Decision making with doubt regarding the consequences of an action *

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Abstract

Anscombe & Aumann (1963) improved the model developed by von Neumann & Morgenstern (1944) suggesting that the outcome of an act can be a lottery. They showed that if the preference relation of a decision maker obeys several axioms then the latter behaves as if they were maximizing some expected utility. We slightly depart from their definition of acts and consider that, in a given state of Nature, the result of an act is a possibility distribution over outcomes rather than a probability distribution. We then extend the work by **?** to a more general setting. One can consider our contribution as a refinement of a model developed by Ghirardato (2001): the latter models the consequence of an act as a list of possible outcomes. We add to the list of possible outcomes some structure, namely a qualitative structure. We show that if the preference relation a decision maker may have obeys several restrictions, their choices ensue from the maximization of a – qualitative – expected utility.

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1 Introduction

Many a decision is not trivial and requires a deep analysis of its aspects from the way it can be implemented to the effects it may have. If, for example, we consider investments decisions, Dixit & Pindyck (1994, p.3) distinguish three important characteristics, the second of them being (emphasis original) the

> "*uncertainty* over the future reward from the investment. The best you can do is to assess the probabilities of the alternative outcomes that can mean greater or smaller profit (or loss) for your venture."

The specification of the various rewards of the investment is by itself a difficult task. We will not tackle the subject and will take the set of rewards as granted. The assessment of the various probabilities is a second step in the analysis of the decision. In some situations, such as the evaluation of the final outcome of scientific search[1], it can be very difficult to build a probability distribution over the various outcomes mentioned earlier. In such a setting, that requires a tool different from probability distribution, we suggest to use qualitative scales. Before introducing the possibility theory, firstly introduced by Zadeh (1978), we put forward a brief review of the settings frequently referred to in decision theory.

Economists, at least since Knight (1921), have been used to distinguish risk from uncertainty. In a nutshell, a risky situation is such that the randomness inhering in the problem can be reduced to a probability distribution whereas in an uncertain setting, this randomness cannot be resumed *via* a probability distribution. Such a distinction was reformulated later by Keynes (1937)

> "By 'uncertain' knowledge, let me explain, I do not mean merely to distinguish what is known for certain from what is only probable. The game of roulette is not subject, in this sense, to uncertainty; nor is the prospect of a Victory bond being drawn. Or, against, the expectation of life is only slightly uncertain. Even the weather is only moderate uncertain. The sense in which I am using the term is that in which the prospect of a European war is uncertain, or the price of copper and the rate of interest twenty years hence, or the obsolescence of a new invention, or the position of private wealth-owners in the social system in 1970. About these matters there is no scientific basis on which to form any calculable probability whatever. We simply do not know."

The study of decision making under risk was formally carried out by von Neumann & Morgenstern (1944). They provide an axiomatic setting that allow to rationalize the intuition formulated by Daniel Bernoulli to answer the St. Petersburg Paradox. A few years later, Savage (1972) extended this work and

[1] Would anyone have guessed that the so-called *Entscheidungsproblem* raised by David Hilbert in 1928 would lead Alan Turing to make that decisive move toward the creation of computers?

fulfilled the problem of decision making under uncertainty. Since these two seminal works, various contributions have lead decision making theorists to develop a wide range of models that allow them either to advice people when the latter face a choice or to explain why they picked out a given alternative. The so-called expected utility approach is dominantly used by economists when they study decision making. It specifies that the decision maker (DM), when they want to value an action, uses some probability distribution over the contingencies they are dealing with and a utility function that converts the various outcomes into monetary payoffs; the evaluation of an action is nothing but the expected value of the utility of its outcomes with respect to that probability distribution. In order to deal with some inconsistencies pointed out, among others, by Allais (1953), Ellsberg (1961) and Mossin (1968), several contributions have been later developed. Schmeidler (1989) shows that a DM whose preferences conform to the axioms he considers evaluates their actions in a expected-utility fashion, using a capacity instead of a probability distribution. Gilboa & Schmeidler (1989) consider a situation such that the DM have in mind a list of possible scenarios; each of them corresponds to a probability distribution on the set of the states of the world. Thus, facing a random variable, the DM can compute as many expected values as scenarios. The authors provide a rationale for them to evaluate the random variable as the smallest of its expected values. Ghirardato (2001) extends the framework studied by Savage (1972) to situations where the DM does not know precisely the outcomes of their actions in a given state of the world and is merely able to make out a list of potential outcomes. They know the outcome will lay somewhere in the list but cannot specify which one of those will eventually occur. The author shows that the axioms studied by Savage (1972) plus two additional ones that are specific to his setting lead the DM to evaluate their actions again through a Choquet integral. Lastly, Jaffray & Jeleva (2004) study situations where the implications of a given action are completely known and understood – *analyzed* in their terminology – on an particular event and more vague and imprecise on the complement of that event. The authors show that if the DM's preferences obey some rules then the valuation attributed to a given action should only depend on the analyzed event itself, the expected value of the utility of the outcomes provided by the action on the analyzed event and the worst and the best outcomes of the action on the non analyzed event.

Most of these contributions consist in a relaxing of the properties imposed to the preference relation among acts the DM is endowed with. Besides, all of them consider actions – acts in Savage (1972)'s terminology – as mappings from $\mathcal{S}$, the set of states of the world, to some set $\mathcal{X}$ that can be the set of outcomes $\mathcal{C}$ itself or some other set derived from the latter such as the power set of $\mathcal{C}$ or the set of all simple probability distributions on set of $\mathcal{C}$. We address two comments to such a formalism. Firstly, it is possible that the DM understands the course of action they have to implement to realize a given action yet they cannot specify the outcome that will result from their action. For example, the DM may understands what they have to do if they want to invest some of their money in company A. A much more difficult task for them is to know what will be the precise value of their portfolio fourteen months from now. Secondly, it is possible that the DM thinks that the list of states of the world they have in mind is somewhat coarse. However, for lack of time, of ressource, of intellectual abilities, they cannot refine these contingencies. In the previous example, the DM may have in mind a few economic indicators that can determine the value of their portfolio yet will it be enough for them to make accurate anticipations? These two remarks may prevent one from modeling the actions as acts *à la* Savage (1972). Indeed, this requires to assign to any action in any state of the world a unique outcome. Furthermore, attributing a list of outcomes as suggested by Ghirardato (2001) means that none of the outcomes considered as possible plays a particular role, that they are all on an equal footing. However, even if the quality of the information acquired by the DM is low, it can nonetheless helps the latter to sort the elements of the list. The aim of the paper is to take into account those two remarks; it suggests to consider an act as a mapping from $\mathcal{S}$, the set of the states of the world, to Δ, the set of the *possibility distributions* over the set of outcomes $\mathcal{C}$ with the following interpretation : given (1) their knowledge, their understanding of the implications of a given action and (2) the occurrence of a given state of the world, the DM attributes to each outcome a degree of *possibility* that varies from *total impossibility* to *total possibility.* In other words, an act is supposed to induce, in any state of the world $s \in \mathcal{S}$, a ranking over the various outcomes $c \in \mathcal{C}$. In a way, this amounts to give an ordinal structure to Ghirardato (2001)'s lists. Anscombe & Aumann (1963) suggest this idea yet these authors require the outcome of an act, in any state of the world, to be a probability distribution – a *lottery ticket* in their terminology – over $\mathcal{C}$. Dealing with possibility distributions rather than probability distributions is, to our mind, less demanding for it does not require the weight attributed to every outcome to be a real number lying between 0 and 1 nor the sum of those weights to be equal to 1.

The organization of this paper is the following: we first state the central concept of this paper, that is to say possibility theory. Then we introduce some axioms that allow us to derive from preferences over acts a valuation of the induced possibility distributions in an expected-utility fashion. Finally we then show how to aggregate these state-wise evaluations in a consistent way; this constitutes the main result of the paper. For sake of legibility, the paper only contains axioms and propositions. All the proofs are in the appendix.

2 Possibility Theory: A – Short – Introduction

Let $\mathcal{S}$ be a non empty set the elements of which are called *states of Nature.* Basically, a state of Nature describes the state of affairs and depicts, given the pieces of information collected by the decision maker, all the relevant aspects of the problem they are facing. For sake of tractability, the set $\mathcal{S}$ will be supposed finite, that is $\mathcal{S} = \{1, \ldots, s, \ldots, S\}$. Let $\mathcal{C}$ be another non empty set the elements of which are called *outcomes*. Lastly, let Λ be finite set equipped with a binary relation $>_\Lambda$ that is assumed to be a total order. 0_Λ (resp. 1_Λ) denotes the minimal (resp. maximal) element of Λ. To make things easier, we will consider that Λ is some finite subset of $[0; 1]$ containing 0 and 1 and supplied with the usual ordering $>$ on real numbers.

Definition 2.1. *A* possibility distribution on $\mathcal{C}$ *is an application* $\delta : \mathcal{C} \mapsto \Lambda$ *such that there exists some outcome* $c \in \mathcal{C}$ *satisfying* $\delta(c) = 1$.

Any outcome $c \in \mathcal{C}$ satisfying $\delta(c) > 0$ is said possible according to δ ; c is said totally possible according to δ if $\delta(c) = 1$. On the contrary, any element $c \in \mathcal{C}$ satisfying $\delta(c) = 0$ will be said impossible according to δ. Moreover, for any couple of outcomes $(c, c') \in \mathcal{C}^2$, the inequality $\delta(c) \geq \delta(c')$ means that according to δ, outcome c is judged more possible than outcome c'.

Definition 2.2. *Let δ be a possibility distribution. The support of δ, $\underline{\delta}$, is the subset of outcomes that δ considers, to some extent or another, possible, that is*

$$\forall\, c \in \mathcal{C},\; c \in \underline{\delta} \Leftrightarrow \delta(c) > 0$$

The core of δ, $\overset{\circ}{\delta}$, is the subset of outcomes that δ considers totally possible that is

$$\forall\, c \in \mathcal{C},\; c \in \overset{\circ}{\delta} \Leftrightarrow \delta(c) = 1$$

Obviously, $\overset{\circ}{\delta} \subset \underline{\delta}$. Moreover, the former (hence both of them) is never equal to the empty set.

Notation 2.1. *For any outcome $c \in \mathcal{C}$, c^* is a shortening for the possibility distribution* $\tilde{c} \in \mathcal{C} \mapsto \begin{cases} 1 & \textit{if } \tilde{c} = c \\ 0 & \textit{else} \end{cases}$.

Definition 2.3. *We denote by $\mathcal{D}$ the set of all possibility distributions on $\mathcal{C}$ with finite support. Succinctly*

$$\mathcal{D} \triangleq \left\{ \delta : \mathcal{C} \mapsto \Lambda \,\middle|\, \overset{\circ}{\delta} \neq \emptyset,\; \#\left(\underline{\delta}\right) < +\infty \right\}.$$

Though strange it may seem, the condition $1 \in \delta(\mathcal{C})$, that can be rewritten as $\max_{c\in\mathcal{C}} \delta(c) = 1$, is nothing but the qualitative counter-part of the condition that a probability distribution must add up to one, that is if p is a probability distribution on a finite set $\mathcal{A}$ then it must be true that $\sum_{\alpha\in\mathcal{A}} p(\alpha) = 1$.

In a probabilistic framework such as the von Neumann & Morgenstern (1944) setting, one can mix probability distributions. In the same way we can define a mixture of two possibility distributions that we name *possibilistic mixture*.

Definition 2.4. *[Possibilistic mixture of distributions] Let $(\delta,\, \delta') \in \mathcal{D}^2$ be a couple of possibility distributions and $(\lambda,\; \lambda') \in \Lambda^2$ a couple of scalars such that* $\max(\lambda,\; \lambda') = 1$. *The application $(\lambda|\delta;\; \lambda'|\delta')$ is defined by*

$$\forall\, c \in \mathcal{C},\; (\lambda|\delta;\; \lambda'|\delta')(c) = \max\left[\min(\lambda, \delta(c)),\; \min(\lambda',\; \delta'(c))\right].$$

This definition allows us to define the application $(1|\delta_1;\, \lambda_2|\delta_2;\, \ldots;\, \lambda_n|\delta_n)$ by iteration: if $N \in \mathbb{N}^*$ is a nonnegative integer, $(\delta_n)_{n=1\ldots N} \in \mathcal{D}^N$ are N possibility distributions and $(\lambda_n)_{n=2\ldots N} \in \Lambda^{N-1}$ $N-1$ scalars such that $\lambda_2 \geq \ldots \geq \lambda_N$ then

$$(1|\delta_1;\, \lambda_2|\delta_2; \ldots;\, \lambda_N|\delta_N) = (1|\delta_1;\, \lambda_2|\,(1|\delta_2;\, \lambda_3|\delta_3;\, \ldots;\, \lambda_N|\delta_N))\,.$$

We will sometimes need to precise the elements that characterize a given possibility distribution, that is the subset $\underline{\delta}$ and the various $\delta(c)$, for $c \in \underline{\delta}$. For that reason, we adopt the following conventions the aim of which is simply to ensure that such a decomposition is unique and coherent with definition 2.4.

Notation 2.2. *Any possibility distribution $\delta \in \mathcal{D}$ will sometimes be written $(\lambda_1|c_1;\, \lambda_2|c_2;\, \ldots; \lambda_p|c_p)$ with the following conventions :*

- *for any $(i,\, j) \in [|1;p|]^2$, $c_i = c_j$ if and only if $i = j$;*
- *$\underline{\delta} = \{c_1,\, c_2, \ldots,\, c_p\}$;*
- *$\lambda_1 = 1$;*
- *for all $i =\in [|1;p-1|]$, either $\lambda_i > \lambda_{i+i}$ or $\lambda_i = \lambda_{i+i}$ and $c_{i+i} \succ_{\mathcal{C}} c_i$.*

Remark 2.1. *if we write $(\lambda_1|{c_1}^*;\, \lambda_2|{c_2}^*;\, \ldots; \lambda_p|{c_p}^*)$ we allow the case where the λ's are not sorted in a decreasing fashion. In other words, there exists a permutation $\sigma\ : [1;\, p] \to [1;\, p]$ such that $(\lambda_1|{c_1}^*;\, \lambda_2|{c_2}^*;\, \ldots; \lambda_p|{c_p}^*)$ is equal to $\left(\lambda_{\sigma(1)}|c_{\sigma(1)};\, \lambda_{\sigma(2)}|c_{\sigma(2)};\, \ldots; \lambda_{\sigma(p)}|c_{\sigma(p)}\right)$ where $\lambda_{\sigma(1)} = 1$ and $\lambda_{\sigma(i)} \geq \lambda_{\sigma(i+1)}$ for all $i \in [|1;p-1|]$. In due time, such a writing will turn out to be convenient.*

3 The Model

As an introduction for this section, we shall provide the reader with a definition that will be used throughout the paper.

Definition 3.1. *Let Γ be a non-empty set supplied with a binary relation $\succeq_\Gamma$. Let $(\Phi, \geq_\Phi)$ be a non-empty bounded, linearly ordered space. The mapping $\Psi : \Gamma \to \Phi$ represents $\succeq_\Gamma$ with respect to $(\Phi, \geq_\Phi)$ if, and only if, for all couple $(\gamma,\, \gamma') \in \Gamma^2$, $\Psi(\gamma) \geq_\Phi \Psi(\gamma')$ if and only if $\gamma \succeq_\Gamma \gamma'$.*

A binary relation $\succeq_\Gamma$ generally admits several representations with respect to some couple $(\Phi, \geq_\Phi)$. Indeed, if $\Psi : \Gamma \to \Phi$ represents $\succeq_\Gamma$ with respect to $(\Phi, \geq_\Phi)$ then $\Psi' = \zeta \circ \Psi$, where ζ is an increasing mapping from Φ to Φ, also represents $\succeq_\Gamma$ with respect to $(\Phi, \geq_\Phi)$.

As put in the introduction, we slightly depart from the traditional definition of acts. We formally express our conception of acts in the following paragraph.

Definition 3.2. *An* act *is an application $f : \mathcal{S} \mapsto \mathcal{D}$. $\mathcal{F} \triangleq \mathcal{D}^{\mathcal{S}}$ denotes the set of all acts.*

Thus, an act will generate, in any state of Nature s, a possibility distribution on $\mathcal{C}$ which means that one outcome – at least – will be judged totally possible and we allow the case that another or more will also be considered as being possible, perhaps to a smaller extent. It is worth noting the main difference from a setting *à la* Anscombe & Aumann (1963). The two authors allow the outcome of a given act in a given space to be a probability distribution over some given set. Thus the set of all acts $\mathcal{F}$ is in their framework a convex subset of a linear space. Here the set of acts does not enjoy such a property thus many a mathematical theorem will fail to be valid in our setting.

Notation 3.1. *For any act $f \in \mathcal{F}$ and any state of Nature $s \in \mathcal{S}$, we slightly abuse notations and write f_s instead of $f(s)$. Moreover, the act f will sometimes be noted $[f_1, \ldots, f_s, \ldots, f_S]$.*
Lastly, if $\delta \in \mathcal{D}$ is a possibility distribution, $[f_{-s}, \delta]$ is a shortening for the act $[f_1, \ldots, f_{s-1}, \delta, f_{s+1}, \ldots, f_S]$

In a framework *a la* Savage (1972) the set of outcomes judged totally possible in any state of Nature $\mathcal{S}$ is reduced to a singleton and all the outcomes outside this singleton are judged impossible. In a framework *a la* Ghirardato (2001) the set of outcomes judged totally possible in any state of Nature $\mathcal{S}$ is some subset C of $\mathcal{C}$ and all the outcomes outside the subset C are judged impossible. If we introduce $\mathcal{F}^{Sav}$ (resp. $\mathcal{F}^{Ghir}$) the set of all acts *a la* Savage (resp. *a la* Ghirardato) the following are true

$$\mathcal{F}^{Sav} = \left\{ f \in \mathcal{F} / \forall s \in \mathcal{S}, \exists!\, c_s \in \mathcal{C}, f_s = {c_s}^* \right\},$$

$$\mathcal{F}^{Ghir} = \left\{ f \in \mathcal{F} / \forall s \in \mathcal{S}, \exists!\, C_s \subset \mathcal{C}, f_s = \max_{c \in C_s} c^* \right\}$$

and

$$\mathcal{F}^{Sav} \subsetneq \mathcal{F}^{Ghir} \subsetneq \mathcal{F}.$$

Contrary to Savage, we allow for situations in which the decision maker is not able to assess the precise outcome of their action in a given state of Nature. This idea was introduced by Ghirardato yet this author does not suggest any rating of the various outcomes considered as possible if a given state of Nature occurs. We explicitly introduce some qualitative orderings of these outcomes.

Given an act f and a state of Nature s, we say that the decision maker perfectly knows the outcome of act f in state s if and only if there exists a unique outcome $c_0 \in \mathcal{C}$ such that $f_s = {c_0}^*$. We say that the decision maker is completely ignorant of the outcome of act f in state s if and only if, for any outcome $c \in \mathcal{C}$, $f_s(c) = 1$.

The set $\mathcal{F}$ is endowed with a binary relation $\succeq_{\mathcal{F}}$ with the classical interpretation : for any couple of acts $(f, f') \in \mathcal{F}^2$, $f \succeq_{\mathcal{F}} f'$ means that act f is at least as good as act f'. We can derive from $\succeq_{\mathcal{F}}$ two binary relations $\succ_{\mathcal{F}}$ and $\sim_{\mathcal{F}}$ in the following way : f is strictly better than f', a situation denoted by $f \succ_{\mathcal{F}} f'$, if and only if it is true that $f \succeq_{\mathcal{F}} f'$ but it is false that $f' \succeq_{\mathcal{F}} f$. On the other hand, f is as good as f', a situation denoted by $f \sim_{\mathcal{F}} f'$, if and only if it simultaneously holds that $f \succeq_{\mathcal{F}} f'$ and $f' \succeq_{\mathcal{F}} f$.

3.1 Evaluation of a Possibility Distribution

The binary relation $\succeq_{\mathcal{F}}$ will be assumed to satisfy several axioms. The first of them makes it a preference relation, at least from an economic point of view.

Axiom 3.1 (Weak order). *$\succeq_{\mathcal{F}}$ is a transitive and total.*

We respectively denote by $\succ_{\mathcal{F}}$ and $\sim_{\mathcal{F}}$ the asymmetric and the symmetric part of $\succeq_{\mathcal{F}}$. Moreover we can derive from $\succeq_{\mathcal{F}}$ two other binary relations: $\succeq_{\mathcal{D}}$ defined on $\mathcal{D}$ and $\succeq_{\mathcal{C}}$ defined on $\mathcal{C}$.

Definition 3.3. *Let $(\delta,\, \delta') \in \mathcal{D}^2$ be a couple of possibility distributions on $\mathcal{C}$ and f and f' the two acts such that, for all $s \in \mathcal{S}$, $f_s = \delta$ and ${f_s}' = \delta'$. Thus*

$$\delta \succeq_{\mathcal{D}} \delta' \Leftrightarrow f \succeq_{\mathcal{F}} f'$$

Definition 3.4. *Let $(c,\, c') \in \mathcal{C}^2$ be a couple of outcomes. Then*

$$c \succeq_{\mathcal{C}} c' \Leftrightarrow c^* \succeq_{\mathcal{D}} c'^*$$

Consider a couple of possibility distributions $(\delta,\ \delta') \in \mathcal{D}^2$. You can create a third possibility distribution that associates to any outcome $c \in \mathcal{C}$ the maximum between the possibility degree of c according to δ and the possibility degree of c according to δ'. In general, this possibility distribution will be different from the most preferred possibility distribution (according to $\succeq_{\mathcal{D}}$). Besides, these last two possibility distributions will not usually be equivalent. To avoid any possible confusion, we use the following convention.

Notation 3.2. *Let $(\delta,\, \delta') \in \mathcal{D}^2$ be two possibility distributions. We denote $\delta \vee \delta'$ the highest possibility distribution according to $\succeq_{\mathcal{D}}$; if δ and δ' are equivalent, $\delta \vee \delta'$ is any of the two. Moreover, $\max(\delta, \delta')$ is the possibility distribution $(1|\delta;\, 1|\delta')$ that is to say the possibility distribution that*

Obviously, we can extend this definition to families of possibility distributions the size of which is greater or equal to two in the following way.

Notation 3.3. *Let $N \in \mathbb{N}^*$ be a nonnegative integer. Let $(\delta_n)_{n=1\,\ldots\,N} \in \mathcal{D}^N$ be a family of possibility distributions. $\bigvee\limits_{n=1\,\ldots\,N} \delta_n$ is any of the possibility distributions $\tilde{\delta} \in \{\delta_n,\, n = 1\,\ldots\,N\}$ such that, for all $n' = 1\,\ldots\,N$, $\tilde{\delta} \succeq_{\mathcal{D}} \delta_{n'}$. $\max\limits_{n=1\,\ldots\,N} \delta_n$ is a shorthand for the possibility distribution $(1|\delta_1;\, 1|\delta_2;\, \ldots;\, 1|\delta_n)$.*

The following axiom will rule out degeneracy of $\mathcal{D}$. It also proves useful if $\mathcal{C}$ is infinite as it implies that the former set is bounded with respect to $\succeq_{\mathcal{C}}$.

Axiom 3.2 (Boundedness and non degeneracy of $\mathcal{C}$)**.** *There exist two outcomes* $\underline{c}$ *and* $\overline{c}$ *such that, for any outcome* $c \in \mathcal{C}$*, the following holds*

$$\overline{c} \succeq_{\mathcal{C}} c \succeq_{\mathcal{C}} \underline{c}.$$

with at least one strict inequality.

We now introduce an axiom that can be interpreted in the following way: consider a possibility distribution δ and an outcome c. From δ, build the possibility distribution δ' by substituting c to an outcome c' arbitrarily chosen in $\underline{\delta}$. The axiom requires that your preference between δ and δ' should only be driven by your preference between c and c'.

Axiom 3.3 ($\mathcal{C}$-Independence)**.** *For any couple* $(c, c') \in \mathcal{C}^2$*, for any* $\delta \in \mathcal{D}$*, for any couple* $(\lambda, \lambda') \in \Lambda^2$ *with* $\max(\lambda, \lambda') = 1$*,*

$$c \succeq_{\mathcal{C}} c' \Rightarrow (\lambda|\delta;\ \lambda'|c^*) \succeq_{\mathcal{D}} (\lambda|\delta;\ \lambda'|c'^*).$$

The following axioms deals with the richness of the scale Λ. Roughly speaking, it means that the outcome $c \in \mathcal{C}$ can be mapped to the set Λ, that the sets $\mathcal{C}$ and Λ are commensurable.

Axiom 3.4 (Richness of Λ)**.** *For any outcome* $c \in \mathcal{C}$ *there exists* $\lambda \in \Lambda$ *such that*

$$c^* \sim_{\mathcal{D}} (1|\underline{c}^*; \lambda|\overline{c}^*).$$

In order to introduce the last axiom of this section, we need to give a convenient label to the elements of a specific family of distributions.

Notation 3.4. *The possibility distribution* $(1|\underline{c}^*; \lambda|\overline{c}^*)$ *is denoted* δ^λ*.*

We now impose the following ordering on the family $\left(\delta^\lambda\right)_{\lambda \in \Lambda}$.

Axiom 3.5 (The more possible $\overline{c}$ the better)**.** *For any* $(\lambda, \lambda') \in \Lambda^2$

$$\lambda > \lambda' \Rightarrow \delta^\lambda \succ_{\mathcal{D}} \delta^{\lambda'}.$$

This axiom can receive – at least – two different interpretations. As far as δ^λ is concerned, the worst outcome $\underline{c}$ is totally possible and the best outcome is all the more possible than λ is close to 1. Thus an increase in λ amounts to a greater uncertainty : the less λ, the more the outcome is bound to be $\underline{c}$. On the other hand, the greater λ is, the more possible the best outcome is. This axiom means that this elation effect compensate, and even overcome the increase in uncertainty.

Before stating the first proposition, we should insist on a particular consequence of axioms 3.4 and 3.5 that may not be conspicuous. The two axioms imply that the agent is optimistic: even if the worst outcome is judged totally possible, any increase in the possibility degree of the best outcome is always positively judged. Instead of those axioms, we could have postulate the following.

Axiom 3.3′ *For any outcome* $c \in \mathcal{C}$ *there exists* $\lambda \in \Lambda$ *such that*

$$c^* \sim_{\mathcal{D}} (\lambda|\underline{c}^*; 1|\overline{c}^*).$$

Axiom 3.4′ (The less possible $\underline{c}$ the better). *For any* $(\lambda, \lambda') \in \Lambda^2$

$$\lambda > \lambda' \Rightarrow \delta^{\lambda'} \succ_{\mathcal{D}} \delta^{\lambda}.$$

In that case, the agent is pessimistic: although the best outcome is totally possible, the agents focus on the worst one and negatively values any increase of the possibility degree of the latter. We show in the appendix how the results are modified if we substitute axioms 3.3′ and 3.4′ to axioms 3.4 and 3.5.

We can now state our first proposition.

Proposition 3.1 (Dubois et al. (1998)). *If* $\succ$ *satisfies axioms 3.1, 3.2, 3.3, 3.4 and 3.5 then there exist*

1. *a linearly ordered utility scale* $(\Lambda', \geq_{\Lambda'})$ *with* $\inf(\Lambda') = 0_{\Lambda'}$ *and* $\sup(\Lambda') = 1_{\Lambda'}$
2. *a mapping* $u : \mathcal{C} \to \Lambda'$ *such that* $u(\underline{c}) = 0_{\Lambda'}$ *and* $u(\overline{c}) = 1_{\Lambda'}$ *;*
3. *an onto order preserving function* $g : \Lambda \to \Lambda'$ *such that* $g(0) = 0_{\Lambda'}$ *and* $g(1) = 1_{\Lambda'}$

such that, with

$$\mathcal{U}(\delta) = \max_{c \in \mathcal{C}} \min[g \circ \delta(c), u(c)]$$

the mapping $\mathcal{U}$ *represents* $\succeq_{\mathcal{D}}$ *with respect to* $(\Lambda', \geq_{\Lambda'})$.

Remark 3.1. *If we formally substitute* $\oplus$ *to* max *and* $\otimes$ *to* min *then the proposition can be rewritten as*

$$\delta \succeq_{\mathcal{D}} \delta' \Leftrightarrow \bigoplus_{c \in \mathcal{C}} g \circ \delta(c) \otimes u(c) \geq_{\Lambda'} \bigoplus_{c \in \mathcal{C}} g \circ \delta'(c) \otimes u(c).$$

Thus the qualitative evaluation of a possibility distribution is very close to the quantitative evaluation of a probability distribution commonly used in decision theory, e.g. in a setting à la von Neumann & Morgenstern (1944).

3.2 Evaluation of an Act

Through the function $\mathcal{U}$, the previous proposition suggests a way to evaluate a possibility distribution. In this section, we show a representation theorem valid not only for possibility distributions but also for acts.

The first axiom is very close to assumption 1 in Anscombe & Aumann (1963). Its meaning is that if two acts only differs in one state, then the preference between the two only depends on the preference of the possibility distributions that differentiate the two acts.

Axiom 3.6 (Separability)**.** *For all* $f \in \mathcal{F}$*, for all* $s = 1 \ldots S$*, for all* $(\delta,\, \delta') \in \mathcal{D}^2$

$$\delta \succeq_{\mathcal{D}} \delta' \Rightarrow [f_{-s},\, \delta] \succeq_{\mathcal{F}} [f_{-s},\, \delta'].$$

Among other things, axioms 3.2, 3.3 and 3.6 imply that the set $\mathcal{F}$ is bounded with respect to $\succeq_{\mathcal{F}}$.

Notation 3.5. *Let* $s \in \mathcal{S}$ *be a state and* $\lambda \in \Lambda$ *be a scalar. The act* $\underline{f}^s(\lambda)$ *is defined as*

$$\underline{f}^s(\lambda) = [\underline{f}_{-s},\, \delta^\lambda] = [\underline{c}^*,\, \ldots,\, \underset{\underset{s-1}{\uparrow}}{\underline{c}^*},\, \underset{\underset{s}{\uparrow}}{\delta^\lambda},\, \underset{\underset{s+1}{\uparrow}}{\underline{c}^*},\, \ldots,\, \underline{c}^*].$$

Definition 3.5 (Possibilistic mixture of acts)**.** *Let* $N \in \mathbb{N}^*$ *be a nonnegative integer. Let* $(f_1,\, \ldots,\, f_N) \in \mathcal{F}^N$ *be* N *acts and* $(\lambda_2,\, \ldots,\, \lambda_N) \in \Lambda^{N-1}$ *be* $N-1$ *scalars such that* $\lambda_2 \geq \ldots \geq \lambda_N$*. We define the act* $f' \triangleq (1|f_1;\, \lambda_2|f_2;\, \ldots;\, \lambda_N|f_N)$ *in the following fashion*

$$f' = (1|f_1;\, \lambda_2|f_2;\, \ldots;\, \lambda_N|f_N) \Leftrightarrow \forall\, s \in \mathcal{S},\, {f_s}' = (1|f_{1s};\; \lambda_2|f_{2s};\, \ldots;\, \lambda_N|f_{Ns}).$$

The next axiom states that if two acts are considered as equivalent, then if each of them is possibilistically mixed with a third act then the two mixed acts remain equivalent.

Axiom 3.7 ($\mathcal{F}$-Independence)**.** *For all* $(f,\, f',\, f'') \in \mathcal{F}^3$*, for all* $(\lambda,\, \lambda') \in \Lambda^2$ *such that* $\max(\lambda,\, \lambda') = 1$

$$f \sim_{\mathcal{F}} f' \Rightarrow (\lambda|f;\lambda'|f'') \sim_{\mathcal{F}} (\lambda|f';\lambda'|f'').$$

There is an obvious connection between axiom 3.3 and axiom 3.7 as established by the following lemma.

Lemma 3.1. *Axioms 3.1, 3.6 and 3.7 imply axiom 3.3.*

Proof: see subsection 5.1.

To carry on our presenting of axioms, we need to introduce a particular class of constant acts.

Notation 3.6. *Let* $\lambda \in \Lambda$ *be a scalar.* f^λ *is the (constant) act such that, for all state* $s \in \mathcal{S}$*,* ${f^\lambda}_s = \delta^\lambda$*.*

The last axiom means that the sets $\mathcal{S}$ and Λ are commensurable, that there are enough scalars to evaluate the various acts.

Axiom 3.8 ($\mathcal{S}$-Commensurability)**.** *Let* $s \in \mathcal{S}$ *be a state. There exists a scalar* $\tilde{\lambda}_s \in \Lambda$ *such that* $\underline{f}^s(1) \sim_{\mathcal{F}} f^{\tilde{\lambda}_s}$*.*

Proposition 3.2. *If axioms 3.1, 3.2, 3.4, 3.5, 3.6, 3.7 and 3.8 are satisfied then there exist*

1. *a linearly ordered utility scale* $(\Lambda'', \geq_{\Lambda''})$ *with* $\inf(\Lambda'') = 0_{\Lambda''}$ *and* $\sup(\Lambda') = 1_{\Lambda''}$

2. *a mapping* $V\ : \mathcal{D} \to \Lambda''$ *which represents* $\succeq_{\mathcal{D}}$ *on* $(\Lambda'', \geq_{\Lambda''})$ *;*

3. *a mapping* $L\ : \mathcal{S} \to \Lambda''$ *such that* $L^{-1}(1_{\Lambda''}) \neq \emptyset$

such that, with

$$\mathcal{V}(f) = \max_{s\in\mathcal{S}} \min[L(s),\, V(f_s)]$$

the mapping $\mathcal{V}$ *represents* $\succeq_{\mathcal{F}}$ *with respect to* $(\Lambda'', \geq_{\Lambda''})$.

Proof: see page 15.

L can be seen as a possibility distribution on the various states of the world. The different values this application take are completely determined by $\succeq_{\mathcal{F}}$. For that reason, it reflects the DM's preferences.

One of the consequences of propositions 3.1 and 3.2 is that an act is evaluated in a Savagean fashion as suggested by the following proposition.

Proposition 3.3. *If* $\succ$ *satisfies axioms 3.1 3.2 and 3.4 to 3.8 are satisfied then there exist*

1. *a linearly ordered utility scale* $(\widetilde{\Lambda}, \geq_{\widetilde{\Lambda}})$ *with* $\inf(\widetilde{\Lambda}) = 0_{\widetilde{\Lambda}}$ *and* $\sup(\widetilde{\Lambda}) = 1_{\widetilde{\Lambda}}$*;*

2. *a mapping* $v\ : \mathcal{C} \to \widetilde{\Lambda}$ *which represents* $\succeq_{\mathcal{C}}$ *on* $(\widetilde{\Lambda}, \geq_{\widetilde{\Lambda}})$ *;*

3. *a mapping* $\mathcal{L}\ : \mathcal{F} \to \widetilde{\Lambda}^{\mathcal{C}}$ *;*

such that, with

$$\mathcal{V}'\ : f \in \mathcal{F} \mapsto \max_{c\in\mathcal{C}} \min[\mathcal{L}_f(c),\, v(c)] \in \widetilde{\Lambda},$$

the mapping $\mathcal{V}'$ *represents* $\succeq_{\mathcal{F}}$ *with respect to* $(\widetilde{\Lambda}, \geq_{\widetilde{\Lambda}})$*. Moreover,* $\mathcal{L}_f$ *can be taken of the form* $\mathcal{L}_f(c) = \max_{s\in\mathcal{S}} \min[L(s),\, \kappa \circ f_s(c)]$*, for all* $c \in \mathcal{C}$*, where*

1. $L\ : \mathcal{S} \to \widetilde{\Lambda}$ *is a mapping such that* $L^{-1}(1_{\widetilde{\Lambda}}) \neq \emptyset$*;*

2. $\kappa\ : \Lambda \to \widetilde{\Lambda}$ *is an onto order preserving function such that* $\kappa(0) = 0_{\widetilde{\Lambda}}$ *and* $\kappa(1) = 1_{\widetilde{\Lambda}}$.

One possible interpretation of theorem 3.3 is the following : when the DM has to value act f, they begin evaluating the possibility, the plausibility of the outcomes the act may lead to. This provides them with the possibility distribution $\mathcal{L}_f$. The value they attribute to the act is the Sugeno integral of u, the utility function over $\mathcal{C}$, with respect to the possibility distribution $\mathcal{L}_f$.

4 Conclusion

In this paper we developed a model in which we introduced a restrictive notion of doubt, precisely questioning regarding the consequences of the action one can implement. Defining suitably acts and preferences among them, we showed that if the preference relation abide to several axioms, the choices can be represented by a maximization of a qualitative expected utility. We consider axiom 3.4 as an important limitation to our work. Our model is fit for choices where there are approximately as many consequences as scalars to ponder them. However axiom 3.4 rules out for example any situation in which there is continuum of consequences – more precisely an infinite number of equivalence classes in $\mathcal{C}$ – while the number of scalars is finite. Such situations of choice, although not systematic, are common in real life and cannot be treated with our model.

We consider our model as an attempt to relax one strong assumption imposed by Savage (1972), the univocal mapping from the set of the states of the world to the state of the consequences induced by an act. The fuzziness we introduce appears to be, at least to us, widespread and natural whenever one has to choose. Checking in the lab if such a rough model can explain the choices made is an issue that should be tackled in future work. We let the question open for the time being.

5 Appendix : Proofs and in-between lemmas

5.1 Results and proofs of subsection 2

Lemma 5.1. *For all possibility distributions* $(\delta,\, \delta') \in \mathcal{D}^2$*, for all scalar* $\lambda \in \Lambda$*, the application* $\delta'' := (1|\delta;\, \lambda|\delta')$ *is a possibility distribution. Moreover* $\underline{\delta''} = \underline{\delta} \cup \underline{\delta'}$*. Besides if* $\lambda = 1$ *then* $\overset{\circ}{\delta''} = \overset{\circ}{\delta} \cup \overset{\circ}{\delta'}$ *else* $\overset{\circ}{\delta''} = \overset{\circ}{\delta}$*.*

5.2 Results and proofs of subsection 3.1

Lemma 5.2. $\succeq_{\mathcal{D}}$ *is a complete weak order on* $\mathcal{D}$*.*
$\succeq_{\mathcal{C}}$ *is a complete weak order on* $\mathcal{C}$*.*

The proof is immediate and left to the reader. $\succeq_{\mathcal{D}}$ and $\succeq_{\mathcal{C}}$ are thus two preference relations.

Lemma 5.3 ($\mathcal{D}$-Boundedness)**.** *If axioms 3.2 and 3.3 are satisfied then, for any triplet* $\delta \in \mathcal{D}$*,*

$$\overline{c}^* \succeq_{\mathcal{D}} \delta \succeq_{\mathcal{D}} \underline{c}^*.$$

Proof Let δ be a possibility distribution. There exist an nonnegative integer $k \in \mathbb{N}^*$, k outcomes $(c_1,\, c_2,\, \ldots,\, c_k) \in \mathcal{C}^k$ and $k-1$ scalars $(\lambda_2,\, \ldots,\, \lambda_k) \in \Lambda^{k-1}$ such that $\delta = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_k|c_k)$. For all k axiom 3.2 implies $\overline{c} \succeq_{\mathcal{C}} c_k$ thus $(1|\overline{c}^*;\, \lambda_2|\overline{c}^*;\, \ldots;\, \lambda_k|\overline{c}^*) \succeq_{\mathcal{D}} \delta$ which is tantamount to $\overline{c}^* \succeq_{\mathcal{D}} \delta$. □
An analogous proof shows that $\delta \succeq_{\mathcal{D}} \underline{c}^*$. ■

Lemma 5.4 ($\mathcal{D}$-Independence)**.** *If axiom 3.3 is satisfied then, for any triplet* $(\delta,\, \delta',\, \delta'') \in \mathcal{D}^3$*, for any couple* $(\lambda,\, \lambda') \in \Lambda^2$ *with* $\max(\lambda,\, \lambda') = 1$*,*

$$\delta \succeq_{\mathcal{D}} \delta' \Rightarrow (\lambda|\delta;\, \lambda'|\delta'') \succeq_{\mathcal{D}} (\lambda|\delta';\, \lambda'|\delta'').$$

Proof We will show the result by induction on the size of the support of the distribution δ''.
If $\#\left(\underline{\delta}\right) = 1$ then there exists some $c \in \mathcal{C}$ such that $\delta'' = c^*$. The result is thus a direct application of axiom 3.3. □
Let $k \in \mathbb{N}^*$. Assume that, for any possibility distribution δ'' the support of which contains k elements, for any scalar $\lambda \in \Lambda$, $(1|\delta;\, \lambda|\delta'') \succeq_{\mathcal{D}} (1|\delta';\, \lambda|\delta'')$.
Let δ'' be a possibility distribution. Assume that there are $(k+1)$ elements in $\underline{\delta''}$. Thus there exist $(k+1)$ outcomes $(c_i)_{1\leq i\leq k+1}$ and k scalars $(\lambda_i)_{i=2\ldots k+1}$ such that $\delta'' = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_{k+1}|c_{k+1})$. Introduce the possibility distribution $\delta''' = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_k|c_k)$. It is straightforward to check that $\delta''' = (1|\delta'';\, \lambda_{k+1}|{c_{k+1}}^*)$. Let $\lambda \in \Lambda$ be a scalar. Applying our hypothesis we can write that if $\delta \succeq_{\mathcal{D}} \delta'$ then $(1|\delta;\, \lambda|\delta''') \succeq_{\mathcal{D}} (1|\delta';\, \lambda|\delta''')$. Let $\lambda'_{k+1} = \min(\lambda,\, \lambda_{k+1})$. By applying axiom 3.3 we obtain

$$(1|(1|\delta;\, \lambda|\delta''');\lambda'_{k+1}|{c_{k+1}}^*) \succeq_{\mathcal{D}} (1|(1|\delta';\, \lambda|\delta''');\lambda'_{k+1}|{c_{k+1}}^*).$$

It is then easy to check that $(1|(1|\delta;\, \lambda|\delta''');\lambda'_{k+1}|{c_{k+1}}^*) = (1|\delta;\, \lambda|\delta'')$ hence $\delta \succeq_{\mathcal{D}} \delta' \Rightarrow \forall\, \lambda \in \Lambda,\, (1|\delta;\, \lambda|\delta'') \succeq_{\mathcal{D}} (1|\delta';\, \lambda|\delta'')$. ■

Lemma 5.5. *(Reduction of possibility distributions) If axioms 3.3 and 3.4 are satisfied then, for any distribution $\delta \in \mathcal{D}$, there exists a scalar $\lambda \in \Lambda$ such that*

$$\delta \sim_{\mathcal{D}} \delta^{\lambda}$$

Proof Once again, we show the result by induction on the size of the support of the distribution δ.
If $\overset{\circ}{\delta} = 1$ then there exists some $c \in \mathcal{C}$ such that $\delta = c^*$. The result is a direct application of axiom 3.4. □
If $\overset{\circ}{\delta} = 2$ then there exists $(c, c') \in \mathcal{C}^2$ and $\lambda' \in \Lambda$ such that $\delta = (1|c;\, \lambda'|c')$ Thanks to axiom 3.4 we know that there exists $(\mu, \mu') \in \Lambda^2$ such that $(1|c) \sim_{\mathcal{D}} \delta^{\mu}$ and $c'^* \sim_{\mathcal{D}} \delta^{\mu'}$. By applying twice the independence axiom we obtain $(1|c; \lambda'|c') \sim_{\mathcal{D}} (1|\delta^{\mu}; \lambda'|\delta^{\mu'})$. It is easy to show that $(1|\delta^{\mu}; \lambda'|\delta^{\mu'}) = \delta^{\lambda}$ with $\lambda = \max(\mu, \min(\mu', \lambda'))$. □
Let $k \in \mathbb{N}^*$. Assume that, for any possibility distribution δ' the support of which contains k elements, there exists $\lambda' \in \Lambda$ such that $\delta' \sim_{\mathcal{D}} \delta^{\lambda'}$. Let δ be a possibility distribution. Assume that there are $(k+1)$ elements in $\overset{\circ}{\delta}$. Thus there exist $(k+1)$ outcomes $(c_i)_{1 \le i \le k+1}$ and k scalars $(\lambda_{i+1})_{1 \le i \le k}$ such that $\delta = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_{k+1}|c_{k+1})$. Introduce the possibility distribution $\delta' = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_k|c_k)$. It is straightforward to check that $\delta = (1|\delta';\, \lambda_{k+1}|{c_{k+1}}^*)$. We know that there exists $\lambda' \in \Lambda$ such that $\delta' \sim_{\mathcal{D}} \delta^{\lambda'}$. Moreover, axiom 3.4 ensures the existence of some μ_{k+1} such that $\delta^{\mu_{k+1}} \sim_{\mathcal{D}} {c_{k+1}}^*$ Thus, by applying twice lemma 5.4, it turns out that $(1|\delta^{\lambda'};\, \lambda_{k+1}|\delta^{\mu_{k+1}}) \sim_{\mathcal{D}} \delta$ which means that $\delta \sim_{\mathcal{D}} \delta^{\lambda}$, where $\lambda = \max(\lambda', \min(\lambda_{k+1}, \mu_{k+1}))$. ■

Corollary 5.1. *If axioms 3.3, 3.4 and 3.5 are satisfied then $\overline{c}^* \sim_{\mathcal{D}} \delta^1$*

Proof Let $\lambda \in \Lambda$ be such that $\overline{c}^* \sim_{\mathcal{D}} \delta^{\lambda}$. Suppose *ab absurdo* that $1 > \lambda$. Given axiom 3.5, $\delta^1 \succ_{\mathcal{D}} \overline{c}^*$ which is a contradiction to lemma 5.3. Thus $\lambda = 1$. ■

Corollary 5.2. *If axioms 3.3, 3.4 and 3.5 are satisfied then, for any distribution $\delta \in \mathcal{D}$, there exists a unique scalar $\lambda \in \Lambda$ such that $\delta \sim_{\mathcal{D}} \delta^{\lambda}$.*

Proof Axiom 3.5 amounts to

$$\forall\, (\lambda, \lambda') \in \Lambda^2, \delta^{\lambda} \succeq_{\mathcal{D}} \delta^{\lambda'} \Leftrightarrow \lambda \geq \lambda'.$$

Thus $\delta^{\lambda} \sim_{\mathcal{D}} \delta^{\lambda'}$ is equivalent to $\lambda = \lambda'$. Moreover, the simple implication of the axiom turns out to be an equivalence.
This allows us to define the mapping $l\ : \mathcal{D} \to \Lambda$ in the following way.

Definition 5.1. *Let $\delta \in \mathcal{D}$ be a possibility distribution; $l(\delta)$ is the only scalar such that $\delta \sim_{\mathcal{D}} \delta^{l(\delta)}$.*

Thanks to axiom 3.5, l respects $\succeq_{\mathcal{D}}$.

Proof of theorem 3.1 $\sim_{\mathcal{D}}$ is obviously an equivalence relation. It will thus generate a partition of the set $\mathcal{D}$. For any possibility distribution δ, let $\|\delta\|$ be the equivalence class of δ, that is the subset $\{\delta' \,/\, \delta \sim_{\mathcal{D}} \delta'\} \subset \mathcal{D}$ and introduce $\Lambda' = \{\|\delta^\lambda\| \,/\, \lambda \in \Lambda\}$. Axiom 3.5 implies that Λ' is isomorphic to $\mathcal{D}/_{\sim_{\mathcal{D}}}$, the set of all the equivalence classes. If we endow Λ' with the binary relation $\geq_{\Lambda'}$ defined as

$$\forall \left(\|\delta^\lambda\|, \|\delta^{\lambda'}\|\right) \in \Lambda' \times \Lambda', \, \|\delta^\lambda\| \geq_{\Lambda'} \|\delta^{\lambda'}\| \Leftrightarrow \lambda \geq \lambda'$$

then axiom 3.5 also implies that Λ' is bounded and totally ordered. Let $0_{\Lambda'} = \inf \Lambda'$ and $1_{\Lambda'} = \sup \Lambda'$. □

According to lemmas 5.3 and 5.5, $0_{\Lambda'} = \|\delta^0\|$ and $1_{\Lambda'} = \|\delta^1\|$. Consider the application $g : \lambda \in \Lambda \mapsto \|\delta^\lambda\| \in \Lambda'$. g is clearly an increasing onto mapping. Define the mapping U from $\mathcal{D}$ to Λ' as $U = g \circ l$. For any outcome c, define $u(c)$ as $u(c) = U(c^*)$. A direct consequence of this definition is that $u(c) = g(\lambda)$, where λ is such that $c^* \sim_{\mathcal{D}} \delta^\lambda$. Besides $u(\underline{c}) = 0_{\Lambda'}$ and $u(\overline{c}) = 1_{\Lambda'}$. The former is immediate, the latter is implied by corollary 5.1. □

Let $(c_1, c_2) \in \mathcal{C}^2$ be a couple of outcomes and $\lambda \in \Lambda$ be a scalar. Given axiom 3.4 there exist two scalars $(\lambda_1, \lambda_2) \in \Lambda^2$ such that ${c_1}^* \sim_{\mathcal{D}} \delta^{\lambda_1}$ and ${c_2}^* \sim_{\mathcal{D}} \delta^{\lambda_2}$. Applying lemma 5.4 we can write that $(1|{c_1}^*;\, \lambda|{c_2}^*) \sim_{\mathcal{D}} (1|\delta^{\lambda_1};\, \lambda|\delta^{\lambda_2})$ that is $(1|{c_1}^*;\, \lambda|{c_2}^*) \sim_{\mathcal{D}} \delta^{\lambda'}$ where $\lambda' = \max[\lambda_1,\, \min(\lambda,\, \lambda_2)]$. Thus $U\left[(1|{c_1}^*;\, \lambda'|{c_2}^*)\right] = \max[u(c_1),\, \min[g(\lambda),\, u(c_2)]]$. □

Let $(\delta,\, \delta') \in \mathcal{D}^2$ be a couple of possibility distributions. According to lemma 5.5 there exist $(\lambda,\, \lambda') \in \Lambda^2$ such that $\delta \sim_{\mathcal{D}} \delta^\lambda$ and $\delta' \sim_{\mathcal{D}} \delta^{\lambda'}$. From lemma 5.4 we deduce that $\delta \vee \delta' \sim_{\mathcal{D}} \delta^\lambda \vee \delta^{\lambda'}$ that is $\delta \vee \delta' \sim_{\mathcal{D}} \delta^{\lambda''}$ where $\lambda'' = \max(\lambda,\, \lambda')$. Thus

$$U[\delta \vee \delta'] = g[\max(\lambda,\, \lambda')] = \max[g(\lambda),\, g(\lambda')] = \max[U(\delta),\, U(\delta')].$$

It is then straightforward to check that for all nonnegative integer $N \in \mathbb{N}$, for all $(\delta_1,\, \delta_2,\, \ldots,\, \delta_N) \in \mathcal{D}^N$, $U\left(\bigvee_{n=1\ldots N} \delta_n\right) = \max_{n=1\ldots N} U(\delta_n)$. □

Finally, let $\delta \in \mathcal{D}$ be a possibility distribution, $\delta = (1|c_1;\, \lambda_2|c_2;\, \ldots;\, \lambda_k|c_k)$. For all $i = 2 \ldots k$ introduce $\delta_i = (1|c_1;\, \lambda_i|c_i)$. Thus $\delta = \bigvee_{i=2\ldots k} \delta_i$. This implies that $U(\delta) = \max_{i=2\ldots k} \max[u(c_1),\, \min[g(\lambda_i),\, u(c_i)]]$ that is $U(\delta) = \max_{i=1\ldots k} \min[g(\lambda_i),\, u(c_i)]$. ■

Remark 5.1. *The proof of theorem 3.1 is nothing but an adaptation of a proof established by Dubois et al. (1998).*

Remark 5.2. *Let $(\delta,\, \delta') \in \mathcal{D}^2$ be a couple of possibility distribution and $(\lambda,\, \lambda') \in \Lambda^2$ a couple of scalars such that $\max(\lambda,\, \lambda') = 1$. Define $\delta'' = (\lambda|\delta,\, \lambda'|\delta')$. Then*

$$\begin{aligned} U(\delta'') &= \max_{c \in \mathcal{C}} \min[g \circ \delta''(c),\, u(c)] \\ &= \max_{c \in \mathcal{C}} \min\{\max[\min[g(\lambda), g \circ \delta(c)],\, \min[g(\lambda'), g \circ \delta'(c)]],\, u(c)\} \\ &= \max_{c \in \mathcal{C}} \max\{\min[\min[g(\lambda), g \circ \delta'(c),\, u(c)],\, \min[g(\lambda'), g \circ \delta'(c),\, u(c)]\}. \end{aligned}$$

Thus $U[(\lambda|\delta,\, \lambda'|\delta')] = \max\{\min[g(\lambda),\, U(\delta)],\, \min[g(\lambda'),\, U(\delta')]\}$.

5.3 Results and proofs of subsection 3.2

Proof of lemma 3.1 Assume that axioms 3.1, 3.6 and 3.7 are satisfied. Let $\delta \in \mathcal{D}$, $(c,\, c') \in \mathcal{C}^2$ and $(\lambda,\, \lambda') \in \Lambda^2$ be such that $\max(\lambda,\, \lambda') = 1$ and $(\lambda|\delta;\, \lambda'|c'^*) \succ_{\mathcal{D}} (\lambda|\delta;\, \lambda'|c^*)$. Then axiom 3.6 implies

$$[(\lambda|\delta;\, \lambda'|c'^*),\, \ldots,\, (\lambda|\delta;\, \lambda'|c'^*)] \succ_{\mathcal{F}} [(\lambda|\delta;\, \lambda'|c^*),\, \ldots,\, (\lambda|\delta;\, \lambda'|c^*)]$$

that is

$$(\lambda|[\delta,\, \ldots,\, \delta];\, \lambda'|[c'^*,\, \ldots,\, c'^*]) \succ_{\mathcal{F}} (\lambda|[\delta,\, \ldots,\, \delta];\, \lambda'|[c^*,\, \ldots,\, c^*]).$$

Given axiom 3.1, $\succ_{\mathcal{F}}$ is a strict order thus axiom 3.7 insures that $c' \succ_{\mathcal{C}} c$. ■

Lemma 5.6. *If axioms 3.2, 3.3, 3.5 and 3.6 hold, then, for any act $f \in \mathcal{F}$, there exists S scalars $(\lambda_1, \ldots, \lambda_S) \in \Lambda^S$ such that*

$$f \sim_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}].$$

Proof Let $f \in \mathcal{F}$ be an act. For all $s \in \mathcal{S}$, corollary 5.2 allows us to write that $f_s \sim_{\mathcal{F}} \delta^{\lambda_s}$, where $\lambda_s = l(f_s)$. Applying axiom 3.6, we have

$$f \sim_{\mathcal{F}} [\delta^{\lambda_1},\, f_2,\, \ldots,\, f_S] \sim_{\mathcal{F}} [\delta^{\lambda_1},\, \delta^{\lambda_2},\, f_3,\, \ldots,\, f_S] \sim_{\mathcal{F}} \ldots \sim_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}]$$

Corollary 5.3. *If axioms 3.2, 3.3, 3.5 and 3.6 hold, then the acts $\overline{f} = [\overline{c}^*,\, \overline{c}^*,\, \ldots,\, \overline{c}^*]$ and $\underline{f} = [\underline{c}^*,\, \underline{c}^*,\, \ldots,\, \underline{c}^*]$ are respectively the most preferred and the least preferred acts that is*

$$\forall\, f \in \mathcal{F},\, \overline{f} \succeq_{\mathcal{F}} f \succeq_{\mathcal{F}} \underline{f}.$$

Proof Let $f \in \mathcal{F}$ be an act. Lemma 5.6 guarantees the existence of S scalars $(\lambda_1, \ldots, \lambda_S) \in \Lambda^S$ such that $f \sim_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}]$. Axiom 3.5 implies, for all state $s \in \mathcal{S}$, that $\overline{\delta} \succeq_{\mathcal{D}} \delta^{\lambda_s} \succeq_{\mathcal{D}} \underline{\delta}$. Applying axiom 3.6, we obtain

$$[\overline{\delta},\, \delta^{\lambda_2},\, \ldots,\, \delta^{\lambda_S}] \succeq_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}] \succeq_{\mathcal{F}} [\underline{\delta},\, \delta^{\lambda_2},\, \ldots,\, \delta^{\lambda_S}]$$

hence

$$[\overline{\delta},\, \overline{\delta},\, \delta^{\lambda_3},\, \ldots,\, \delta^{\lambda_S}] \succeq_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}] \succeq_{\mathcal{F}} [\underline{\delta},\, \underline{\delta},\, \delta^{\lambda_3},\, \ldots,\, \delta^{\lambda_S}].$$

Finally, after applying the procedure $S-2$ times, we can write

$$[\overline{\delta},\, \ldots,\, \overline{\delta}] \succeq_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_S}] \succeq_{\mathcal{F}} [\underline{\delta},\, \ldots,\, \underline{\delta}].$$

that is $\overline{f} \succeq_{\mathcal{F}} f \succeq_{\mathcal{F}} \underline{f}$. Note that using notation 3.6, we could have written f^0 instead of $\underline{f}$ and f^1 instead of $\overline{f}$. ■

Lemma 5.7. *Assume that axioms 3.3, 3.4 and 3.6 hold. For all act $f \in \mathcal{F}$ there exist S scalars $(\lambda_1,\, \ldots,\, \lambda_S) \in \Lambda^S$ such that*

$$f \sim_{\mathcal{F}} [\delta^{\lambda_1},\, \ldots,\, \delta^{\lambda_s},\, \ldots,\, \delta^{\lambda_S}].$$

Proof Let $f \in \mathcal{F}$ be an act. If axioms 3.3, 3.4 and 3.6 hold then ,for all $s = 1 \ldots S$, lemma 5.5 guarantees the existence of a unique scalar λ_s such that $f_s \sim_{\mathcal{D}} \delta^{\lambda_s}$. Then $f \sim_{\mathcal{F}} [\delta^{\lambda_1}, \ldots, \delta^{\lambda_s}, \ldots, \delta^{\lambda_S}]$. ■

Corollary 5.4. *If axioms 3.3, 3.4 and 3.6 hold then, for all act $f \in \mathcal{F}$ there exist S scalars $(\lambda_1, \ldots, \lambda_S) \in \Lambda^S$ such that*

$$f \sim_{\mathcal{F}} \Big(1|\underline{f}^1(\lambda_1); \ldots; 1|\underline{f}^s(\lambda_s); \ldots; 1|\underline{f}^S(\lambda_S)\Big).$$

Lemma 5.8. *Let $\lambda \in \Lambda$ be a scalar and $s \in \mathcal{S}$ be a state. Then*

$$\underline{f}^s(\lambda) = (1|f^0; \lambda|\underline{f}^s(1)).$$

Lemma 5.9. *Assume that axioms 3.3, 3.4 3.6, 3.7 and 3.8 hold. For all act $f \in \mathcal{F}$ there exists a scalar $\lambda \in \Lambda$ such that $f \sim_{\mathcal{F}} f^\lambda$.*

Proof Let s be a state. Axiom 3.8 ensures that there exists a scalar $\tilde{\lambda}_s \in \Lambda$ such that $\overline{f}^s(1) \sim_{\mathcal{F}} f^{\tilde{\lambda}_s}$. Given axioms 3.7 and 3.8,

$$\underline{f}^s(\lambda) \sim_{\mathcal{F}} (1|f^0; \lambda|f^{\tilde{\lambda}_s}) = f^{\min(\lambda,\, \tilde{\lambda}_s)}.$$

For all $s \in \mathcal{S}$, introduce $\overline{\lambda}_s = \min(\lambda_s, \tilde{\lambda}_s)$. Lemma 5.7 is thus tantamount to

$$f \sim_{\mathcal{F}} \left(1|f^{\overline{\lambda}_1}; \ldots; 1|f^{\overline{\lambda}_s}; \ldots; 1|f^{\overline{\lambda}_S}\right).$$

Thus, if $\lambda = \max_{s \in \mathcal{S}} \overline{\lambda}_s = \max_{s \in \mathcal{S}} \min(\lambda_s, \tilde{\lambda}_s)$ then $f \sim_{\mathcal{F}} f^\lambda$. ■

Corollary 5.5. $\overline{f} \sim_{\mathcal{F}} f^1$ *and* $\underline{f} \sim_{\mathcal{F}} f^0$*.*

Proof Let $\lambda \in \Lambda$ be such that $\overline{f} \sim_{\mathcal{F}} f^\lambda$. Assume that $1 > \lambda$. Due to axioms 3.5 and 3.6 we have $f^1 \succeq_{\mathcal{F}} f^\lambda$. Besides, corollary 5.3 implies that $\overline{f} \succeq_{\mathcal{F}} f^1$ hence the first equality of the corollary. □

Let $\lambda' \in \Lambda$ be such that $\underline{f} \sim_{\mathcal{F}} f^{\lambda'}$. Assume that $\lambda' > 0$. Due to axioms 3.5 and 3.6 we have $f^{\lambda'} \succeq_{\mathcal{F}} f^0$. Besides, corollary 5.3 implies that $f^0 \succeq_{\mathcal{F}} \underline{f}$ hence the second part of the corollary. ■

Proof of theorem 3.2 Let Λ'' be the quotient space $\mathcal{F}/_{\sim_{\mathcal{F}}}$. For any act f, $\|f\| = \{\tilde{f}/f \sim_{\mathcal{F}} \tilde{f}\}$ denotes its equivalence class. If we equipped Λ'' with the binary relation $\geq_{\Lambda''}$ defined as

$$\forall\, (\|f\|, \|f'\|) \in \Lambda'' \times \Lambda'',\, \|f\| \geq_{\Lambda''} \|f'\| \Leftrightarrow \forall\, (\tilde{f}, \tilde{f}') \in \|f\| \times \|f'\|,\, \tilde{f} \succeq_{\mathcal{F}} \tilde{f}'$$

then axioms 3.1, 3.2, 3.3 and 3.6 imply that Λ'' is bounded and totally ordered. Let $0_{\Lambda''} = \inf \Lambda''$ and $1_{\Lambda''} = \sup \Lambda''$. One particular consequence of lemma 5.3 is that $0_{\Lambda''} = \|\underline{f}\|$ and $1_{\Lambda''} = \|\overline{f}\|$. □

Introduce the applications $h\ : \lambda \in \Lambda \mapsto \|f^\lambda\| \in \Lambda''$ and $L\ : s \in \mathcal{S} \mapsto h(\tilde{\lambda}_s)$, where, for all $s \in \mathcal{S}$, $\tilde{\lambda}_s \in \Lambda$ is such that $\underline{f}^s(1) \sim_{\mathcal{F}} f^{\tilde{\lambda}_s}$. Axiom 3.8 ensures the existence of such scalars. Moreover, if axiom 3.6 holds then it is easy to check that h is increasing. □

Define $\mathcal{V}(f) = h(\lambda)$ where λ is defined by lemma 5.7. Such a definition is consistent. Indeed, if there exist two scalars λ and λ' then, due to the symmetry and the transitivity of $\sim_{\mathcal{F}}$, $f^{\lambda} \sim_{\mathcal{F}} f^{\lambda'}$ which implies that $h(\lambda) = h(\lambda')$. Besides, introduce $V = h \circ l$, where l is characterized by definition 5.1. Since (1) l respects $\succeq_{\mathcal{D}}$ and (2) h respects $>$ on Λ then V respects $\succeq_{\mathcal{D}}$. Moreover, corollary 5.5 implies that $V(\overline{c}^*) = 1_{\Lambda''}$ and $V(\underline{c}^*) = 0_{\Lambda''}$. Lastly, it holds

$$\mathcal{V}(f) = h\left(\max_{s \in \mathcal{S}} \min(\lambda_s,\, \tilde{\lambda}_s)\right) = \max_{s \in \mathcal{S}} \min[h(\lambda_s),\, h(\tilde{\lambda}_s)]$$

that is $\mathcal{V}(f) = \max\limits_{s \in \mathcal{S}} \min[L(s),\, V(f_s)]$. □

In order to conclude, all we have to do is to check that $L^{-1}(1_{\Lambda''}) \neq \emptyset$. We have $1_{\Lambda''} = \mathcal{V}(\overline{f}) = \max_{s \in \mathcal{S}} \min[L(s),\, h(1)] = \max_{s \in \mathcal{S}} L(s)$. The finiteness of $\mathcal{S}$ completes the proof. ■

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