymmetric. To prove it, we consider a framework with complete information.
This analysis is relevant for two main reasons. On the one hand, complete information is more realistic and corresponds to events which occur for instance in a local service club (like in a voluntary organization) or in a show business dinner. Potential bidders are acquaintances or know one another well. On the other hand, our model keeps the qualitative predictions of a private value model under incomplete information in which bidders are *ex ante* asymmetric that is to say different bidders' values are drawn from different distributions. Furthermore, we also analyze second-price all-pay auction. Finally, we show that individual minimum bids could improve the relative revenue performance of first-price all-pay compared to first-price winner-pay auction.

KEYWORDS: All-pay auctions, charity, complete information, externalities

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1 Introduction

More and more voluntary organizations wish to raise money for charity purposes through a partnership with firms. Charity auctions have been held in the United States for many years now. However, in China this phenomenon has emerged recently and is in strong progress[1]. In this kind of auction, an object (for example a key case with a zero value or an item given by a luxury brand) is sold. The proceeds then go to charity. Most of these auctions are planned and organized in charity dinners where only wealthy or famous people can participate. Beyond the item value, the valuations of potential bidders depend on their interest for this voluntary organization (their altruism or philanthropy) and also show some kind of conformism "to be seen as the most wealthy and generous". For instance, in China's traditional society, charity auctions were not put forward. The participants preferred to keep a low profile about their bids. However, time has changed: the rich and famous now show their wealth through their involvement in charity auctions. According to the *Beijing Review*:

> *With the development of society, more rich people are emerging. They have their own lifestyle [...] Some day, behind the rich lifestyle, people will find that it is only by offering their love and generosity that they can realize their true class.*

Thus, through charity auctions, potential bidders can build their position in their social class. Everybody wishes, independently of the winner's identity, to raise the highest revenue. Potential bidders make a trade-off between giving money for the fund-raising and keeping it for another personal use. Contrary to non-charity auctions, here the amount paid is "never lost". A wealthy investor, who bought a Dior perfume for 60 000 yuans (about 6 000 euros or 7 700 dollars) – with a reserve price of 20 000 yuans – recently said in the *Beijing Review*:

> *I would never buy perfume for this amount normally, but this time it is for charity. I feel very happy.*

In fact, the money raised will be used to finance a public good. Every participant of the charity auction may take advantage of it, independently of the winner's identity. More precisely, the money raised by each potential bidder impacts the utility of all participants as they take advantage of an externality on the amount of the money raised for the public good or the charity purpose.

Under complete information, these kinds of auctions can be compared to the work of Ettinger (2002) who analyzed a general winner-pay auction framework with two kinds of non-linear externalities. One of them does not depend on the winner's identity and can be applied to charity auctions where only the winner pays. Moreover, he shows there is no "revenue equivalence" with these externalities. Maasland & Onderstal (2006) investigate winner-pay auctions with this kind of linear externalities in an independent private signals model. Their paper can also be applied to charity. They find similar qualitative predictions as Ettinger (2002): the second-price can outperform the first-price winner-pay auction. In their recent paper, Goeree et al. (2005) analyze charity auctions in the symmetric independent-private-value-model. They show, given the externality, that all-pay auctions raise more money for charity than winner-pay auctions and lotteries. In particular, they determine that the optimal fund-raising mechanism is given by an

[1] For example, in 2004, at the Formula One Grand Prix opening dinner party in Shanghai (China), *an auction was held of racing suits and crash helmets used by famous racing drivers* (*Beijing Review*, 2005).

all-pay auction: the lowest-price all-pay auction with an entry fee and a reserve price. Their paper completes and generalizes the work of Engers & McManus (2006), who find similar results for a sufficiently high number of bidders. Contrary to Goeree et al. (2005), a psychological effect comes into play: the winner benefits from a higher externality with his own bid, the others' bids having a lower effect on him. Moreover they show that an English (button) auction yields the same outcome as the sealed-bid second-price winner-pay auction.
The predictions of Goeree et al. (2005) and Engers & McManus (2006) have been tested experimentally with contradictory results. Onderstal & Schram (2006) have experimented the Goeree et al. (2005)'s result in a laboratory with 180 students. They are the first to conduct a lab experiment for charity auctions in an independent private value setting. Their results are close to the theoretical predictions: in charity auction, the revenue raised with all-pay auctions is higher than with other mechanisms. Carpenter et al. (2004) have tested the predictions of Engers & McManus (2006) in a field experiment. Similar objects are sold in four American pre-schools through three different mechanisms which are all-pay auctions, first-price and second-price winner-pay auctions. They study the determinants of the bidders' behavior and the revenue raised. Contrary to the theoretical predictions, all-pay auctions do not produce higher revenues than the winner-pay auctions. Therefore, if auction theory about charity is confirmed in the laboratory, it is not the case in the field. The main explanation for the gap between theory and field experiment can be a non-participation effect, due to the unfamiliarity with these mechanisms and their complexity: the participants didn't know the all-pay design and few took part in second-price auctions on the Internet.

This paper has two main goals. First of all, the paper of Goeree et al. (2005) is revisited with the introduction of asymmetric valuations under complete information. Although an incomplete information setting is more realistic, this model keeps the qualitative predictions of a private values model under incomplete information and suppose that the bidders are *ex ante* asymmetric, that is to say different bidders' values are drawn from different distributions. Moreover, as we saw before, a lot of charity auctions are conducted among rich people during charity dinners. These events could occur in a local service club (like the Rotary club[2] or another type of voluntary organization) or during a show business dinner. Potential bidders are acquaintances or know one another well. Consequently, a complete information environment is well suited for these kinds of situation. The purpose of this paper is to determine whether or not winner-pay auctions can raise higher revenue for charity when the asymmetry between bidders is strong.

In his recent paper, Konrad (2006) introduces externalities in all-pay auctions in a complete information setting. He analyzes the competition between firms with this framework when a firm owns a large part of one of its rivals. The equilibrium properties change particularly if these two firms are the strongest on the market. Indeed, the strongest firm takes advantage of his ownership.

There is a wide literature about all-pay auctions. The seminal paper is the famous *auction dollar game* paper of Shubik (1971). These auctions may be used to illustrate many economic, social and political issues as they have the same structure as a contest or a tournament. Hillman & Samet (1987) and Hillman (1988) were the first one to apply them to lobbying models

[2] *The Rotary club is a worldwide organization of business and professional leaders that provides humanitarian services, encourages high ethical standards in all vocations, and helps build goodwill and peace in the world.* There are about 32 000 clubs in 200 countries and geographical areas and 1,000 clubs in France like Paris, but also in small town like Niort. http://www.rotary.org/

in which some groups of interest give a bribe to the decision maker in order to obtain a market or a political favor under complete information. Baye et al. (1993) studied the case where the decision maker excludes the lobbyists with the highest value in the lobbying process to maximize the rents. Daguspta (1986) has applied them to R&D competition, Konrad (2004), Che & Gale (1998) and Sahuguet & Persico (2006) to political campaigns. In these papers, the contests take place in an all-pay auction in effort framework. The agent with the highest effort wins the competition while the others are not awarded for their efforts.

Two cases have been distinguished in the literature under complete information. On the one hand, bidders have homogeneous valuations and give identical values to the objects. Hillman & Samet (1987) have first characterized a unique symmetric Nash equilibrium for this framework. However, later on, Baye et al. (1996) have defined the set of Nash equilibria. According to their study, there exists a continuum of asymmetric Nash equilibria. In each equilibrium, at least two agents bid on the same support with mixed strategies while others bid on a subset of the support of these two agents and have an atom at zero. All these equilibria lead to the same revenue. On the other hand, Hillman & Riley (1989) determine a unique equilibrium when bidders have heterogeneous values. Nevertheless, the result also holds if at least the three highest valuations are heterogeneous. The two bidders with the highest valuations bid a positive amount on the same support and one of them has an atome at zero. Others do not participate. Alternatively, Baye et al. (1996) show if the second and the third highest valuations are the same, there is a unique symmetric equilibrium but also a continuum asymmetric equilibrium. More recently, Vartiainen (2006) characterized all-pay auctions for bidders with linear and nonlinear cost functions. The bidders' valuations are normalized to 1 while the cost functions are asymmetric and depend on their abilities (similar to Moldovanu & Sela (2001)).

All-pay auctions have also been characterized under incomplete information. Weber (1985) was the first one to study independent private value all-pay auctions with this framework. Amann & Leininger (1996) characterize the equilibria for two asymmetric bidders. They demonstrate that when the degree of uncertainty on the values decreases, pure strategies tend to the Nash equilibrium found under complete information. Krishna & Morgan (1997) consider the general framework of affiliated values. They determine the equilibria for all-pay auctions and a new *linkage principle* for mechanisms in which the winner is not the only one to pay. This permits them to compare the revenues of all-pay and winner-pay auctions. Lizzeri & Persico (2000) study the existence and uniqueness of the equilibrium in all-pay auctions with two bidders and affiliate values when there is a reserve price.

Other papers characterize equilibrium with caps under complete information (Che & Gale (1998), Kapplan & Wettstein (2006) and Che & Gale (2006)), and with constrained budget or caps under incomplete information (Che & Gale (1996), Gavious et al. (2002), and Sahuguet (2006)). One of them considers risk aversion in all-pay auction: Fibich et al. (2006).

Amann & Leininger (1996), Krishna & Morgan (1997) and Lizzeri & Persico (2000) do not consider only first-price all-pay auctions but also second-price all-pay auctions (or wars of attrition). A war of attrition is the oral or dynamic version of the second-price all-pay auction. Smith (1974) was the first one to work on the equilibrium in a war of attrition framework with two bidders under complete information. The war of attrition under complete information has also been studied by Hendricks et al. (1988) and incomplete information by Bulow & Klemperer

(1999), among others. Vartiainen (2006) have characterized the second-price all-pay auction with abilities under complete information when there are more than two bidders.

Following the work of Vartiainen (2006), we analyze all-pay auctions for charity as a mechanism. This approach relies on a general model which can be applied to both first and second-price all-pay auctions. In our setting, every bidder takes as much advantage of his own bid as of his rival's bid thanks to the externalities. Additionally, we assume that the altruism and the valuations of the bidders are ranked in the same order. We discuss this assumption and its consequences.

First-price all-pay auction equilibrium is characterized and the expected revenue computed; but there is no pure strategy Nash equilibrium. As in a case without externality, only the two bidders with the highest valuations are active. In order to raise money for charity, we set up an optimal lobbying policy based on two steps. The first step consists in making the active bidder with the lowest valuation aware of the charity auction. Once the updated-valuation of the bidder with the lowest initial valuation is equal to the highest valuation, the goal is to make both agents sensitive to the auction so as to keep their valuations equal. Indeed, it is important not to work only on the sensitiveness of the bidder with the highest valuation so as to avoid disastrous consequences in terms of revenue. We also show the existence of a Nash equilibrium with non-linear externality.

The equilibrium is also characterized and the expected revenue computed for the second-price all-pay auction. In that case, the pure strategy Nash equilibria are degenerated. That is why we find the mixed strategy Nash equilibrium. We discuss our results by comparing them to Ettinger (2002) who analyzes winner-pay auctions with externalities that do not depend on the identity of the winner and which could be applied to charity auctions. Even if the second-price all-pay auction raises more money than the other designs, the revenue of the first-price all-pay auction can be dominated by the winner-pay auctions contrary to the results of Goeree et al. (2005). Indeed, beyond a certain threshold of asymmetry in the bidders' valuations, winner-pay auctions raise more money for charity than the first-price all-pay auctions. We can also revisit this result by an analysis of the bidders' altruism.

In the last section, we evaluate the impact of individual minimum bids on first-price all-pay and first-price winner-pay auctions. We assume the auctioneer knows the bidders' valuations. This assumption is relevant in a charity dinner which takes place in an isolated environment or in a local service club. The auctioneer gets informations through the board of directors of the service club as he does not belong to this environment. Minimal bids could improve the relative revenue performance of first-price all-pay auction compared to winner-pay auction. Indeed, minimal bids can offset the effects of asymmetry in the bidders' valuations.

2 The model

Following the work of Vartiainen (2006) with linear cost functions, we analyze all-pay auctions for charity as a mechanism. This approach relies on a general model which can be applied to both first and second-price all-pay auctions. Yet, our approach is different. Moreover, in our case, every bidder takes as much advantage of his own bid as of his rival's bid thanks to introduction of the externalities.

In a charity dinner, an indivisible object (or prize) is sold through an all-pay auction. This prize is allocated to one of the potential bidders $N = \{1, ..., n\}$ contingents upon their bids

$x = (x_1, ..., x_n) \in \mathbb{R}_+^n$. As the bidders usually meet each other in these kinds of events, the willingness to pay and the valuation ranking of each bidder, $v_1 > v_2 > ... > v_n$, are common knowledge. An all-pay auction is a pairwise (a, t), a being the allocation rule and t the payment rule.

Allocation Rule. The allocation rule $a = (a_1, ..., a_n) : \mathbb{R}_+^n \longrightarrow [0, 1]^n$ is such that the winner i gets the object if and only if $a_i(x) = 1$ given the bids and $\sum_{i=1}^n a_i(x) = 1$ for all x. The object is allocated to the highest bidder such that

$$\begin{cases} a_i(x) = \frac{1}{\#Q(x)} \text{ if } i \in Q(x) \\ a_i(x) = 0 \text{ otherwise} \end{cases}$$

where $Q(x) := \{j | j = \arg\max\{x_k, k \in N\}\}$ is the collection of the highest bids.

Payment Rule. The payment rule $t = (t_1, ..., t_n) : \mathbb{R}_+^n \longrightarrow \mathbb{R}_+^n$ represents for each bidder i his transfer $t_i(x)$ to the charity organization for all of the bids x. This payment rule is contingent upon the all-pay design. In fact, in a first-price all-pay auction, each bidder pays his own bid

$$t_i(x) = x_i \;\; \forall i \in N$$

while in the second-price all-pay auction the winner pays the second highest bid and the losers their own bid

$$\begin{aligned} t_i(x) &= x^{(2)} \text{ if } i \in Q(x) \\ t_i(x) &= x_i \text{ otherwise} \end{aligned}$$

with $x^{(2)}$ the second order statistic of sample $(x_1, ..., x_n)$.

The bidders wish to raise the maximum of money for charity. Every bidder takes advantage of his own participation in the charity auction and of the others' participations as well. In other words, the money raised by each potential bidder impacts the utility of all of the participants including himself. Thus, the bidder's utility function includes an externality which depends on the amount of money raised for the public good or the charity purpose. Denote $h_i(t(x))$ the externality that the bidder i takes advantage of. This is a function with only one argument $\sum_{j=1}^{n} t_j(x)$. Indeed, the externality is independent of the winner's identity and only takes into account the amount raised. Like Goeree et al. (2005) and other papers about charity auctions, we make a linearity assumption on the form of the externality price:

$$h_i(t(x)) = h_i(t_1(x), ..., t_n(x)) = \alpha_i \sum_{j=1}^{n} t_j(x)$$

where $\alpha_i \geq 0$ is the threshold of the bidder i's altruism for the charity purpose. Thus, the bidder i's utility is given by

$$U_i(x) = v_i a_i(x) - t_i(x) + \alpha_i \sum_{j=1}^{n} t_j(x)$$

Assumption 1 (*A*1)**.** *$U_i(x)$ is a continuous and differentiable function in all of his arguments.*

Thus, $h_i(t(x))$ is also continuous and differentiable in all of his arguments.

Assumption 2 (*A*2)**.** $\forall x_i \geq 0$ $\dfrac{\partial U_i}{\partial t_i(x)}(x) < 0$ *equivalent to* $\alpha_i \sum_{j=1}^{n} \dfrac{dt_j(x)}{dt_i(x)} < 1$.

This assumption reminds that the bidder has a strict preference to keep one euro for his own use rather than to give it to the charity auction. This is the limit to the bidders' altruism to give money for charity[3]. The altruism threshold changes with the payment rule. Indeed, the bidder i's transfer can be a function of his opponents' bid. Thus, a change in the payment rule leads to a new altruism threshold: in first-price it is $\alpha_i < 1$ while in second-price $\alpha_i < 1/2$.

Assumption 3 *(A3)*. $\alpha_1 \geq \alpha_2 \geq ... \geq \alpha_n$

This assumption suggests that the altruism level and the value of the bidders are ranked in the same order. Thus, we assume the bidder with the highest valuation is also the bidder who is the most concerned by the charity purpose. As a consequence, he is the one who takes the most advantage of the money raised for charity because of the externality effect. Assumption $A3$ allows us to select the equilibrium for the first-price auction. This assumption does not have necessarily important consequences on the results but if it does, this would be discussed.

Denote $F_i(x) \equiv \mathbb{P}(X_i \leq x)$ the cumulative distribution functions such as the bidder i decides to take a bid inferior to x. We denote $f_i(x)$ the density associated and $F_i(0)$ the probability that bidder i bids 0. When $F_i(0) \neq 0$, bidder i bids zero with a probability strictly positive. When $F_i(0) = 1$, bidder i always bids zero which means that he does not participate to the auction. $F_1, ..., F_n$ can be interpreted as the bidding strategies where the support is $\mathbb{R}_+$. Thus, the expected utility of bidder i is given by:

$$\mathbb{E}U_i(x_i, X_{-i}) = \int_{\mathbb{R}_+^{n-1}} \left(v_i a_i(x) - (1-\alpha_i)t_i(x) + \alpha_i \sum_{\substack{j=1 \\ j\neq i}}^{n} t_j(x) \right) \prod_{j\neq i} dF_j(x_j) \tag{1}$$

$$\begin{aligned} &= v_i \prod_{j\neq i} F_j(x_j) - (1-\alpha_i) \int_{\mathbb{R}_+^{n-1}} t_i(x) \prod_{j\neq i} dF_j(x_j) \\ &\quad + \alpha_i \int_{\mathbb{R}_+^{n-1}} \sum_{\substack{j=1 \\ j\neq i}}^{n} t_j(x) \prod_{j\neq i} dF_j(x_j) \end{aligned} \tag{2}$$

with $X_{-i} = (X_1, ..., X_{i-1}, X_{i+1}, ..., X_n)$. To go from (1) to (2) we can notice that $\#Q(x) = 1$ and $\#Q(x) > 1$ are disjoints. Thus, when $\#Q(x) > 1$ the value of the integral is zero: at least one of the support is an atom.

3 First-price All-Pay Auction

In this section, we study the most popular all-pay auction design, *i.e.* the first-price all-pay auction. Every bidder pays his own bid, but only the one with the highest bid wins the object. We first analyze the auction with two bidders, and then we extend the model to n bidders.

3.1 Two Bidders

Given assumption $A2$, there is no pure strategy Nash equilibrium. This is a well known result when there is no externality.

[3] If $\alpha_i \sum_{j=1}^{n} \frac{dt_j(x)}{dt_i(x)} = 1$ then the bidder is indifferent between giving one euro for charity or investing it in an another activity.

Let us assume that $x_i \geq x_j$ and consider some general externality (not necessarily linear) given by $h_i(x_i, x_j)$. This externality need not verify $A3$. In such a framework, two cases can occur. First, if bidder j can overbid, then his best reply is $x_i+\varepsilon$, for $\varepsilon > 0$ such that $v_j-(x_i+\varepsilon)+h_j(x_i, x_i+\varepsilon) \geq -x_j + h_j(x_i, x_j)$. Hence, it is impossible that $x_i \geq x_j$. Second, if j cannot overbid, then his best reply consists in offering zero since, given assumption $A2$, $h_j(x_i, 0) > -x_j + h_j(x_i, x_j)$. Consequently, i's best reply is to offer $\varepsilon > 0$. As a result, the equilibrium is unstable and there is no pure strategy Nash equilibrium.

As we noticed in the last section, assumption $A2$ implies that $\alpha_i < 1$.

Lemma 1. *There is no pure strategy Nash equilibrium. The equilibrium (or equilibria) is (are) in mixed strategies and with no mass point.*

If bidder i offers x_i, then j will offer less with probability $F_j(x_i)$ and will offer more with probability $1 - F_j(x_i)$. Whatever the outcome, bidder i benefits from the sum of all bids, including his. This is what we call an externality. When computing his expected utility, he takes the amount payed by his opponent into account. Indeed, he considers it as a mean. The bidders' expected utilities when 1 offers x_1 and 2 offers x_2 are given by,

$$\begin{cases} \mathbb{E}U_1(x, X_2) = F_2(x)v_1 - (1-\alpha_1)x + \alpha_1 \mathbb{E}X_2 \\ \mathbb{E}U_2(x, X_1) = F_1(x)v_2 - (1-\alpha_2)x + \alpha_2 \mathbb{E}X_1 \end{cases}$$

A potential bidder takes part to the auction if his expected utility is equal to or higher than the externality he enjoys when his bid is zero. Otherwise, he could benefit his own externality without taking part to the auction. Formally, a bidder takes part to the auction if

$$\mathbb{E}U_i(x, X_j) \geq \alpha_i \mathbb{E}X_j$$

with $\alpha_i \mathbb{E}X_j$ bidder i's expected reservation utility when he takes part to the auction.

Lemma 2. *Bidders have the same maximum and minimum bids at the equilibrium. In particular, the minimum bid is zero.*

Lemma 1 and 2 are the same as Hillman & Riley (1989)'s proposition 1 for the case without externality. Thus, we do not write the proof for lemma 1 and leave out most of the proof for lemma 2. Now, we must define the maximum bid. In order to do so, we must prove that bidders have the same maximum at the equilibrium.

Let us assume that, on the contrary, $\max x_i > \max x_j = \tilde{x}_j$. For all bid $x_j < \tilde{x}_j$ made by bidder j, bidder i will offer $\tilde{x}_j + \varepsilon$, with $\varepsilon > 0$, with probability 1. Then, bidder j will decide to offer zero. Thus, there is no Nash equilibrium. It follows that $\max x_i = \max x_j = \tilde{x}_j$.

It is the lowest price at which one of the bidders is indifferent between taking part to the auction or not. We call the lowest price at which a given bidder is ready to take part to the auction his *indifference pricing*. i's *indifference pricing* is noted $\tilde{x}_i$ and satisfies $\mathbb{E}U_i(\tilde{x}_i) = \alpha_i \mathbb{E}X_j$. We know from assumption $A3$ that $\tilde{x}_1 > \tilde{x}_2$.

Remark 1. *If assumption $A3$ had not been made, there would exist a variety of values for v_1, v_2, α_1 and α_2 at which the maximum bid would have been $\tilde{x}_2$. Indeed, in that case, we would have $\tilde{x}_1 > \tilde{x}_2$ if and only if $v_1 - v_2 > \alpha_2 v_1 - \alpha_1 v_2$. Thus, we would have to consider two different equilibria.*

Through the proof of proposition 1, we display a more general outcome than with the result of

this proposition. In fact, to obtain the result of proposition 1 we need to impose assumption A3, which is not the case for the proof. In this proof, we assume that bidder i determines the maximum offer, noted $\tilde{x}_i$. Then, bidder j will take part to the auction for sure.
Furthermore, assumption A3 affects the expected revenue. Indeed, with A3, the revenue that is raised is always higher in an auction with externalities than in an auction with no externality, which would not always be the case without assumption A3.

Proposition 1. *Let $\tilde{x}_i = \dfrac{v_i}{1-\alpha_i}$ bidder i's adjusted-value. There is a unique Nash equilibrium and the mixed strategies are given by*

$$F_1(x) = \frac{1-\alpha_2}{v_2}x \ \ \forall x \in \left[0, \frac{v_2}{1-\alpha_2}\right] \ \text{ and } F_2(x) = 1 - \frac{1-\alpha_1}{1-\alpha_2}\frac{v_2}{v_1} + \frac{1-\alpha_1}{v_1}x \ \ \forall x \in \left(0, \frac{v_2}{1-\alpha_2}\right]$$

The expected revenue is given by $\mathbb{E}R = \dfrac{1}{2}\dfrac{v_2}{1-\alpha_2}\left(\dfrac{1-\alpha_1}{1-\alpha_2}\dfrac{v_2}{v_1} + 1\right)$.

In the appendix we prove this result using the proof for the case without externality (proposition 2 of Hillman & Riley (1989)). *i's indifference pricing* defines his adjusted-value. From lemma 2, the lowest adjusted-value specifies the bidders' maximum bid. Moreover, the bidder with the highest adjusted-value (i.e. bidder 1) offers a bid in the interval $[0, \frac{v_2}{1-\alpha_2}]$ and his competitor bids in the interval $(0, \frac{v_2}{1-\alpha_2}]$. We know from Lemma 1 that the cumulative distribution functions are continuous (with no mass point). The bidders' mixed strategies are uniform distributions and are supported on $[0, \frac{v_2}{1-\alpha_2}]$ given that bidder 2 (the bidder with the lowest adjusted-value) takes part to the auction with probability

$$1 - F_2(0) = \frac{1-\alpha_1}{1-\alpha_2}\frac{v_2}{v_1}$$

Corollary 1. *The bidder with the highest adjusted-value obtains a payoff $U_1^\star = v_1 - \frac{1-\alpha_1}{1-\alpha_2}v_2 + \frac{\alpha_1}{2}\frac{1-\alpha_1}{v_1}\left(\frac{v_2}{1-\alpha_2}\right)^2$ and his competitor gets $U_2^\star = \frac{v_2}{2}\frac{\alpha_2}{1-\alpha_2}$.*

Contrary to the case with no externality, the low bidder gets a positive payoff. That is a consequence of externalities: bidders take an advantage of the competitors' behavior.

Remark 2. *Let us assume that assumption A3 is not satisfied and that the difference between α_1 and α_2 is high enough for bidder 1's adjusted-value to be ranked second. Then bidder 1 can get a lower payoff than in the case with no externality if and only if his altruism level is lower than $\tilde{\alpha} \equiv 2\frac{v_1-v_2}{3v_1-2v_2}$. We notice that this threshold does not depend on his rival's altruism level, while the changes in the ranking of the adjusted-values is only due to the difference between the players' altruism levels.*

We can notice here that there are two opposite effects. Because of the externalities, the value of one euro that is invested in the auction is less than one euro. Thus, it is possible that the bidders choose more aggressive offers. However, every bidder knows that his competitor is more agressive and that this will affect one's probability of winning.Given an increasing of his competitor's aggressiveness, the bidder's best reply can be increasing or decreasing.

Example 1. *Let us consider two bidders with external effects $\alpha_1 = \alpha_2 = \frac{1}{2\min \tilde{x}_i}$. We note that $A1 - A3$ are satisfied. Furthermore, $\tilde{x}_1 > \tilde{x}_2$, $\tilde{x}_1 = \frac{v_1}{v_2}\left(v_2 + \frac{1}{2}\right)$ et $\tilde{x}_2 = v_2 + \frac{1}{2}$.*
Thus, we can determine

$$F_1(x) = \frac{2}{2v_2+1}x, \ \ F_2(x) = 1 - \frac{v_2}{v_1} + \frac{2v_2}{(2v_2+1)v_1}x, \mathbb{E}R = \frac{2v_2+1}{4}\left(\frac{v_2}{v_1}+1\right)$$

The bidders' payoffs are $U_1^\star = v_1 - v_2 + \frac{1}{2}\frac{v_2}{v_1}$ and $U_2^\star = \frac{1}{4}$

3.2 n Bidders

Bidder i's expected utility with n potential competitors is given by

$$\mathbb{E}U_i(x_i, X_{-i}) = \Pi_{j\neq i}^n F_j(x_j)v_i - (1-\alpha_i)x_i + \alpha_i \sum_{j\neq i} \mathbb{E}X_j$$

As in the two bidders case, the bidder with the greatest *indifference pricing* does not offer a higher bid than the second adjusted-value[4]. Given $A1 - A3$, only two bidders take part to the auction.

Proposition 2. *If $v_1 > v_2 > v_3 \geq v_i \ \forall i > 2$ and $A1 - A3$ are satisfied, there is a unique Nash equilibrium and the bidders' strategies are mixed. In this equilibrium, only the two bidders with the highest adjusted-values participate actively i.e. do not bid zero for sure.*

Note that all the results with two bidders remain true.

Proof. Let us assume that a third bidder takes part to the auction. His expected utility is equal to or higher than $\alpha_3\mathbb{E}X_1 + \alpha_3\mathbb{E}X_2$. Given his two rivals' mixed strategies, it follows that $F_1(x_3)F_2(x_3)v_3 \geq (1-\alpha_3)x_3$, which is equivalent to $\tilde{x}_1(\tilde{x}_3 - \tilde{x}_2) \geq \tilde{x}_3(\tilde{x}_2 - x_3)$. As $\tilde{x}_2 > \tilde{x}_3$ and $\tilde{x}_3 \geq x_3$, there is a contradiction. This result can be generalized to a game with n bidders. However, it does not lead to a unique solution. To show that there is a unique solution, here we could apply[5] Baye et al. (1990)'s lemma 14': $\tilde{x}_i = 0 \ \forall i > 2$. ■

In order to raise money for a charity auction, a good lobbying policy consists in inducing bidders to equal their adjusted-values. In other words, one should make the low[6] bidder increase his adjusted-value or the high bidder decrease his. It is well known reducing the asymmetry that exists between bidders tends to increase competition, and thus leads to a higher rent for the auction.

Corollary 2. *Inducing the highest bidder only to care about charity leads to a lower rent.*

An optimal lobbying policy consists in making the low bidder aware of the charity auction and increases his adjusted-value. Once the updated-value of the low bidder is equal to the adjusted-value of the high bidder, the second step is to make both agents sensitive to the auction so as to keep their adjusted-values equal.

Proof. Computations. ■

It is important not to work only on the sensitiveness of the bidder with the highest valuation in order to avoid disastrous consequences in terms of revenue. Indeed, it could make the low bidder less sensitive to the auction and thus the rent might be low. On the opposite, inducing the low bidder only to care about the auction could increase his adjusted-value and his maximum bid.

When the bidders have the same adjusted-value, they get an identical probability to win $F(x) = \frac{x}{v}$ for $x \in [0, v]$. Finally, the optimal level of altruism (α_1, α_2) that gives the maximum revenue for the auction is given by $\alpha_2 = 1 - \frac{v_2}{v_1}(1-\alpha_1)$.

[4]The third bidder is the one whose *indifference pricing* (or adjusted-value) is ranked third. Given the values' ranking and assumption $A3$, this is consistent with the choice of indexes.

[5]Actually, the proof of this lemma has to be slightly changed and be adapted to our setting. As the modifications are of minor importance, we do not give the details of the proof.

[6]The low and high bidders are respectively the bidders with the second and the first highest values.

Thus, as opposed to Baye et al. (1993), in charity auctions it is not conceivable to exclude bidders with higher values. Furthermore, caps *à la* Che & Gale (1998) would have a similar impact as the lobbying policy we suggested.

3.3 Non-linear Externalities

We extend our result to non-linear externalities. We consider two bidders only, such that the expected utility is given by,

$$\mathbb{E}U_1(x_1, X_2) = F_2(x_1)\,(v_1 + \mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) - x_1) + (1 - F_2(x_1))(\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \geq x_1) - x_1)$$
$$\mathbb{E}U_2(x_2, X_1) = F_1(x_2)\,(v_2 + \mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \leq x_2) - x_2) + (1 - F_1(x_2))(\mathbb{E}_{X_1}(h_2(X_1, x_2)\backslash X_1 \geq x_2) - x_2)$$

with $\mathbb{E}_{X_2}(h_1(x_1, X_2)\backslash X_2 \leq x_1) = \dfrac{1}{F_2(x_1)}\displaystyle\int_0^{x_1} h_1(x_1, x_2)dF_2(x_2)$
It can also be written as

$$\begin{cases} \mathbb{E}U_1(x_1, X_2) = F_2(x_1)v_1 - x_1 + \mathbb{E}_{X_2}h_1(x_1, X_2) \\ \mathbb{E}U_2(x_2, X_1) = F_1(x_2)v_2 - x_2 + \mathbb{E}_{X_1}h_2(X_1, x_2) \end{cases}$$

Bidder i takes part to the auction if his expected utility is higher than his reservation utility:

$$\mathbb{E}U_i(x_i, X_j) \geq \mathbb{E}_{X_j}h_i(0, X_j)$$

Proposition 3. *Given $A1 - A2$ and given that the two bidders have a common support $[0, b]$, the mixed strategy equilibrium exists.*

The expected utility's derivative is a Fredholm equation of the second type. The existence of a solution depends on a condition made on the kernel (the kernel being the externality here). Nonetheless, given that the solution is a distribution function defined on a closed and convex set of continuous distribution functions, we are able to show its existence by using the second Schauder's theorem without this standard condition. The sketch of this proof is similar to the one used by Anderson et al. (1998) (proposition 2). The solution seems to be unique only in very specific cases, as said in the literature about Fredholm equations[7].

4 Second-price All-Pay Auction

In a second-price all-pay auction, the payement rule is the following: the winner pays the second highest bid and others pay their own bid. Our purpose is now to determine bidders' strategies and revenues. In the next section, we will compare the rents obtained in first-price and second-price auctions, as well as winner-pay and all-pay auctions. As a result, we will know which of these designs is the best to raise money for charity.

As before, we first analyze the two bidders case. It is not necessary to find each agent's probability distribution's support in order to determine the mixed strategy Nash equilibrium. Actually, we only need to assume that each bidder i's offer, x_i belongs to a strategy space $X_i \subseteq [0, +\infty)$. For the same reasons as in lemma 2, the bidders' minimum valuations is zero. As noticed before, assumption $A2$ allows us to write that $\alpha_i < 1/2$.

[7] Kanwal (1971) has written a very complete book about these questions while Ledder (1996) gives a simple method and finds another condition to prove the solution's uniqueness.

4.1 Two Bidders

The strategies' supports are no mass points and are continuous. If two bidders have a mass point, a deviation increases their probability to win. Furthermore, if one bidder has a mass point, his rival will never choose an action below this point. Thus, this bidder's mass point can only be zero. The expected utility given by (2) is

$$\mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1-2\alpha_i)x)dF_j(x) - (1-2\alpha_i)x_i(1-F_j(x_i))$$

In the second-price all-pay auction with two bidders, the payment rule leads to $t_1(x) = t_2(x)$. Thus, when a bidder wins he pays his rival's bid. Additionally, each bidder benefits from two externalities, one of which is associated to his own bid, and this other one of which is associated to his rival's bid.

Proposition 4. *There is a unique mixed strategy Nash equilibrium. Bidder i's strategy is given by an exponential distribution defined as follows,*

$$F_i \sim \mathcal{E}\left(\frac{1-2\alpha_j}{v_j}\right) \text{ and } \mathbb{E}R = \frac{v_1}{1-2\alpha_1} + \frac{v_2}{1-2\alpha_2}$$

Example 2. *We use example 1's hypotheses. Two bidders have the same externality such that* $h(x) = \dfrac{x_1 + x_2}{2\min\tilde{x}}$ *with* $\min\tilde{x} = \min_{i=1,2} \tilde{x}_i$. *Thus,*

$$F_i(x) = 1 - exp\left(\frac{\min\tilde{x} - 2x}{v_j \min\tilde{x}}\right) \quad \mathbb{E}R = (v_1 + v_2)\frac{\min\tilde{x}}{\min\tilde{x} - 1}$$

Remark 3. *As for now, we have exclusively studied mixed strategy equilibria. Yet, there are also pure strategy Nash equilibria. In the two bidders case, we find two equivalent equilibria. Note that these equilibria are degenerated as in the situations without externalities.*

As before, we note $\tilde{x}_i$ bidder i's maximum bid, such that $\tilde{x}_1 > \tilde{x}_2$.
Bidder i's expected utility is given by

$$U_i(x) = \begin{cases} v_i + (2\alpha_i - 1)x_j & \text{if } x_i > x_j \\ \dfrac{v_i}{2} + (2\alpha_i - 1)x_i & \text{if } x_i = x_j \\ (2\alpha_i - 1)x_i & \text{if } x_i < x_j \end{cases}$$

Let x_i be bidder i's offer.
First case : $x_2 \geq x_1$
If $\tilde{x}_1 \geq x_2$, bidder 2 wins the auction and his competitor earns a payoff $U_1 = (2\alpha_1 - 1)x_1 < 0$. Thus, bidder 1 deviates and offers $x_2 + \varepsilon$ in order to win the auction, which is contradictory to the initial hypothesis.
If $x_2 > \tilde{x}_1$, offering more than bidder 2's bid is a dominated strategy for the bidder 1. Then his best reply is to bid zero. Thus, $(0, \beta)$ with $\beta \in (\tilde{x}_1; +\infty)$ is an equilibrium.

Second case : $x_1 > x_2$
It is completely symmetric to the former analysis. As a result, there exists a second equilibrium $(\beta, 0)$ with $\beta \in (\tilde{x}_2; +\infty)$
Finally, there are two pure strategy Nash equilibria,

$$(0, \beta_1) \text{ with } \beta_1 \in (\tilde{x}_1, +\infty)$$
$$(\beta_2, 0) \text{ with } \beta_2 \in (\tilde{x}_2, +\infty)$$

The revenue earned for the auction is zero.

4.2 n Bidders

It is more difficult to find the equilibrium with n bidders. We note $G_i(x) = \prod_{j\neq i} F_j(x)$. It follows that the expected utility (2) can be written

$$\begin{aligned}\mathbb{E}U_i(x_i, X_{-i}) = &\int_0^{x_i} (v_i - (1-\alpha_i)x)dG_i(x) - (1-\alpha_i)x_i(1-G_i(x_i)) \\ &+ \alpha_i \sum_{l\neq i} \int_{\mathbb{R}_+} x_l \Big(1 - \mathbb{1}_{x_i \leq x_l} \prod_{k\neq l,i} F_k(x_l)\Big) dF_l(x_l) \qquad (3) \\ &+ \alpha_i \sum_{l\neq i} \Bigg(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} \sum_{k\neq l,i} x_k \prod_{\substack{m\neq i,k,l \\ k\neq l}} F_m(x_k)dF_k(x_k)dF_l(x_l) + x_i \prod_{m\neq i,l} F_m(x_i)(1-F_l(x_i)) \Bigg)\end{aligned}$$

(3) comes from (2). The transition from one to the other is explained in proposition 5's proof, in appendix page 13. The first line's two terms represent bidder i's payoff condition to his winning or losing the auction, given the externality that arises from his own action. The other lines represent the externalities that come from his competitors' actions (whether they lose or win).

The first of those two lines describes the situation when bidder l ($l \neq i$) loses the auction. In the last line bidder l wins the auction; on this line, we distinguish situations where bidder i's offer is the second highest offer from situations in which it is not. Each bidder's offer can be the second highest bid and we hold account of it (sign sum under the integral). The bidder who makes an offer between bidder i and bidder l's offers puts forward the second highest bid. The other part gives the amount of money that bidder l will have to paid when i offers the second highest bid. Indeed, $\prod_{m\neq i,l} F_m(x_i)(1-F_l(x_i))$ is the probability that every bidder except l makes a lower bid than i . This probability is multiplied by the sum offered by the bidder i.

Note that this expression of expected utility is not valid unless there are at least four bidders. In order to study the three bidders case, it is necessary to (slightly) change the third line. To do this, we must stop computations to the second line of term B_I in the appendix. Thus, this term is writing $\alpha_i \sum_{l\neq i} \left(\int_{\mathbb{R}_+} \int_{x_i}^{x_l} x_k dF_k(x_k)dF_l(x_l) + x_i F_k(x_i)(1-F_l(x_i)) \right)$, where k is not i, neither l. We do not explain this calcul more.

Proposition 5. *If $v_1 > v_2 > v_i$ $\forall i > 2$ and $A1 - A3$ are verified, only two bidders among n participate actively to the auction.*

The bidders' mixed strategies are given by the proposition 4. The weakness of this result is we do not know which bidders are going to participate. Thus, it could happen that the two bidders with the highest values participate or the ones with the lowest values[8]. There are some consequences on the expected revenue.

[8]Other equilibria could also exist with two more bidders. However, as we will see later, they cannot have more implications than the equilibria discussed before regarding the revenue comparisons. Indeed, the second-price all-pay auction outperforms the other auction designs as long as the bidder with the highest adjusted-value takes part to the auction with another bidder. On the contrary, when this bidder does not take part to the auction, the ranking of the expected revenue raised in the second-price all-pay auction can be lower than the other auction designs.

5 Revenue Comparisons

In this section, we investigate the performance of the revenues and the expected revenues obtained with the different designs.

We consider here that the two bidders have the same altruism level i.e. $\alpha_1 = \alpha_2 = \alpha$. Hence, the bidder with the highest value is also the one with the highest adjusted-value. The expected revenue becomes

$$\mathbb{E}R^{AP1} = \frac{1}{2}\frac{v_2}{1-\alpha}\left(\frac{v_2}{v_1}+1\right) \text{ et } \mathbb{E}R^{AP2} = \frac{v_i + v_j}{1-2\alpha} \;\; i,j \in N$$

Indexes AP_i and WP_i correspond to i^{st}-price all-pay and winner-pay auctions. If bidders are complete altruists, i.e. $\alpha^{AP1} \longrightarrow 1$ and $\alpha^{AP2} \longrightarrow 1/2$, the expected revenues diverge as Goeree et al. (2005) predicted. Thus, the altruism level is an essential element to determine the expected revenue. When bidders' altruism levels are the same, the rent for the auction is at least equal to the rent one would obtain with non-altruistic bidders.

We can notice that the second-price all-pay auction gives a higher rent than other auction designs as long as the bidder with the highest adjusted-value takes part to the auction. To show that, we use Ettinger (2002)'s results about winner-pay auctions with externality. On the contrary, when this bidder does not take part to the auction, the ranking of the expected revenue raised in the second-price all-pay auction depends on the asymmetry between bidders' valuations.

As a consequence, the second price all-pay auction seems more adapted than others to raise money for charity. Yet, if our setting is suited to charity dinners in complete information (for example dinners organized by a local Rotary Club) first-price all-pay auction contradicts Goeree et al. (2005)'s results. We sum up our results[9] in this table:

$v_1 > v_2 > v_3 > v_i \; \forall i > 3$	R^{WP1}	R^{WP2}	$\mathbb{E}R^{AP1}$	$\mathbb{E}R^{AP2}$
$\alpha > 0$	v_2	v_1	$\frac{1}{2}\frac{v_2}{1-\alpha}\left(\frac{v_2}{v_1}+1\right)$	$\frac{v_1+v_i}{1-2\alpha},\ i \neq 1$
$\alpha = 0$	v_2	v_2	$\frac{v_2}{2}\left(\frac{v_2}{v_1}+1\right)$	$v_1 + v_i,\ i \neq 1$

Table 1: Revenues and expected revenues for every design

We notice that with homogeneous values, we find the same results as Goeree et al. (2005) does. In particular, the first-price all-pay auction rent dominates the winner-pay auction rent. In order to analyze the impact of asymmetry on rents, we use the following definition.

Definition. *The level of asymmetry between bidders' valuations will be considered "high" if* $v_1 - v_2 > 2\alpha v_1$, *"medium" if* $2\alpha v_1 > v_1 - v_2 > 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$ *and "low" if* $v_1 - v_2 < 2\alpha v_1 - v_1 + v_2\frac{v_2}{v_1}$.

Proposition 6. *We assume that* $\alpha_i = \alpha \; \forall i$ *and that the bidder with the highest adjusted-value takes part to the second-price all-pay auction. Then, this design gives the highest revenues:*

$$\mathbb{E}R^{AP2} > R^{WP2} \geq R^{WP1} \text{ and } \mathbb{E}R^{AP2} > \mathbb{E}R^{AP1}$$

All other things being equal, $\mathbb{E}R^{AP1} > R^{WP2}$ *if and only if the level of asymmetry between valuations is "low",* $R^{WP2} > \mathbb{E}R^{AP1} > R^{WP1}$ *if and only if this level is "medium", and* $R^{WP1} > \mathbb{E}R^{AP1}$ *if and only if it is "high".*

[9] Here, in the second-price all-pay auction without externality, we consider only equilibria where only two bidders take part to the auction.

Proof. Computations. ■

The second part of this proposition can be interpreted in two independent ways.

- First of all, given α, the (first-price) all-pay auction is dominated by the first-price winner-pay auction when asymmetry is "high". Furthermore, this all-pay auction raises more money than the second-price winner-pay auction when asymmetry is "low".

- Given v_1 and v_2, first-price all-pay auction is dominated by first and second-price winner-pay auctions when the bidders' altruism level is less than $\frac{1}{2}(1-\frac{v_2}{v_1})$ and $1-\frac{1}{2}\frac{v_2}{v_1}(\frac{v_2}{v_1}+1)$. In particular, the threshold above which this all-pay auction raises more money than the first-price winner-pay auction is less than $\frac{1}{2}$.

The more asymmetry increases, the more the level of the altruism must also increase for the first-price all-pay auction to give a higher rent than winner-pay auctions. The two graphs below show the limits (in terms of rent domination) for the first-price all-pay auction. We use two parameters: altruism level and the asymmetry among bidders' values (from left to right, $\frac{v_2}{v_1}$ varies from 0.9 to its limit in zero with a 0.1 step).

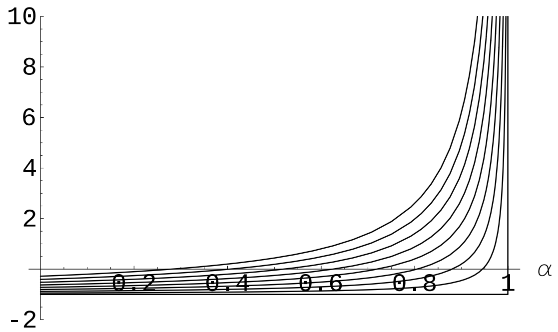


Figure 1: $\mathbb{E}R^{AP1} > R^{WP2}$

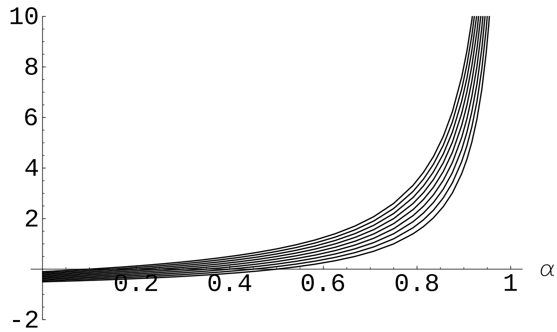


Figure 2: $\mathbb{E}R^{AP1} > R^{WP1}$

6 Minimum Bids Imposed

In this section, we determine the impact of minimum bids imposed on rent for two auction designs: first-price all-pay and winner-pay auction. In the rest of the paper, we will note $t_i(x) = x_i$ for all $i \in N$. Moreover, we analyze only the two bidders case (who have the highest valuations). Indeed, only these two bidders participate in all-pay auction as in the second section.

The value rankings and externality parameters (given by $A3$) are kept. While the bidders know the values and these rankings, the charity auction organizer imposes an individual bid on everybody: bidder i has to offer a bid at least equal to tv_i so as to take part to the auction. This implies that the auctioneer also knows the bidders' value, so that he can impose a rate t on them. This assumption is not unrealistic. This phenomenon could occur in a local service club (like a local Rotary club) or during a show business dinner. Indeed, the auctioneer could obtain this kind of informations through the staff of the local community or because he is himself a member or a friend of the participants.

As expected, there is no pure strategy Nash equilibrium[10]. In order to find the strategies and the probability of entry, we focus on the situation where every bidder wants to participate. As in section 2, we first define the distribution functions' supports.

Lemma 3. *At equilibrium, the bidders' minimum bids are asymmetric. They are tv_1 for bidder 1 and tv_2 for bidder 2. In fact, the latter's density is equal to zero on the support $(tv_2, tv_1]$.*

With probability one, bidder i's offer will be at least equal to tv_i. We conclude that $\min x_i \geq tv_i$. Now, let us assume that $\min x_1 = x > tv_1$. Then $\mathbb{P}(X_1 < \{x\}) = 0$, because bidder 1 never makes any offer in the interval (tv_1, x). His competitor offers either tv_2 or $x + \varepsilon$ for $\varepsilon > 0$, a bid between these two values being strictly dominated. Then, if bidder 1 bids $x - \varepsilon$ his probability of winning is not affected. Thus, his minimum bid is tv_1. Moreover, bidding in the interval $(tv_2, tv_1]$ is strictly dominated for bidder 2. Hence, $\mathbb{P}(tv_2 < X_2 \leq tv_1) = 0$ and if he bids $tv_2 < x \leq tv_1$ he loses for sure. When he offers $x = tv_2$ he does not affect his probability of winning but increases his payoff by $A2$. Furthermore, he increases his probability of winning by bidding $x = tv_1 + \varepsilon$ for $\varepsilon > 0$. Bidder 2's density function is zero on the interval $(tv_2; tv_1]$.

Lemma 4. *At equilibrium, bidders offer the same maximum bid $\bar{x} = (1 - \alpha_2 t)\tilde{x}_2$. Every bidder has a mass point for his minimum bid and a mass point can never be on $(tv_1, \bar{x}]$.*

For similar reasons as the ones pointed out in section 3, all bidders' maximum bids are equal. Additionally, even if the payoff functions are the same, that is to say

$$\mathbb{E}U_1(x, X_2) = F_2(x)v_1 - (1 - \alpha_1)x + \alpha_1 \mathbb{E}X_2, \quad \mathbb{E}U_2(x, X_1) = F_1(x)v_2 - (1 - \alpha_2)x + \alpha_2 \mathbb{E}X_1$$

the expected level of the bidders' reservation utilities are changed. Indeed, as the minimum bids are positive, bidder i's reservation utility is $\alpha_i \mathbb{E}X_j + \alpha_i tv_i$: he participates to the auction if he gets at least $\alpha_i \mathbb{E}X_j$ (as before) plus the reward of his own minimum bid. Hence, the maximum bid is equal to the lowest of the two bidders' *indifference pricing*. At his *indifference pricing*, bidder i is indifferent between taking part to the auction or not, that is to say to offer tv_i. Thus, the maximum bid is $\bar{x} = (1 - \alpha_2 t)\tilde{x}_2$.
Given the former analysis, bidder 2 has a mass point on tv_2. Bidder 2's strategy space is $\{tv_2\} \cup (tv_1; \bar{x}]$. For similar reasons as in section 3 and for the case without externality, having a mass point on the bidders' common strategy set is dominated for every bidder[11] (since they deviate).

For now, we only consider the bidders' common strategy set, that is to say $(tv_1; \bar{x}]$. A bidder's equilibrium payoff is a constant function on his whole strategy set. Hence,

$$F_2(x)v_1 - (1 - \alpha_1)x + \alpha_1 \mathbb{E}X_2 = v_1 - (1 - \alpha_1)\bar{x} + \alpha_1 \mathbb{E}X_2 \tag{4}$$

for all $x \in (tv_1; \bar{x}]$. The left member of this equation is the bidder 1's expected utility for all bids in $(tv_1; \bar{x}]$, while the right member is bidder 1's payoff when he bids $\bar{x}$. In the same way, bidder 2's bid is such that

$$F_1(x)v_2 - (1 - \alpha_2)x + \alpha_2 \mathbb{E}X_1 = v_2 - (1 - \alpha_2)\bar{x} + \alpha_2 \mathbb{E}X_1 \tag{5}$$

[10] To see this, let us assume that $x_1 \geq x_2$. As before, we have to consider two situations. First, bidder 2 can overbid. It contradicts the initial assumption. If he cannot overbid, given $A2$, his best reply is to offer tv_2. Hence, bidder 1 bids tv_1. The equilibrium is unstable.

[11] We give here a well-known argument (see for instance Che & Gale (1998)) to support this idea. If only one bidder has a mass point on the support that is common to both bidders, his competitor's density function below this mass point is equal to zero. Hence, he is going to move and his mass point will be the support's lower bound. This action does not affect his probability of winning, but it increases his payoff if he wins. In a similar way, if bidders have a mass point, deviating increases their probability of winning. Consequently, the result follows.

and thus belongs to the interval $\{tv_2\} \cup (tv_1; \bar{x}]$.

In particular, for all bids in the interval $(tv_1, \bar{x}]$ and for $\alpha_1 = \alpha_2$, we find that

$$v_2(1 - F_1(x)) = v_1(1 - F_2(x))$$

As bidder 2 has a mass point on tv_2, the limit in tv_1 gives us the following result[12]

$$F_1(tv_1) = 1 - \frac{v_1}{v_2} + \frac{v_1}{v_2} F_2(tv_1)$$

Using (4) and (5), it is easy to determine the bidders's distribution functions. We specify them in proposition 7 below. As $F_2(tv_1)$ is not equal to $1 - \frac{v_2}{v_1}$ (the value in zero without any externality and minimum bids imposed) bidder 1 has indeed a mass point on tv_1. The bidders' distribution functions are drawn below.

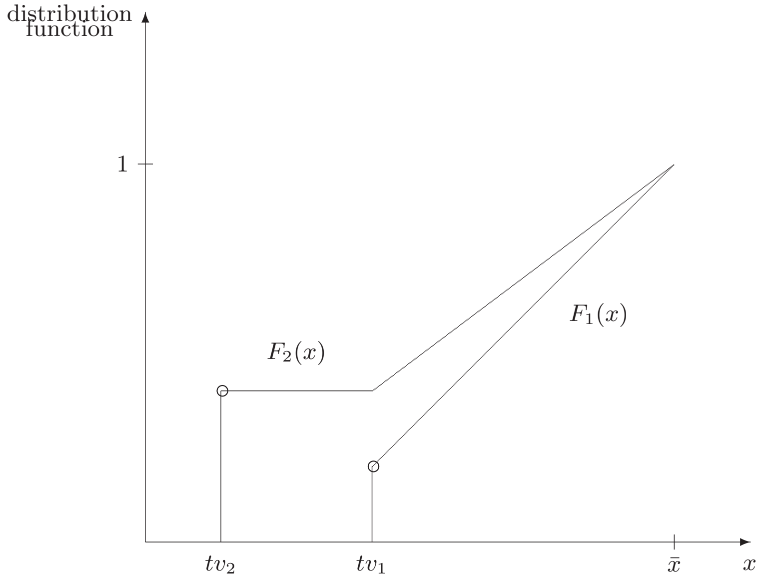


Figure 3: Cumulative distribution functions at the equilibrium

Proposition 7. *Given the bidders' adjusted-values, $(1 - \alpha_1 t)\tilde{x}_1$ and $(1 - \alpha_2 t)\tilde{x}_2$, there is a unique Nash equilibrium. The bidders' strategies for all $x \in (tv_1; \bar{x}]$ are*

$$F_1(x) = \alpha_2 t + \frac{x}{\tilde{x}_2} \text{ and } F_2(x) = 1 + \frac{x - \bar{x}}{\tilde{x}_1}.$$

Every bidder has one point mass: it is tv_1 for bidder 1 and tv_2 for bidder 2.

A bidder's decision is given by his probability to participate,

$$1 - F_1(tv_1) = 1 - \alpha_2 t + \frac{tv_1}{\tilde{x}_2} \text{ and } 1 - F_2(tv_2) = \frac{tv_1 - \bar{x}}{\tilde{x}_1}$$

Additionally, if the maximum bid $\bar{x}$ is inferior to bidder 1's minimum bid $\bar{x} \leq tv_1$, offering a higher bid than their minimum bid is dominated for all bidders. Hence, $\mathbb{E}R = t(v_1 + v_2)$ for all $t \geq \bar{t}$ where $\bar{t} \equiv \dfrac{\tilde{x}_2}{v_1 + \alpha_2 \tilde{x}_2}$.

Here, we consider the case where $0 \leq t < 1$ only[13].

[12] As $\mathbb{P}(tv_2 < X_2 \leq tv_1) = 0$ it follows that $\lim_{x \to tv_1} F_2(x) = F_2(tv_1) = F_2(tv_2)$.

[13] $t > 1$ is not appropriate here. Indeed, the minimum bid of one bidder could be higher than the maximum bid.

Proposition 8. *Given the distribution functions $F_1(.), F_2(.)$ at equilibrium, the expected revenue raised for charity is*

$$\mathbb{E}R = \begin{cases} \bar{x}^2 \dfrac{\tilde{x}_1 + \tilde{x}_2}{2\tilde{x}_1\tilde{x}_2} + (tv_1)^2 \dfrac{\tilde{x}_1 - \tilde{x}_2}{2\tilde{x}_1\tilde{x}_2} + t^2 v_1 \alpha_2 + tv_2 \left(1 + \dfrac{tv_1 - \bar{x}}{\tilde{x}_1}\right) & \text{if } t < \bar{t} \\ t(v_1 + v_2) & \text{otherwise} \end{cases}$$

Proof. We only have to compute the expected revenue associated to every bidder when $t < \bar{t}$:

$$\begin{aligned} \mathbb{E}R_i &= \int_{tv_1}^{\bar{x}} x f_i(x)dx + tv_i F_i(tv_1) \\ &= \bar{x} \int_{tv_1}^{\bar{x}} f_i(y)dy - \int_{tv_1}^{\bar{x}} \int_{tv_1}^{x} f_i(y)dydx + tv_i F_i(tv_1) \\ &= \bar{x}(F_i(\bar{x}) - F_i(tv_1)) - \int_{tv_1}^{\bar{x}} F_i(x) - F_i(tv_1)dx + tv_i F_i(tv_1) \\ &= \bar{x} - \int_{tv_1}^{\bar{x}} F_i(x)dx + (tv_i - tv_1)F_i(tv_1) \end{aligned}$$

Hence, $\mathbb{E}R_1 = \dfrac{\bar{x}^2 + (tv_1)^2}{2\tilde{x}_2} + t^2 v_1 \alpha_2$ and $\mathbb{E}R_2 = \dfrac{\bar{x}^2 - (tv_1)^2}{2\tilde{x}_1} + tv_2 \left(1 + \dfrac{tv_1 - \bar{x}}{\tilde{x}_1}\right)$ ■

We must analyze the impact of all t values on the rent. This will allow us to determine whether imposing a minimal bid to every bidder permits to improve the first-price all-pay auction's efficiency compared to the first-price winner-pay auction or not. In order to do so, we assume that bidders have the same altruism attitude, such that $\alpha = \alpha_1 = \alpha_2$. We analyze only the revenue achievement for $t \leq \bar{t}$. After an increase in t, there are two contradictory effects. The bidders' support's lower bound increases while its upper bound decreases. As a consequence, the expected revenue can increase or decrease. The result depends on which effect dominates the other.

First of all, let us assume that the asymmetry between the bidders' values is considered "high" such that $v_1 - v_2 > 2\alpha v_1$. As a consequence, the all-pay auction expected revenue is increasing in t. The low altruism level of the bidders offsets the impact of t on the bidders' maximum bid, so that the effect on the lower bound dominates. As was pointed out before, when $t = 0$ the all-pay auction expected revenue is strictly dominated by the first-price winner-pay auction revenue[14]. Given this result, the all-pay auction gives a higher revenue than the winner-pay auction for a value of t that offsets the impacts of asymmetry. The graph below illustrates this result for $v_1 = 20$ and $v_2 = 5$. Each curve is the expected revenue when asymmetry is considered "high" and for a specific value of t. The lower envelope curve is given by $t(v_1 + v_2)$. The first-price winner-pay auction revenue is given by the dashed curve.

[14]The first-price auction gives revenue v_2 with a rate t inferior to $\frac{v_2}{v_1} < \bar{t}$. For higher rates, the revenue becomes $tv_1 < t(v_1 + v_2)$.

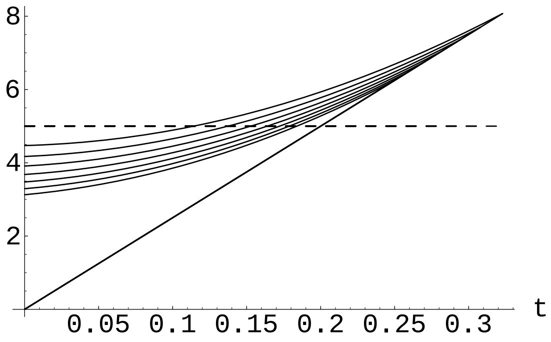


Figure 4: Expected revenue with a "high" asymmetry

It is obvious that situations where asymmetry is "medium" or "low" give the same result: winner-pay auction raise more money than all-pay auction. Yet, it is interesting to draw the expected revenues associated to those asymmetry levels. Here, the decreasing effect of the support's upper bound is higher than the increasing effect of the lower bound below a given value of t, where dynamics is reversed.

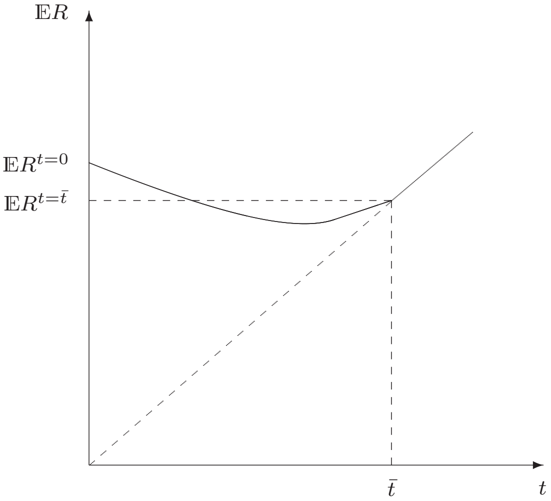


Figure 5: $\mathbb{E}R$ for $\alpha > \max\{\frac{v_1}{v_1+v_2}, \frac{v_1-v_2}{2v_1}\}$

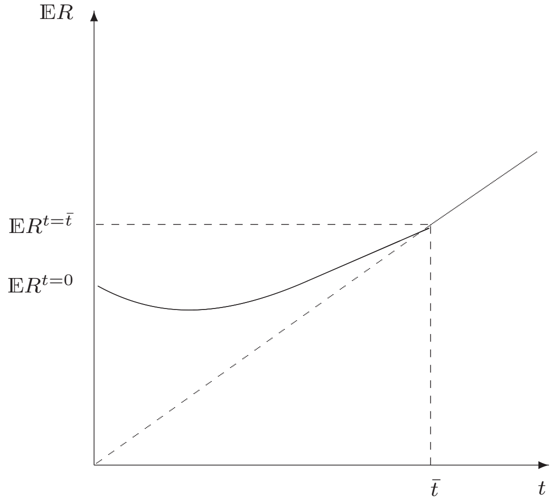


Figure 6: $\mathbb{E}R$ for $\frac{v_1}{v_1+v_2} > \alpha > \frac{v_1-v_2}{2v_1}$

Proposition 9. *Imposing a minimal bid to every bidder permits to improve the first-price all-pay auction's efficiency compared to the first-price winner-pay auction. There is a threshold t above which the all-pay auction dominates the winner-pay auction when the values' asymmetry is considered "high".*

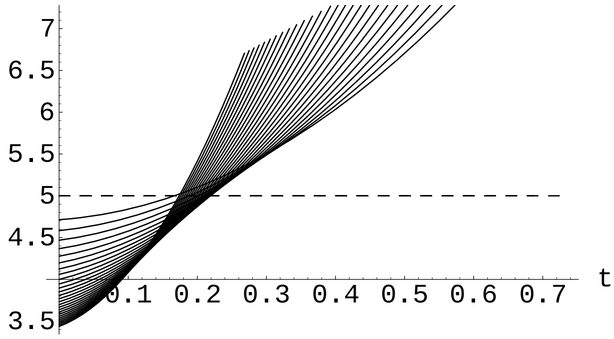
Example 3. *We focus again on the example 1: two bidders benefit the same externality* $\alpha_1 = \alpha_2 = \frac{1}{2\min \tilde{x}_i}$. *Hence, the two bidders' maximum bid is* $\bar{x} = v_2 + \frac{1-t}{2}$ *and the bidders' mixed strategies are*

$$F_1(x) = \frac{2x+t}{2v_2+1} \text{ and } F_2(x) = 1 + \frac{v_2(2x-2v_2+t-1)}{v_1(2v_2+1)}$$

Furthermore, the expected revenue when $t < \frac{2v_2+1}{2v_1+1}$ *is*

$$\mathbb{E}R = \frac{1}{2v_1(2v_2+1}\left[(v_2+\frac{1-t}{2})^2(v_1+v_2)+(tv_1)^2(v_1-v_2)+2(tv_1)^2+1tv_1v_1(2v_2+1)+4tv_2^2(tv_1-v_2-\frac{1-t}{2})\right]$$

The graphic below gives all the charts of expected revenue with $t < \bar{t}$, $v_2 = 5$ and v_1 increasing[15] *from 7 to 20 with a 0.5 step. Example 1 (without minimum bids imposed) is equivalent to the situation when $t = 0$. Thus, when the values of t are high enough, we can notice that all-pay auction is better than winner-pay auction with "high" asymmetry.*



7 Conclusion

All-pay auctions with externalities that are independent of the winner's identity but functions of the amount raised have other applications in economy.
Here, we focused on the team theory. The next illustration could be connected to other forms of team works (particularly in firms) leading to social promotion. Let's consider, a team sport like basket-ball. Every year during the American championship of basket-ball (the NBA) or the all-stars game finals, the most valuable player (MVP) is elected. During such games, every player makes the highest effort to win the event but also to be elected the MVP of the game. Each player takes advantage of the team's effort to win the game and thus can be elected MVP thanks to the externality of the total amount of the efforts made. v_i represents the player's value for the MVP title. Therefore, his effort x_i has two gaols: to win the game and be elected MVP. When a player is not elected MVP, he takes advantage of the externality by winning the game. As a player tries to win the game by making the highest effort, he helps also his team mates to be elected MVP .

This work could be completed by an experiment. In fact, only two experiments have been implemented until now with opposite results. We have already cited them in the introduction: Onderstal & Schram (2006) and Carpenter et al. (2004). Onderstal & Schram (2006) find similar results to Goeree et al. (2005). However, our results are quite different from Goeree et al. (2005)'s because of the introduction of asymmetric valuations. That is why, it would be interesting to test our prediction with the introduction of asymmetry between the bidders' valuations: all-pay auction can be dominated by winner-pay auction. That could also be the occasion to test the impact of altruism on agents behavior. Finally, theoritical and experimental works should be lead about the form of the externalities that we considered here linear.

In a recent paper, Edlin (2005) displays a tax credit method to incite people to give more for charity purposes. He suggests to deduce the agents' donations to charity organizations from their income tax (limited to a certain percent of their income). The agents are free to choose

[15] $v_1 \geq 7$ ensures that the asymmetry between values is high.

the organization they want to help. This method should improve all-pay auctions for charity and lets an open question for futur researches.

Appendix

proof of proposition 1. If we divide the bidders' expected utility by $1-\alpha_i$, we almost obtain the same bidders' expected utility as in the case without externality given by Hillman & Riley (1989). However, after this operation has been made, there remains an important difference between the bidder's expected utility we find and the one Hillman & Riley (1989) find in the case without externality. Indeed, there is a constant in their function while our function has an externality $\alpha_j \frac{\mathbb{E}X_j}{1-\alpha_i}$. Thus, we only have a constant in our function at the equilibrium. By this division the result follows as in Hillman & Riley (1989):

$$F_j(x) = \frac{1-\alpha_i}{v_i}x \qquad \forall x \in \left[0, \frac{v_i}{1-\alpha_i}\right]$$
$$F_i(x) = 1 - \frac{1-\alpha_j}{1-\alpha_i}\frac{v_i}{v_j} + \frac{1-\alpha_j}{v_j}x \quad \forall x \in \left(0, \frac{v_i}{1-\alpha_i}\right]$$

Participant k's expected payoff $(k=i,j)$ is given by

$$\begin{aligned}\mathbb{E}R_k &= \int_0^{\tilde{x}_i} x dF_k(x) + 0.F_k(0)\mathbb{1}_{k=i} \\ &= \tilde{x}_i \int_0^{\tilde{x}_i} dF_k(y)dy - \int_0^{\tilde{x}_i}\int_0^x dF_k(x) \\ &= \tilde{x}_i - \int_0^{\tilde{x}_i} F_j(x)dx\end{aligned}$$

that is to say $\mathbb{E}R_j = \frac{1}{2}\frac{v_i}{1-\alpha_i}$ and $\mathbb{E}R_i = \frac{1}{2}\frac{1-\alpha_j}{v_j}\left(\frac{v_i}{1-\alpha_i}\right)^2$

Hence

$$\mathbb{E}R = \frac{1}{2}\frac{v_i}{1-\alpha_i}\left(\frac{1-\alpha_j}{1-\alpha_i}\frac{v_i}{v_j}+1\right)$$

■

Proof of proposition 3. By lemma 2, the two players make their bids on the common support $[0,b]$. The set of equilibria in mixed strategies is completely characterized by a Nash equilibria where only pure strategies which are better responses to the others strategies are played with a strictly positive probability. All of these strategies lead to the same expected utility. Next, we denote $\lambda = \frac{1}{v_i}$ and ignore the suffix.

Let T be an operator such as $T : F(x) \longmapsto TF(x)$ and

$$TF(x) \equiv \lambda x - \lambda \int_0^b h(x,y) f(y)dy + \text{ constant} \tag{6}$$

As F is a continuous function, we restrict our study to the set of continuous functions on $[0,b]$ denoted $C[0,b]$. Especially, we consider $D = \{F \in C[0,b] \backslash ||F|| \leq 1\}$ with $||.||$ the supremum norm. The set D, which includes all of the continuous distribution functions, is closed and convex but not bounded. Indeed, it's an infinite-dimensional unit ball. Thus, to prove that (6) has a solution, we apply the following Schauder's second theorem:

Theorem (Schauder, 1930)**.** *If D is a closed convex subset of a normed space and E is a relatively compact subset of D, then every continuous mapping of D to E has a fixed-point.*

To apply this theorem, we need to prove two parts. First, that $T(D) \equiv E = \{TF \backslash F \in D\}$ is relatively compact[16]. Second, T is a continuous mapping from D to E.

Showing that E is relatively compact is equivalent to showing that E is equicontinuous (Ascoli's theorem) on $[0, b]$. Let's show that E is equicontinuous. We need to show that$\forall \varepsilon, \exists \eta, \forall F \in E$ such that $|TF(x_1) - TF(x_2)| < \varepsilon$ when $|x_1 - x_2| < \eta$.

$$\begin{aligned}
|TF(x_1) - TF(x_2)| &= \left| \lambda(x_1 - x_2) - \lambda \int_0^b [h(x_1, y) - h(x_2, y)] f(y) dy \right| \\
&\leq \lambda \left[|x_1 - x_2| + \left| \int_0^b [h(x_1, y) - h(x_2, y)] f(y) dy \right| \right] \\
&\leq \lambda |x_1 - x_2| \left[1 + \frac{|\sup_{y \in [0,b]} [h(x_1, y) - h(x_2, y)]|}{|x_1 - x_2|} \right] \\
&< \lambda \eta \left[1 + \frac{|\sup_{y \in [0,b]} [h(x_1, y) - h(x_2, y)]|}{|x_1 - x_2|} \right]
\end{aligned}$$

The function h is continuous and bounded on $[0, b]$. $[0, b]$is a compact which explains the result of the last line. Denoted $\kappa \equiv |\sup_{y \in [0,b]} [h(x_1, y) - h(x_2, y)]|$. Thus, $|TF(x_1) - TF(x_2)| < \varepsilon$ for $\eta = \varepsilon \frac{|x_1 - x_2|}{\lambda(|x_1 - x_2| + \kappa)}$.

Now, let's prove the continuity of T. The operator T is continuous if, for all F_1, F_2 and for all $\varepsilon > 0$, there exists a $\eta > 0$ such that $|TF_1(x) - TF_2(x)| < \varepsilon$ when $|F_1 - F_2| < \eta$. Let us write $F_1(x) = F_2(x) + g(x)$ with $-\eta < g(x) < \eta \ \forall x \in [0, b]$. Henceforth

$$\begin{aligned}
|TF_1(x) - TF_2(x)| &= \left| - \lambda \int_0^b h(x, y)(f_1(y) - f_2(y)) dy \right| \\
&\leq \lambda \int_0^b |h(x, y)||g'(y)| dy \\
&\leq h(b, b) \lambda \int_0^b |g'(y)| dy \\
&< h(b, b) \lambda \eta
\end{aligned}$$

To go from the first to the second line, notice that $F_1'(x) - F_2'(x) = g'(x)$. We use the fact that h is a continuous function on $[0, b]$ bounded by a maximum $h(b, b)$ to go to the third line. Hence, the difference between TF_1 and TF_2 is inferior to $\varepsilon > 0$ when $\eta = \frac{\varepsilon}{\lambda h(b,b)}$. ∎

Proof of proposition 4. All mixed strategies at the equilibrium lead to the same expected utility. Thus, we can completely characterize the set of equilibrium in mixed strategies. In particular, the expected utility is zero for $x_i = 0$:

$$\mathbb{E}U_i(x_i, X_{-i}) = \int_0^{x_i} (v_i - (1 - 2\alpha_i)x) dF_j(x) - (1 - 2\alpha_i)x_i(1 - F_j(x_i)) = 0$$

Hence the Volterra integral equation

$$f_j(x)v_i = (1 - 2\alpha_i)(1 - F_j(x)) \tag{7}$$

[16]A space is relatively compact when his closed span is compact.

The solution is given by

$$F_j(x) = 1 - k_j exp\left(-\frac{(1-2\alpha_i)x}{v_i}\right) \quad x \in X_j \;\; k_j \in \mathbb{R}$$

F_j is a distribution function defined on X_j where the minima is zero and the maxima noted $\tilde{x}$. As the distribution functions must verify $F_j(0) = 0, F_j(\tilde{x}) = 1$ and $\int_0^{\tilde{x}} f_j(x)dx = 1$, we know that X_j and $[0; +\infty)$ are merged but also that $k_j = 1$. Henceforth,

$$F_j(x) = 1 - exp\left(-\frac{(1-2\alpha_i)x}{v_i}\right) \quad x \in [0; +\infty)$$

■

Proof of proposition 5. By (2) we have the expected utility:

$$\mathbb{E}U_i(x_i, X_{-i}) = v_i \prod_{j \neq i} dF_j(x_j) - (1-\alpha_i) \underbrace{\int_{\mathbb{R}_+^{n-1}} t_i(x) \prod_{j \neq i} dF_j(x_j)}_{A} + \alpha_i \underbrace{\int_{\mathbb{R}_+^{n-1}} \sum_{j \neq i} t_j(x) \prod_{j \neq i} dF_j(x_j)}_{B}$$

A represents bidder i's expected payment when we take into account its own external effect. The term B is the expected payment of bidder i's rivals. $\alpha_i B$ is the sum of the externalities of bidder i's rivals that i takes advantage of.

We can write A again as follow

$$\underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)} \mathbb{1}_{\substack{x_i \geq x_j \\ \forall j \neq i}} \prod_{j \neq i} dF_j(x_j)}_{A_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_i \mathbb{1}_{\substack{\exists k / x_k > x_i \\ k \neq i}} \prod_{j \neq i} dF_j(x_j)}_{A_{II}}$$

The term A_I is i's expected payment when he wins *i.e.* he pays the second highest bid. A_{II} is i's expected payment when he loses. He could then either be the second highest bidder or a lower bidder.

$$\begin{aligned}
A_I &= \int_{\mathbb{R}_+^{n-1}} \sum_{j \neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j,i\}, j \neq i}} \prod_{j \neq i} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \int_{\mathbb{R}_+^{n-2}} \prod_{k \neq i,j} \mathbb{1}_{x_k \leq x_j \leq x_i} \prod_{k \neq i,j} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \left\{ \prod_{k \neq i,j} \int_{\mathbb{R}} \mathbb{1}_{x_k \leq x_j \leq x_i} dF_k(x_k) \right\} dF_j(x_j) \\
&= \int_{\mathbb{R}_+} \sum_{j \neq i} x_j \mathbb{1}_{x_j \leq x_i} \prod_{k \neq i,j} F_k(x_j) dF_j(x_j) \\
&= \int_0^{x_i} x dG_i(x)
\end{aligned}$$

We get the first line from the fact that $x^{(2)} \mathbb{1}_{x_i \geq x_j} = \sum_{j \neq i} x_j \mathbb{1}_{\substack{x_k \leq x_j \leq x_i \\ \forall k \neq \{j,i\}, j \neq i}}$. The independence of the distribution functions explains how we go from the second to the third line. By denoting $dG_i(x) = \sum_{j \neq i} \prod_{k \neq i,j} F_k(x) dF_j(x)$, we obtain the final result.

$$
\begin{aligned}
A_{II} &= \int_{\mathbb{R}_+^{n-1}} x_i(1-\mathbb{1}_{i\in Q(x)})\prod_{j\neq i} dF_j(x_j)\\
&= x_i - x_i\prod_{j\neq i}\int_{\mathbb{R}_+}\mathbb{1}_{i\in Q(x)} dF_j(x_j)\\
&= x_i - x_i\prod_{j\neq i}F_j(x_i)\\
&= x_i(1-G_i(x_i))
\end{aligned}
$$

The independence of the distribution functions, explains how we go from the first line to the second.
B can be written also like

$$
\begin{aligned}
B &= \sum_{l\neq i}\int_{\mathbb{R}_+^{n-1}} t_l(x)\prod_{j\neq i} dF_j(x_j)\\
&= \sum_{l\neq i}\Bigg\{\underbrace{\int_{\mathbb{R}_+^{n-1}} x^{(2)}\mathbb{1}_{\substack{x_l\geq x_k\\ \forall k\neq l}}\prod_{j\neq i} dF_j(x_j)}_{B_I} + \underbrace{\int_{\mathbb{R}_+^{n-1}} x_l\mathbb{1}_{\substack{\exists k/x_l<x_k\\ k\neq l}}\prod_{j\neq i} dF_j(x_j)}_{B_{II}}\Bigg\}
\end{aligned}
$$

We add all of the expected external effects. The case where player $l\neq i$ takes the second higher bid is distinguished from the others.

$$
\begin{aligned}
B_I &= \int_{\mathbb{R}_+^{n-1}}\sum_{k\neq l} x_k\mathbb{1}_{\substack{x_m\leq x_k\leq x_l\\ \forall m\neq\{k,l\}}}\prod_{j\neq i} dF_j(x_j)\\
&= \int_{\mathbb{R}_+^{n-1}}\sum_{k\neq l} x_k\prod_{m\neq\{k,l\},k\neq l}\mathbb{1}_{x_m\leq x_k\leq x_l}\prod_{j\neq i} dF_j(x_j)\\
&= \int_{\mathbb{R}_+^{n-1}}\sum_{k\neq i,l} x_k\prod_{\substack{m\neq i,k,l\\ k\neq l}}\mathbb{1}_{x_m\leq x_k\leq x_l} dF_m(x_m)\mathbb{1}_{x_i\leq x_k\leq x_l} dF_k(x_k)dF_l(x_l)\\
&\quad + \int_{\mathbb{R}_+^{n-1}} x_i\prod_{m\neq i,l}\mathbb{1}_{x_m\leq x_i\leq x_l}\prod_{j\neq i} dF_j(x_j)\\
&= \int_{\mathbb{R}_+^2}\sum_{k\neq i,l} x_k\int_{\mathbb{R}_+^{n-3}}\prod_{\substack{m\neq i,k,l\\ k\neq l}}\mathbb{1}_{x_m\leq x_k} dF_m(x_m)\mathbb{1}_{x_i\leq x_k\leq x_l} dF_k(x_k)dF_l(x_l)\\
&\quad + x_i\int_{\mathbb{R}_+}\prod_{m\neq i,l}\left\{\int_0^{x_i} dF_m(x_m)\right\}\mathbb{1}_{x_i\leq x_l} dF_l(x_l)\\
&= \int_{\mathbb{R}_+^2}\sum_{k\neq i,l} x_k\prod_{\substack{m\neq i,k,l\\ k\neq l}} F_m(x_k)\mathbb{1}_{x_i\leq x_k\leq x_l} dF_k(x_k)dF_l(x_l) + x_i\prod_{m\neq i,l}F_m(x_i)(1-F_l(x_i))\\
&= \int_{\mathbb{R}_+}\int_{x_i}^{x_l}\sum_{k\neq i,l} x_k\prod_{\substack{m\neq i,k,l\\ k\neq l}} F_m(x_k)dF_k(x_k)dF_l(x_l) + x_i\Bigg(\prod_{m\neq i,l}F_m(x_i) - G_i(x_i)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
B_{II} &= \int_{\mathbb{R}_+^{n-1}} x_l(1-\mathbb{1}_{l\in Q(x)})\prod_{j\neq i} dF_j(x_j)\\
&= \int_{\mathbb{R}_+^{n-1}} x_l\prod_{j\neq i} dF_j(x_j) - \int_{\mathbb{R}_+^{n-1}} x_l\mathbb{1}_{\substack{x_k\leq x_l\\ \forall k\neq l, l\neq i}}\prod_{j\neq i} dF_j(x_j)\\
&= \int_{\mathbb{R}_+^{n-1}} x_l\prod_{j\neq i} dF_j(x_j) - \int_{\mathbb{R}_+^{n-1}} x_l\prod_{k\neq i,l}\Big(\mathbb{1}_{x_k\leq x_l}dF_k(x_k)\Big)\mathbb{1}_{x_i\leq x_l}dF_l(x_l)\\
&= \int_{\mathbb{R}_+^{n-1}} x_l\prod_{j\neq i} dF_j(x_j) - \int_{\mathbb{R}_+} x_l\mathbb{1}_{x_i\leq x_l}\Big\{\int_{\mathbb{R}_+^{n-2}}\prod_{k\neq i,l}\mathbb{1}_{x_k\leq x_l}dF_k(x_k)\Big\}dF_l(x_l)\\
&= \int_{\mathbb{R}_+} x_l dF_l(x_l) - \int_{\mathbb{R}_+} x_l\mathbb{1}_{x_i\leq x_l}\prod_{k\neq i,l}F_k(x_l)dF_l(x_l)\\
&= \int_{\mathbb{R}_+} x_l(1-\mathbb{1}_{x_i\leq x_l}\prod_{k\neq i,l}F_k(x_l))dF_l(x_l)
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{E}U_i(x_i,X_{-i}) = &\int_0^{x_i}(v_i-(1-\alpha_i)x)dG_i(x) - (1-\alpha_i)x_i(1-G_i(x_i))\\
&+\alpha_i\sum_{l\neq i}\int_{\mathbb{R}_+} x_l(1-\mathbb{1}_{x_i\leq x_l}\prod_{k\neq i,l}F_k(x_l))dF_l(x_l)\\
&+\alpha_i\sum_{l\neq i}\Big(\int_{\mathbb{R}_+}\int_{x_i}^{x_l}\sum_{k\neq i,l}x_k\prod_{\substack{m\neq i,k,l\\ k\neq l}}F_m(x_k)dF_k(x_k)dF_l(x_l) + x_i\prod_{m\neq i,l}F_m(x_i)(1-F_l(x_i))\Big)
\end{aligned}
$$

Next, we will note

$$
G_{il}(x) = \prod_{k\neq i,l}F_k(x) \text{ et } G'_{il}(x) = \sum_{j\neq i,l}\prod_{k\neq i,l,j}F_k(x)dF_j(x)
$$

As the expected utility is constant at the equilibrium, the FPO lead to

$$
v_iG'_i(x) - (1-\alpha_i)(1-G_i(x)) + \alpha_i\sum_{l\neq i}G_{il}(x) - \alpha_i\sum_{l\neq i}G_{il}(x)F_l(x) - \alpha_i x\sum_{l\neq i}G'_{il}(x)F_l(x) = 0
$$

Notice that $(n-1)G_i(x) = \sum_{l\neq i}G_{il}(x)F_l(x)$ and $(n-2)G'_i(x) = \sum_{l\neq i}G'_{il}(x)F_l(x)$ henceforth

$$
v_iG'_i(x) - (1-\alpha_i)(1-G_i(x)) + \alpha_i\sum_{l\neq i}G_{il}(x) - \alpha_i(n-1)G_i(x) - \alpha_i x(n-2)G'_i(x) = 0 \ \ \forall i\in\{1,...,n\}
$$

Hence

$$
(v_i - \alpha_i x(n-2))G'_i(x) + (1-\alpha_i n)G_i(x) = (1-\alpha_i) - \alpha_i\sum_{l\neq i}G_{il}(x) \ \ \forall i\in\{1,...,n\} \qquad \text{(A1)}
$$

This result is true for all $n>3$. The closed characterization of the solution is very difficult. Yet, we can deduce the solution by an alternative way. Indeed, let F_i and F_j be the mixed strategies of the two bidders i and j. We can notice that the derivative of the expected utility of a third bidder k $H_k(x) = \frac{\partial \mathbb{E}U_k}{\partial x}(x_i, X_1, X_2)$ is a monotonous increasing function. Furthermore, $H_k(0) = -(1-\alpha_k)$ and $\lim_{x\to+\infty}H_k(x) = 0$. Thus, given the mixed strategies of i and j, k do not participate.

This result can easily be extended to a number n of bidders. For that, we should use recurrence.

■

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