

# Bubbles and Crashes with Partially Sophisticated Investors\*

Milo Bianchi<sup>†</sup>      Philippe Jehiel<sup>‡</sup>

December 1, 2008

## Abstract

We consider a purely speculative market with finite horizon and complete information. We introduce partially sophisticated investors, who know the true average buy and sell strategies of other traders, but lack a precise understanding of how these strategies depend on the history of trade. In this setting, it is common knowledge that the market is overvalued and bound to crash, but agents hold different expectations about the date of the crash. We define conditions for the existence of equilibrium bubbles and crashes, characterize their structure, and investigate whether bubbles may last longer when the amount of fully rational traders increases.

**Keywords:** Speculative bubbles, crashes, bounded rationality.

**JEL codes:** D84, G12, C72.

---

\*We thank numerous seminar participants and in particular Cedric Argenton, Abhijit Banerjee, Markus Brunnermeier, Mike Burkart, Sylvain Chassang, Gabrielle Demange, Daniel Dorn, Tore Ellingsen, Botond Koszegi, John Moore, Marco Ottaviani, Peter Raupach, Marcus Salomonsson, Johan Walden, Jörgen Weibull, Muhamet Yildiz and Robert Östling for useful discussions. Milo Bianchi gratefully acknowledges financial support from Region Ile-de-France and from the Knut and Alice Wallenberg Foundation.

<sup>†</sup>Paris School of Economics. E-mail: bianchi@pse.ens.fr

<sup>‡</sup>Paris School of Economics and University College London. E-mail: jehiel@enpc.fr

# 1 Introduction

In a speculative bubble, trade occurs at prices above fundamentals *only* because investors expect the selling price to be even higher in the near future (Stiglitz, 1990). Trading activities based on speculative motives seem widespread and have been widely documented (see Garber, 1990; Kindleberger, 2005), but their foundation remains largely unclear. As purely speculative trade must rely on some inconsistent expectations or suboptimal decisions, it cannot arise in standard rational expectation models (Tirole, 1982).

In this paper, we show that speculative bubbles may be explained by considering that a fraction of investors have a *partial* rather than *total* understanding of the investment strategies employed by other investors. Specifically, in our model, all investors understand the aggregate buying and selling pressures that apply on average throughout the entire duration of the market. But, while some investors also understand how these buying and selling pressures precisely depend on the history of trade (these are fully rational traders), others lack such precise knowledge (these are partially sophisticated traders). As a result, some investors form erroneous beliefs about the date of the crash, and they are not able to sell just before it, thereby allowing bubbles and crashes to occur.

Before we present in more detail the features of our model and their implications, we make two observations about bubbles, which will also allow us to explain how our approach differs from other approaches.<sup>1</sup>

The first observation is that assuming traders have private information is neither necessary nor sufficient for the emergence of bubbles. From a theory viewpoint, private information alone cannot explain bubbles, as can be inferred from the no-trade theorems (Milgrom and Stokey, 1982). From an experimental viewpoint, there is substantial evidence that bubbles emerge even in contexts in which by design the structure of the game and the value of future dividends are commonly known to subjects (see Porter and Smith, 2003, for a review). Such an observation leads us to consider a model with complete information in which the structure of fundamentals is commonly

---

<sup>1</sup>More precise references to the literature on bubbles are provided in Section 5.1.

known but traders are heterogeneous in their ability to understand others' trading strategies.

The second observation is that, in real bubble episodes, it seems highly plausible that most agents realize, at least eventually, to be in a speculative market. There are anecdotes about this<sup>2</sup>, and somewhat systematic survey evidence. Shiller (1989), for example, reports that just before the U.S. stock market crash of October 1987, 84% of institutional investors thought that the market was overpriced; 78% of them thought that this belief was shared by the rest of investors and, still, 93% of them were net buyers. Such an observation calls in our view for a new modelling of bounded rationality, allowing (at least a fraction of) agents to have some but not full understanding of the situation. In most existing approaches, agents are assumed to be either fully rational, thereby, in equilibrium, having a complete understanding of the market dynamics, or completely mechanical, and so lacking any understanding that they are in a bubble and that the market may crash. By contrast, in our model, it is common knowledge among agents that they are in a bubble and that the market must crash. Bubbles emerge as all agents believe, rightly (when rational) or wrongly (when partially sophisticated), that they can profit by investing in the speculative market and exiting at the right time.

We now describe in some more detail our framework. We consider a market in which agents can trade an asset with no fundamental value. Trade can occur only for a finite number of periods, as liquidity shocks force a (small) fraction of agents to leave the market in each period. Investors are either fully rational or partially sophisticated. Fully rational investors have rational expectations, as usual. Partially sophisticated investors understand the aggregate buy and sell rates along the duration of the speculative market, without having a precise perception of how these rates vary along the life-cycle of the bubble. Moreover, they adopt the simplest theory of trade

---

<sup>2</sup>For example, Eric Janszen, a leading commentator of speculative phenomena, wrote an article in the middle of the Internet bubble (November 1999) saying: "During the final stages, the mania participants finally admit that they are in a mania. But they rationalize that it's OK because they – only they and not the other participants – will get out in time." (article accessible at [www.bankrate.com](http://www.bankrate.com))

volumes and price dynamics that is compatible with their knowledge, thereby expecting constant buy and sell rates throughout the duration of the speculative market, independently from the history of trades.<sup>3</sup> In equilibrium, these constant rates match the aggregate intensities averaged over time, as resulting from the actual sell and buy strategies. In each period, based on their expectations, agents make the optimal trading decisions.

Before previewing our main results, let us emphasize that this equilibrium is not, in our view, the result of individual learning, whereby the same group of investors repeatedly acts in the same market, since in this case partially sophisticated investors would probably either learn that they get exploited or they would run out of money. We think instead of a process of learning at an historical level, in which a new cohort of investors enters the market in each bubble episode. These investors interpret the current market in light of some historical data about similar episodes. Fully rational investors analyze these data with elaborate statistics and as a result they rightly understand the trade dynamics. Partially sophisticated investors, instead, use a simplified model, able to provide the correct averages but no more detailed statistics.<sup>4</sup> In a sense, they apply a linear model to analyze trade dynamics that are not necessarily linear (as in the spirit of Sargent, 1993).

As it turns out, in our model, optimal investment strategies depend only on investors' expectations about the market dynamics one period ahead. In line with the experimental evidence reported by Haruvy, Lahav and Noussair (2007), these expectations can be described by a simple function of investors' knowledge about past bubble episodes and of the behaviors observed during the current bubble episode.

Such an interaction between historical knowledge and current trends generates a rich pattern of expectations and behaviors. For partially sophis-

---

<sup>3</sup>This assumption may be viewed as being in line with the observation that the date of the crash tends to appear quite similar to many other days. Investors and analysts typically find it hard to understand in what sense that precise day was so special, and fundamentally different from the previous day. Even the systematic analysis by Cutler, Poterba and Summers (1989) concludes that "many of the largest market movements in recent years have occurred on days when there were no major news events."

<sup>4</sup>This may be so because these averages may be easier to understand and remember than more detailed information about say the daily buy and sell rates.

ticated investors, historical knowledge is based on the average strategy observed along a complete bubble episode, which include days in which most people want to buy and days in which most people want to sell. As a result, these investors get a positive surprise after any "good day" and a negative surprise after any "bad day". For these investors, then, a series of good days leads to euphoria and a series of bad days (or one very bad day) leads to panic.

This mechanism builds the foundation of bubbles and crashes in our framework. Along the bubble equilibrium, investors first observe a series of rising prices, due to excess demand for speculative stocks. Partially sophisticated investors interpret such an increase in prices as a sign that the bubble will last longer.<sup>5</sup> Hence, they decide to remain invested longer than they had planned, and, in doing so, they end up overestimating the duration of the bubble. These behaviors can be exploited by fully rational investors, who feed the bubble for a while and exit just before the endogenous crash. Upon observing the massive sale by rational investors, partially sophisticated investors realize it is time to sell (actually, it is too late for most of them), and this indeed leads to the crash.<sup>6</sup> In this way, our framework generates both bubbles and crashes, phenomena which tend to be considered separately in the literature.<sup>7</sup>

Inspired by the efficient market hypothesis, whereby rational investors play a stabilizing role and ensure market efficiency, we then explore the relation between bubbles and the share of rational investors in our setting. We observe that rational investors should be neither too many nor too few for bubble equilibria to arise with the property that, just before the crash, there is a panic phase in which investors realize everyone is trying to sell and the crash is about to occur.<sup>8</sup> We also observe in our basic model that when there

---

<sup>5</sup>This captures a strong regularity documented in Shiller (2000). As the price increases, more people display "bubble expectations", i.e. the belief that, despite the market being overvalued, it will still increase for a while before the crash.

<sup>6</sup>Kindleberger (2005) provide a rich historical account of such periods of euphoria and panic; Brunnermeier and Nagel (2004) and Temin and Voth (2004) document how major investors earn great profits by rightly timing the market.

<sup>7</sup>See Section 5.1 for elaboration on this.

<sup>8</sup>With too many rational agents, bubbles cannot arise because we are too close to a

are more rational investors the maximal duration of a bubble gets smaller. However, by extending our analysis to a setting with uncertainty aversion, we show that bubbles may last longer as the fraction of rational investors increases.<sup>9</sup> Thus, in our setting, whether rational investors have a stabilizing role depends on the attitude of investors toward uncertainty.

The rest of the paper is organized as follows. In Section 2 we describe the model and the solution concept. In Section 3 we analyze bubble equilibria, characterizing the conditions for their existence and showing how the maximal duration of bubbles varies as a function of our parameters. In Section 4 we explore whether rational agents have a stabilizing role. In Section 5 we discuss some related literature, policy implications, and avenues for future research. Omitted proofs are provided in the Appendix.

## 2 The model

Our economy is populated by a unit mass of risk neutral individuals.<sup>10</sup> Initially, a fraction  $K$  of them is endowed with cash, each owning  $w > 0$ , and a fraction  $(1 - K)$  is endowed with stocks, each owning one stock. Cash and stocks are distributed independently across agents. The value of cash is constant over time. Stocks are purely speculative: they pay no dividend, their fundamental value is zero, and their return is given only by changes in the price  $p_t$ . For simplicity, we assume that each agent can hold at most one stock at a time, and each stock is indivisible.<sup>11</sup>

---

rational expectation model. With too few rational agents, boundedly rational traders do not observe any massive sale before the crash occurs and as a result they do not get to understand that the crash is about to occur before it actually does.

<sup>9</sup>In fact, facing less strategic uncertainty, rational agents are more prone to invest than boundedly rational ones. As the share of rational investors increases, more people enter the speculative market, which may induce boundedly rational agents to be more optimistic, thereby allowing to sustain longer bubbles.

<sup>10</sup>Section 4.3 considers the case of uncertainty averse agents.

<sup>11</sup>The substance of our analysis would not change if stocks were perfectly divisible and everyone could spend his entire wealth in stocks. The crucial assumption, as we shall see, is that the entire economy has finite wealth.

## 2.1 Financial market

In each period  $t = 1, 2, \dots$ , individuals can trade. Within each period  $t$ , trading occurs as follows: individuals decide simultaneously whether to submit their orders, a market clearing price  $p_t$  is announced, and orders are cleared.

Borrowing stocks or cash is not allowed, so the investment option for individual  $i$  in period  $t$  is simply  $\{buy, stay\ out\}$  if  $i$  holds cash at  $t$ , or  $\{sell, stay\ in\}$  if  $i$  holds a stock at  $t$ . In addition, in case he submits a trade order, each individual specifies a maximum price  $p_t^b$  at which he is willing to buy or a minimum price  $p_t^s$  at which he is willing to sell. Such reserve prices will ensure that demand and supply are smooth functions of the price.

While individual decisions on whether or not to submit a trade order are endogenously determined in equilibrium, for simplicity we leave reservation prices exogenous. For each individual, these prices write as  $p_t^s = \lambda p_{t-1}$  and  $p_t^b = \lambda p_{t-1} + (1 - \lambda)w$ , where the parameter  $\lambda$  is drawn independently across individuals from a commonly known distribution with smooth density and support on  $[0, 1]$ .<sup>12</sup> The induced distributions of reservation prices for those agents who *may* sell or buy a stock in period  $t$  are described by the cumulative functions  $\Gamma_t$  and  $\Lambda_t$ , respectively. These distributions depend on the history of trades, but from the above assumption their support always lies respectively in  $[0, p_{t-1}]$  and in  $[p_{t-1}, w]$ .

The market clearing price  $p_t$  can be characterized in every period  $t$  as follows. Denote the amount of buy and sell orders at  $t$  by  $B_t$  and  $S_t$ , respectively. These quantities can be written as

$$B_t = \beta_t K_t \text{ and } S_t = \sigma_t (1 - K), \quad (1)$$

where  $K_t$  denotes the amount of people who can buy a stock,  $\beta_t$  the share of those who want to buy, and  $\sigma_t$  the share of those who want to sell in period  $t$ .<sup>13</sup> If  $B_t \geq S_t$ , the price  $p_t$  solves  $B_t [1 - \Lambda_t(p_t)] = S_t$ , which implies  $\Lambda_t(p_t) \geq 0$

<sup>12</sup>A fully specified model may derive such  $\lambda$  from heterogeneous preferences, regarding for example attitudes towards uncertainty.

<sup>13</sup>These rates are in principle the realization of a random variable that aggregates the strategies of every agent at a given point in time. However, since we consider a setting with a continuum of agents, each realization of such variable corresponds to its expected value.

and so  $p_t \geq p_{t-1}$ . If instead  $B_t < S_t$ , the price  $p_t$  solves  $B_t = S_t[1 - \Gamma_t(p_t)]$ , which implies  $\Gamma_t(p_t) > 0$  and so  $p_t < p_{t-1}$ . Hence, we have

$$p_t \geq p_{t-1} \Leftrightarrow B_t \geq S_t. \quad (2)$$

Finally, we assume that, at the end of each period  $t$ , each agent observes the trading price  $p_t$  and the volume of trade  $V_t = \min\{B_t, S_t\}$ , from which he can correctly infer  $B_t$  and  $S_t$ . In what follows, we refer to  $B_t$  and  $S_t$  simply as demand and supply in period  $t$ .

## 2.2 Exit from the market

While the amount of stocks is fixed to  $(1 - K)$  throughout the analysis, the amount of people who can buy stocks changes over time due to the exit of some investors. We define exit from the speculative market in period  $t$  as the sum of stock-holders who sell and cash-holders who decide not to buy in period  $t$ . That is, the amount of exit in period  $t$  is

$$E_t = V_t + (1 - \beta_t)K_t. \quad (3)$$

Selling may either be deliberate or it may be induced by a liquidity shock, which forces some agents to sell the stock immediately and stay out of the market from then on.<sup>14</sup> An agent with stock may be hit by a shock with probability  $z > 0$ , thus the amount of exogenous exit in each period is  $z(1 - K)$ , where we assume that  $z(1 - K) < K$ .<sup>15</sup>

In equilibrium, agents who decide to exit the speculative market never wish to re-enter, and we assume that this is rightly understood by everybody.<sup>16</sup> Accordingly, the amount of people who can buy a stock at  $t$  evolves

---

Accordingly, in what follows, we simplify the notation and ignore the distinction between the expected values of these quantities in a given period and their actual realizations.

<sup>14</sup>Such shocks should not be confused with noise trade. They are simply to avoid the possibility that the speculative market lasts forever.

<sup>15</sup>Assuming, perhaps more naturally, that everyone may be hit by a liquidity shock would not change our results, but it would complicate their derivation.

<sup>16</sup>In Bianchi and Jehiel (2008), we show that, under a (natural) assumption, this is the only consistent case.



as

$$K_{t+1} = K_t - E_t, \quad (4)$$

and, by equation (3), we have

$$K_{t+1} = \beta_t K_t - V_t. \quad (5)$$

From equation (5), it follows that the price  $p_t$  never recovers after having dropped. If in period  $t$  the price drops, it must be due to excess supply in  $t$ , in which case the volume of trade  $V_t$  is equal to the demand  $\beta_t K_t$  and equation (5) yields  $K_{t+1} = 0$ . By equation (4),  $K_t$  can only decrease over time, which implies that  $K_{t+s} = 0$  for all  $s \geq 1$ . Thus, after a price drop, the market closes.

### 2.3 Cognitive abilities and equilibrium

Agents differ in their ability to understand other agents' trading strategies. An agent's type  $\theta$  determines his expectation about the dynamics of trade volumes and the associated prices. Such dynamics depend on demand and supply in each period, which in turn depend on the amount of agents still active in the market together with their buy and sell strategies.

From definition (1), the period  $s$  expectations for an agent of type  $\theta$  about the demand and supply in period  $t$  are given by

$$B_t^{\theta,s} = \beta_t^{\theta,s} K_t^{\theta,s} \text{ and } S_t^{\theta,s} = \sigma_t^{\theta,s} (1 - K) \text{ for every } s \leq t, \quad (6)$$

where  $\beta_t^{\theta,s}$  and  $\sigma_t^{\theta,s}$  are this agent's expected buy and sell rates, and  $K_t^{\theta,s}$  is the expected amount of potential buyers in the market at  $t$ . In order to estimate the latter, agents need to know how many traders are in the market at  $s$ , and how many exit from  $s$  to  $t - 1$ . Recall that, in period  $s$ , agents have observed the history of prices and trade volumes, by which they can correctly infer the amount of exits until  $s - 1$  and so  $K_s$ . Hence,

$$K_s^{\theta,s} = K_s, \text{ for every } \theta \text{ and } s.$$

For  $t > s$ , using equation (4), we have

$$K_t^{\theta,s} = K_s - \sum_{w=s}^{w=t-1} E_w^{\theta,s}, \quad (7)$$

where, by equation (3),

$$E_w^{\theta,s} = \min(B_w^{\theta,s}, S_w^{\theta,s}) + (1 - \beta_w^{\theta,s})K_w^{\theta,s}. \quad (8)$$

Given equations (6), (7) and (8), an agent's expectation about future market dynamics are completely characterized by his expectations about future buy and sell rates. These expectations depend on the agent's type, as we now describe. For simplicity, we consider a setting with only two cognitive types: standard rational agents  $R$  and partially sophisticated agents  $I$ , in proportion  $r$  and  $(1 - r)$ , respectively.

$R$ -types understand perfectly well the patterns of other investors' strategies. Hence, if the actual buy and sell rates arising in equilibrium in period  $t$  are given by  $\beta_t$  and  $\sigma_t$ ,  $R$ -agents' expectations must satisfy

$$\beta_t^{R,s} = \beta_t \text{ and } \sigma_t^{R,s} = \sigma_t \text{ for every } s \leq t. \quad (9)$$

$I$ -agents, instead, expect constant buy and sell rates throughout the duration of the speculative market, where these rates coincide with the actual average rates that prevail throughout the speculative market. Formally, denote with  $T + 1$  the last date in which the speculative market operates, as determined endogenously in equilibrium. The average buy rate  $\bar{\beta}$  and the average sell rate  $\bar{\sigma}$  for the sequence of buy and sell decisions arising in equilibrium are

$$\bar{\beta} = \frac{1}{T + 1} \sum_{t=1}^{T+1} \beta_t \text{ and } \bar{\sigma} = \frac{1}{T + 1} \sum_{t=1}^{T+1} \sigma_t. \quad (10)$$

$I$ -types' expectations are required to correspond to such averages, hence we have

$$\beta_t^{I,s} = \bar{\beta} \text{ and } \sigma_t^{I,s} = \bar{\sigma} \text{ for every } s \leq t. \quad (11)$$

We first note that  $I$ -agents hold  $\beta_t^{I,s}$  and  $\sigma_t^{I,s}$  constant irrespective of the histories of trade. For example, they may think of these rates as resulting from a given distribution of strategies, of which they know only the mean. In this way, even when observing a realization different from the mean, they need not change their theory about the underlying distribution.<sup>17</sup>

Investment decisions are determined as the optimal investment strategies given these expectations and agents' payoffs, which are defined as follows. The payoff of an agent is zero if he holds cash or stock forever;  $(p_s - p_t)$  if he buys a stock at time  $t$  and he sells it at time  $s$ ;  $p_s$  if he initially owns a stock and sells it at  $s$ , and  $-p_t$  if he buys a stock at  $t$  and keeps it forever.<sup>18</sup> After each history of prices and trade volumes, each agent chooses an investment strategy that maximizes his expected payoff. An investment strategy profile specifies an investment strategy for every agent in the economy, which serves to define an equilibrium in our setting.

**Definition 1 (*Equilibrium*):** *An investment strategy profile is an equilibrium if, all along the equilibrium path, each agent's investment strategy maximizes his expected payoff, given the expectations defined in equations (9) and (11).*

Our definition of equilibrium is in the spirit of the rational expectation equilibrium in which, due to the dynamic nature of the interaction, beliefs and investment strategies must be optimally adjusted at every point in time. Note however that our definition only considers the incentives of agents on the equilibrium path, and not the adjustment of beliefs and strategies after a positive mass of agents have made non-equilibrium decisions.<sup>19</sup>

---

<sup>17</sup>We also prefer having in mind that  $I$ -agents are not aware that other investors may have a more accurate understanding of the market dynamics. Otherwise, given that in our model trades mostly occur for speculative reasons,  $I$ -agents may simply decide to stay outside the market if they realize to be less sophisticated than others. Such a conclusion would be altered if all agents thought that some other agents are less sophisticated than they are. We leave the extension of our model to the case of infinitely many cognitive types (that would allow for this) for future research.

<sup>18</sup>For an agent hit by a liquidity shock, payoffs may be described differently. For example, such agent may only care about immediate cash, and place no value on cash in the future.

<sup>19</sup>While one could easily amend the solution concept to cover off-the-path optimizations

## 3 Analysis

### 3.1 Optimal investment strategies

We focus on symmetric equilibria in pure strategies, where all investors of a given type and with a given endowment in period  $t$  follow the same pure strategy. Observe first that the existence of an equilibrium is not an issue, as there is always the non-bubble equilibrium in which every agent exits the speculative market at the very first period.<sup>20</sup> Our interest lies in showing the possibility of bubble equilibria, and characterizing the conditions for such equilibria to exist. Given that the fundamental value of the asset is zero, we define any situation in which trade occurs as a bubble. Conversely, if at some point no one is willing to buy the stock at any price, the speculative market closes. Provided that some trade had occurred, we then say that there is a crash.

The problem faced by an individual of a given type is the same irrespective of whether he has cash or stock. That is, for any agent  $i \in \theta$  with cash and any agent  $j \in \theta$  with stock (who is not hit by a liquidity shock),  $i$  prefers to buy if and only if  $j$  prefers to stay in, and  $i$  prefers to stay out if and only if  $j$  prefers to sell. Intuitively, trade occurs either among people with different needs, as described by the liquidity sellers  $z$ , or among those with different expectations, as described by the different types.

In principle, investment strategies may be very complicated, since each agent may condition his current strategy on the whole history of past trades and on his own past trading decisions. However, as it turns out, our model allows a very simple representation of optimal trading strategies. We start by showing that, expecting all exits from the market to be permanent, no agent wishes to re-enter after having exited, and so equation (4) is indeed

---

and expectations, this would make the notation heavier (in particular, the state variable parameterizing the decisions should no longer be the calendar time  $t$  but the entire history of buy/sell decisions) without adding much economic insight. Moreover, since each individual agent has a negligible weight (there is a continuum of agents), our notion of equilibrium is in the spirit of the Nash equilibrium, where no single agent can on his own move the system away from the equilibrium path.

<sup>20</sup>In this equilibrium,  $I$ -agents' expectations are correct, and their decisions to exit immediately is thus rational.

consistent with equilibrium behaviors.

**Lemma 1** *Expecting exits to be permanent, an agent who exits the speculative market at  $t$  prefers to stay out from then on.*

**Proof.** See Section 6.1 in the Appendix. ■

As noticed after equation (5), the fact that exits are permanent implies that the price never recovers after having dropped. As a result, optimal trading strategies at time  $t$  can be expressed as a function of the expected prices at  $t$  and at  $t + 1$  only. We state the result in the next Proposition.

**Proposition 1** *An agent  $i \in \theta$  prefers to buy/stay in the market at  $t$  if and only if*

$$p_{t+1}^{\theta,t} \geq p_t^{\theta,t}, \quad (12)$$

*or, equivalently, if and only if*

$$B_{t+1}^{\theta,t} \geq S_{t+1}^{\theta,t}. \quad (13)$$

**Proof.** See Section 6.2 in the Appendix. ■

Proposition 1 allows us to write the optimal investment strategy for  $I$ -agents in any period  $t$  simply as a function of the observed amount of people who can buy at  $t$  and of the *constant* expectation about future buy and sell rates. We express this more precisely in the next Corollary.

**Corollary 1** *An agent  $i \in I$  prefers to buy/stay in the market at  $t$  if and only if*

$$K_t \geq W, \quad (14)$$

*where*

$$W \equiv \frac{\bar{\sigma}(1 - K)(1 + \bar{\beta})}{\bar{\beta}^2}. \quad (15)$$

**Proof.** See Section 6.3 in the Appendix. ■

In sum, at any  $t$ , optimal trading strategies for  $I$ -agents are only a function of  $K_t$ , which describes the history of trades, and of the expectation about

future buy and sell rates, as expressed in equation (15). Such expectation, in turn, depends on the equilibrium duration of the bubble, but it remains constant over time.  $R$ -agents' expectation, instead, reflects the true strategies observed along the equilibrium, and so it may vary with  $t$ . In particular, given equation (13), these agents buy/stay in the market in period  $t$  if and only if  $\beta_{t+1}K_{t+1} \geq \sigma_{t+1}(1 - K)$ .

### 3.2 Bubble equilibria

We can now show that, under conditions to be characterized in the next Subsection, there exist equilibria of the following form. Apart from a share  $z$  of stock-holders who sell in each period due to liquidity shocks, investors' strategies are such that in each period  $t \leq T - 1$  everyone tries to enter the speculative market and no one wants to sell; in period  $T$ ,  $I$ -investors buy and  $R$ -investors sell; at  $T + 1$ , everyone tries to sell but no one is willing to buy. The crash then occurs and the market closes. Along these equilibria, called bubble equilibria, the aggregate buy and sell rates are

$$\beta_t = \begin{cases} 1 & \text{for } t \leq T - 1, \\ 1 - r & \text{for } t = T, \\ 0 & \text{for } t = T + 1, \end{cases} \quad (16)$$

and

$$\sigma_t = \begin{cases} z & \text{for } t \leq T - 1, \\ z + r(1 - z) & \text{for } t = T, \\ 1 & \text{for } t = T + 1. \end{cases} \quad (17)$$

According to equation (10),  $I$ -agents are induced to expect the following buy and sell rates:

$$\bar{\beta} = \frac{T - r}{T + 1}, \quad (18)$$

and

$$\bar{\sigma} = \frac{Tz + r(1 - z) + 1}{T + 1}. \quad (19)$$

This allows us to define  $B_s^{\theta,t}$  and  $S_s^{\theta,t}$  according to equations (6), (7), and (8). Besides, given the above specifications, the only variable remaining to

endogenize is the duration  $T$  of the bubble. A major objective of the next analysis is then to characterize the conditions for the existence of such  $T$ , and to understand how  $T$  depends on our exogenous parameters  $K$ ,  $z$  and  $r$ .

Given that the exits from the speculative market are permanent, if  $i$  wants to sell/stay out at  $t$ , then he wants to sell/stay out for every  $s > t$ . Likewise, if  $i$  wants to buy/stay in the market at  $t$ , this reveals that he wanted to buy/stay in the market at each  $s < t$ . Hence, the equilibrium  $T$  is defined simply by three conditions. First, each agent  $i \in R$  has to prefer to buy at  $T - 1$ , so we must have  $p_T > p_{T-1}$ . By equation (13), this condition writes as

$$B_T \geq S_T. \quad (20)$$

Second, each agent  $i \in I$  has to prefer to buy at  $T$ , which, by equation (14), writes as

$$K_T \geq W. \quad (21)$$

Third, each agent  $i \in I$  has to prefer to sell at  $T + 1$ , which, again using equation (14), writes as

$$K_{T+1} < W. \quad (22)$$

The last two conditions also imply that each agent  $i \in R$  prefers to sell at  $T$ , since given  $I$ -agents' behavior, the market crashes at  $T + 1$ . We summarize these observations in the following Proposition.

**Proposition 2** *Any  $T$  satisfying conditions (20), (21) and (22) can be sustained as a bubble equilibrium.*

### 3.2.1 Example

While we postpone a more detailed analysis of conditions (20), (21) and (22) to Section 6.4 in the Appendix, we now highlight their structure with a numerical example. Suppose that  $z = 0.2$ ,  $r = 0.2$  and  $K = 0.9$ .

In Figure 1, the solid curve is the function  $W(T)$ , the solid line is the function  $F(T) = K - z(1 - K)(T - 1)$ , the dashed line is the function  $G(T) = (1 - r)[K - z(1 - K)T] - r(1 - K)$ .  $F(T)$  and  $G(T)$  map the

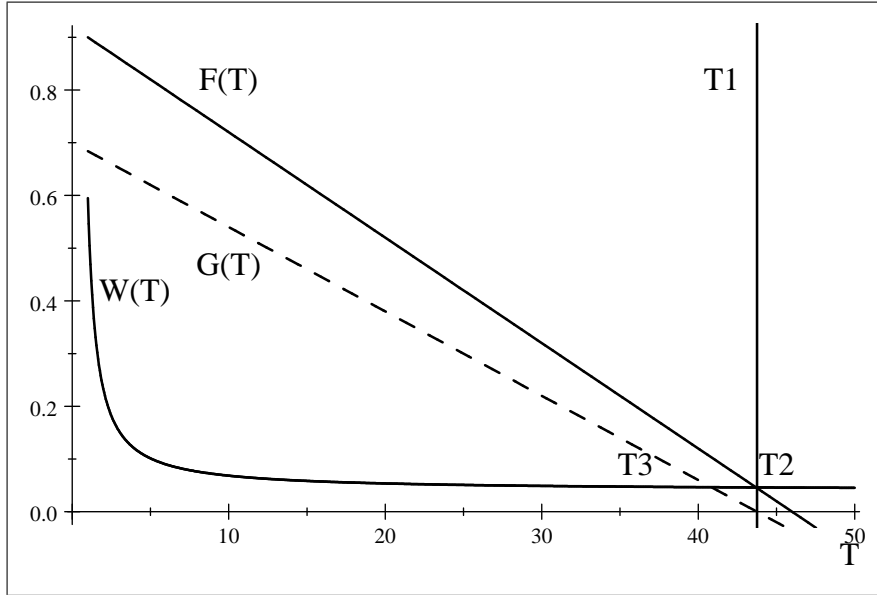


Figure 1: Conditions defining the equilibrium bubble for  $z=0.2$ ,  $r=0.2$  and  $K=0.9$ .

equilibrium  $T$  with the amount of investors who can buy in period  $T$  and  $T + 1$ , respectively. These functions are derived in Section 6.4, and, by construction, they are such that  $F(T) = K_T$  and  $G(T) = K_{T+1}$ . The vertical line plots  $T = T_1$ , as derived from condition (20). In this example, condition (20) is satisfied for  $T \leq T_1$ ; condition (21) for  $T \leq T_2$ , as defined by the intersection of  $W(T)$  and  $F(T)$ ; and condition (22) for  $T > T_3$ , as defined by the intersection of  $W(T)$  and  $G(T)$ . Specifically, substituting our values in equations (20), (21) and (22) we find that, up to integer approximations, they require respectively  $T \leq 42$ ,  $T \leq 43$ , and  $T \geq 41$ . Hence, any  $T \in [41, 42]$  can be a bubble equilibrium.

We now describe how investors' expectations evolve along the bubble equilibrium in this example. Consider the equilibrium in which all rational agents sell at  $T = 42$ . Fully rational agents have rational expectations, and so they expect the crash to occur at  $t = 43$  throughout the duration of the bubble. Instead, as mentioned in Introduction,  $I$ -agents' expectations may change once they observe the actual buy and sell rates realized in each period. On the one hand, after any  $t < T$ ,  $I$ -agents revise their expectation



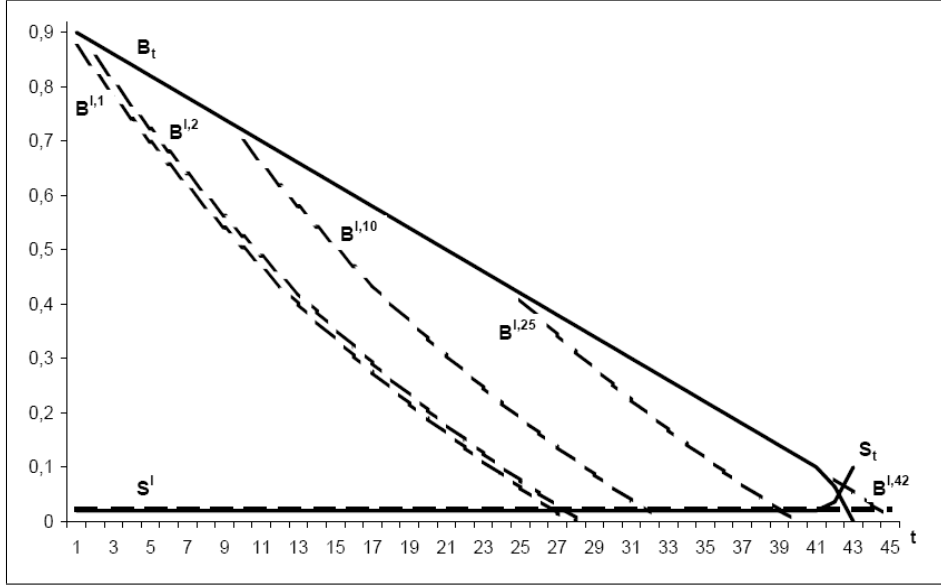


Figure 2: Equilibrium strategies and  $I$ -agents' expectations for  $z=0.2$ ,  $r=0.2$  and  $K=0.9$ .

about the date of the crash upwards. This is so because, in all these periods,  $\beta_t > \bar{\beta}$  and  $\sigma_t < \bar{\sigma}$ . On the other hand, at date  $T$ ,  $I$ -agents revise their expectation about the date of the crash downwards (provided that  $r$  is not too small, see Section 4.2 for details). This is so because  $\beta_T < \bar{\beta}$  and  $\sigma_T > \bar{\sigma}$ .

These patterns are shown in Figure 2. The increasing solid line describes the actual supply observed along the equilibrium  $S_t$  (as determined by equation (17)). This induces  $I$ -agents' expected supply  $S_t^{I,s} = \bar{\sigma}(1 - K)$  for each  $t$  and  $s$  (as determined by equation (19)). This expectation is represented by the dashed constant line  $S^I$ . The decreasing solid line describes the actual demand  $B_t$  (as determined by equation (16)), which determines  $I$ -agents' expected demand according to  $B_t^{I,s} = \bar{\beta}K_t^{I,s}$  for each  $t \geq s$  (as determined by equations (7), (8) and (18)). The decreasing dashed lines  $B^{I,s}$  represent the period  $s$  expectations for  $I$ -agents about the demand in all periods  $t \geq s$ .

The expected date of the crash in each period  $s \leq T$ , which is defined by  $\min_t \{t \mid B_t^{I,s} < S_t^{I,s}\}$ , evolves accordingly. In this example, before any trade takes place,  $I$ -agents expect the crash to occur at  $t = 27$ . At the beginning of the second period, upon having observed fewer exits than expected, they

expect the crash at  $t = 28$ . Similar dynamics occur at each  $s \leq 42$  : for example, at  $s = 10$  they expect the crash at  $t = 32$ ; at  $s = 25$  they expect the crash at  $t = 39$ ; at  $s = 42$ , when they buy for the last time, they expect the crash at  $t = 45$ . Only at the end of period 42, upon observing the massive exit by rational agents (and when it is too late to sell), they realize that the crash will indeed occur at  $t = 43$ .

### 3.3 Existence and maximal duration of a bubble equilibrium

We now ask ourselves when a  $T \geq 1$  satisfying conditions (20), (21) and (22) exists as a function of the parameters  $K$ ,  $z$  and  $r$ . When such  $T$  exists, we say that a bubble equilibrium exists. Intuitively, a bubble is more likely to develop when there is a large amount of cash that could potentially be used to fuel it; when not too many people are hit by shocks that force them to exit the speculative market, and when the number of investors who can correctly predict the date of the crash is not too large. We express this in the following Proposition.

**Proposition 3** *There exists a  $K^*(r, z) < 1$  such that if  $K \geq K^*(r, z)$ , then a bubble equilibrium exists. Such minimal  $K^*(r, z)$  increases in  $r$  and  $z$ .*

**Proof.** See Section 6.5 in the Appendix. ■

As shown in the previous example, and more generally in Section 6.4, there need not be only one  $T$  satisfying conditions (20), (21) and (22). One natural point of interest is the largest  $T$  that can be sustained in equilibrium, the one maximizing  $R$ -agents' profits.

Such largest  $T$  is defined by conditions (20) and (21). The first condition can be explained recalling that, even if no one exits voluntarily from the market, a mass  $z(1 - K)$  of agents sells in each period due to liquidity shocks. Hence, given that the mass of potential buyers decreases over time,  $R$ -agents cannot exit too late if they want to find enough  $I$ -agents who buy their

stocks. Condition (20) can be written as

$$T \leq \frac{K - r}{z(1 - K)(1 - r)} \equiv T_1. \quad (23)$$

Condition (21) instead imposes an upper bound on  $T$  whereby, if  $R$ -agents sell too late,  $I$ -agents would not buy since the amount of cash observed at that stage would be too low. Such an upper bound is defined by

$$T \leq T_2,$$

where  $T_2$  is the largest root solving  $K - z(1 - K)(T - 1) = W$  (see Section 6.4 for details). Accordingly, we define the longest bubble equilibrium as

$$T_{\max} \equiv \min\{T_1, T_2\}.$$

In order to investigate how  $T_{\max}$  varies with our exogenous parameters, the first issue is under which conditions  $T_1$  or  $T_2$  is the constraint defining  $T_{\max}$ . As shown in Section 6.6, when the fraction of rational agents  $r$  is small, the latter constraint binds, while the opposite occurs when  $z$  or  $K$  are small. Irrespective of this, however, the comparative statics are clear: both  $T_1$  and  $T_2$  increase in  $K$  and decrease in  $r$  and  $z$ , as we show in the next Proposition.

**Proposition 4** *The maximal equilibrium bubble  $T_{\max}$  increases in  $K$  and decreases with  $z$  and  $r$ . Moreover,  $T_{\max} \rightarrow \infty$  as  $K \rightarrow 1$  or  $z \rightarrow 0$ .*

**Proof.** See Section 6.7 in the Appendix. ■

Propositions 3 and 4 show that bubbles are supported by large  $K$ , small  $z$  and small  $r$ . These relations are consistent with empirical evidence. The effect of a large  $K$  is in line with the observation that speculative stocks tend to be initially in short supply, and that bubbles are sustained by the large involvement of new investors (see Cochrane, 2002; Kindleberger, 2005). A small probability of shock  $z$  implies that the fraction of potential investors decreases slowly, which is consistent with the fact that bubbles tend to display slow booms and sudden crashes (see Veldkamp, 2005). Finally, the effect of

a small  $r$  echoes the observation that bubble episodes tend to attract a large number of inexperienced investors (see Shleifer, 2000; Kindleberger, 2005). However, as we discuss in the next Section, the relation between bubbles and rationality is not so clear-cut, once we allow for ambiguity aversion.

## 4 Bubbles and rationality

In this Section, we discuss further how the existence and the structure of a bubble equilibrium vary with the share of rational agents in the market.

### 4.1 Rational investors should not be too many

As in standard models, we cannot have bubbles if all investors are fully rational. In particular, in a bubble equilibrium,  $r$  has to be small enough so that all rational agents are able to sell at  $T$ . This condition defines  $T_1$ , as expressed in equation (23). As we must have  $T_1 \geq 1$ , we need

$$r \leq \frac{K - z(1 - K)}{1 - z(1 - K)} \equiv r_{\max}.$$

Hence, we can define a necessary condition for the existence of a bubble equilibrium.

**Proposition 5** *If  $r > r_{\max}$ , then no bubble equilibrium exists.*

### 4.2 Rational investors should not be too few

As expressed in Proposition 3, bubbles are more likely to arise when the share of rational investors  $r$  is low. On the other hand, in the bubble equilibrium characterized above, rational investors play a key role. By exiting at  $T$ , they give a negative shock to the market, which makes  $I$ -investors aware that they had overestimated the length of the bubble and that the crash is about to occur. As a result,  $I$ -investors rush to sell as they realize everyone else is trying to sell. Such final panic phase is a rather common feature of market

crashes (see Kindleberger, 2005), and we now show that it requires  $r$  to be not too small. Specifically, consider the following condition:

$$B_{T+1}^{I,T+1} < S_{T+1}^{I,T+1}, \quad (24)$$

which ensures that, at the beginning of  $T + 1$ , just before the crash occurs, all investors expect the crash to occur next. Together with condition (21), this requires that  $I$ -investors' expectation about the date of the crash changes between period  $T$  and period  $T + 1$ , which in turns requires that some bad shock occurs in period  $T$ . Since the only source of such bad shocks is that  $R$ -investors decide to exit, we need sufficiently many of them. We can state this more precisely with the following Proposition.

**Proposition 6** *In a bubble equilibrium where  $I$ -agents, at the beginning of  $T + 1$ , realize that the market will indeed burst at  $T + 1$ , we must have*

$$r > r_{\min},$$

where  $r_{\min}$  is implicitly defined by the condition  $Tr_{\min} = 1$ .

**Proof.** See Section 6.8 in the Appendix. ■

### 4.3 Uncertainty Aversion

Proposition 4 shows that bubbles are more likely to last longer when the fraction of rational investors is smaller. We now show that this need not be the case if we consider a setting with uncertainty averse agents.<sup>21</sup>

In our model, uncertainty concerns solely the predictions of what other investors do. Hence, the amount of uncertainty faced by each agent depends on his ability to understand other investors' equilibrium strategies. If some  $I$ -agent perceives enough uncertainty, and he prefers to avoid it, he may refrain from investing in the speculative market. On the other hand, since

---

<sup>21</sup>Uncertainty (or equivalently ambiguity) describes situations in which agents' perceptions need not be accurate enough to provide them with a unique probability measure over the possible states of the world.

fully rational agents face no uncertainty, they may be more willing to invest. As a result, the amount of investors in the speculative market is in general increasing with the share of rational agents. This in turn may induce more optimistic expectations and higher demand, thereby allowing to sustain longer bubbles.

In order to formally illustrate this idea, we enrich our setting by assuming that, independently from their cognitive types, investors differ in their attitudes towards ambiguity. Such attitudes however are not relevant for  $R$ -investors, since as noted they face no ambiguity. For  $I$ -investors, instead, we distinguish between ambiguity averse investors  $H$  and ambiguity neutral investors  $L$ , which have mass  $(1-r)h$  and  $(1-r)(1-h)$  respectively. Admitting that their predictions can be mistaken by some  $\varepsilon$ ,  $H$ -agents believe that, in every  $t$ , the actual buy rate  $\beta_t$  will be in the interval  $[\bar{\beta} - \varepsilon, \bar{\beta} + \varepsilon] \cap [0, 1]$  and the actual sell rate  $\sigma_t$  will be in the interval  $[\bar{\sigma} - \varepsilon, \bar{\sigma} + \varepsilon] \cap [0, 1]$ .<sup>22</sup> Furthermore, these agents consider the worst realizations of  $\beta_t$  and  $\sigma_t$ , and, given that, they choose the optimal investment strategy.<sup>23</sup> Hence, in order to be part of speculation, they require a return which compensates the perceived uncertainty.<sup>24</sup> Investors of type  $L$  are instead neutral towards uncertainty (or, alternatively, they do not admit that their predictions can be mistaken). Hence, as in Section 2, such investors only consider the averages  $\bar{\beta}$  and  $\bar{\sigma}$ .

We here consider the special case of  $z \rightarrow 0$  (Bianchi and Jehiel, 2008 provide a more general treatment). In this case,  $T_{\max}$  is defined by  $T_1$ , which may increase in  $r$  since a higher  $r$  reduces the mass of ambiguity averse agents in the market. In fact, the bubble equilibrium is such that  $H$ -agents exit at some  $\tilde{T}$ , by selling to  $L$ - and to  $R$ -agents;  $R$ -agents sell to  $L$ -agents in period

---

<sup>22</sup>The error term  $\varepsilon$  is here taken as given. One could for example endogenize this interval by letting the expected  $\beta_t$  and  $\sigma_t$  lie between the minimum and the maximum buy and sell rates observed along the equilibrium.

<sup>23</sup>Formally, we are assuming that these investors have a set of probability measures over the possible realizations of  $\beta_t$  and  $\sigma_t$ . Investors compute the minimal expected payoffs conditional on each possible prior, and decide the investment strategy corresponding to the maximum of such payoffs. This idea, which may be thought as an extreme form of uncertainty aversion, was formalized by Gilboa and Schmeidler (1989).

<sup>24</sup>Indeed, many authors have invoked ambiguity aversion as a possible resolution of the Equity Premium Puzzle (Chen and Epstein, 2002; Klibanoff, Marinacci and Mukerji, 2005).

$T > \tilde{T}$ ; and in period  $T + 1$  the crash occurs. The smaller the mass of  $H$ -agents, the smaller is the amount of investors who buy stocks at  $\tilde{T}$ , and so the fewer are  $R$ -investors with stocks at  $T$  and the more are  $L$ -investors with cash at  $T$ . Hence, the lower is  $S_T$  and the higher is  $B_T$ , which pushes towards an higher  $T_{\max}$ . This result is expressed in the following Proposition.

**Proposition 7** *If  $z \rightarrow 0$ , there exists a  $\hat{r}(K, h) < 1$  such that  $T_{\max}$  increases in  $r$  for every  $r \leq \hat{r}$ .*

**Proof.** See Section 6.9 in the Appendix. ■

## 5 Discussion

In this Section, we review some of our key ingredients in relation with the existing literature. We then suggest some policy implications of our results, and we conclude with some avenues for extensions.

### 5.1 Related literature

There is a vast literature on speculative bubbles, and we review only some general themes here.<sup>25</sup> Part of the literature builds on the fact that some information, e.g. relative to the value of fundamentals, is dispersed among agents. Bubbles are then generated by adding some extra ingredients. Allen, Morris and Postlewaite (1993), for example, show that bubbles may arise in a finite setting with private information only if one introduces also ex-ante inefficiency, short sale constraints, and lack of common knowledge of agents' trades. Alternatively, bubbles may occur if agents have subjective (and thus erroneous) views about how private information is distributed.<sup>26</sup>

Another stream of literature focuses on the effects of purely mechanical traders (De Long, Shleifer, Summers and Waldmann, 1990a) or of agents who form their expectations about future prices simply by extrapolating from past

---

<sup>25</sup>For a more detailed review, see Bianchi (2007).

<sup>26</sup>This line of reasoning, however, raises the issue of where subjective priors come from, and why they survive in equilibrium (see Dekel, Fudenberg and Levine, 2004, for a critique of the learning foundations of Nash equilibria with subjective priors).

market trends (Cutler, Poterba and Summers, 1990 and De Long, Shleifer, Summers and Waldmann, 1990b). In a phase of rising prices, however, these agents would expect the prices to increase with no bounds, and so they would never understand to be in a bubble nor that the market may crash. As emphasized in the Introduction, we instead focus on agents with enough sophistication to understand that they are in a bubble and that the market must crash.

Moreover, differently from our approach, the literature has typically modeled bubbles and crashes separately. For example, Gennotte and Leland (1990) focus on the role of hedge funds in provoking the crash while taking as given the fact that the market is overvalued. Abreu and Brunnermeier (2003) focus on how coordination issues among rational arbitrageurs may delay the crash while abstracting from the underlying process generating the bubble. On the other hand, De Long, Shleifer, Summers and Waldmann (1990b) explain how feedback trading can generate a bubble but exogenously impose an end period at which the crash occurs. Scheinkman and Xiong (2003) show how overconfidence can sustain speculative trade but do not consider how the crash may occur.

Finally, by emphasizing cognitive heterogeneity, our work is related to a wide literature on the limits to information processing.<sup>27</sup> In particular, our idea of equilibrium follows very closely the spirit of Jehiel (2005) who assumes, in the context of extensive form games, that each player  $i$  is characterized by a partition of the set of nodes where other players move, where each subset of nodes is called analogy class. Player  $i$  assesses only the average behavior of his opponents within each analogy class, and expects this same average behavior to be played at each node within the analogy class. A related idea is developed in static games of incomplete information by Eyster and Rabin (2005).

---

<sup>27</sup>See in particular Higgins, 1996 for an exposition of the idea of accessibility in psychology and Kahneman, 2003 for economic applications. Many authors have explored such themes in strategic interactions (see Rubinstein, 1998 and the references therein and Jehiel, 1995; Jehiel, 2005; Jehiel and Samet, 2007), and financial markets (see Hirshleifer, 2001).



## 5.2 Information and market efficiency

Our analysis of bubbles focuses on a setting in which information is complete but some people face limitations in processing all the relevant aspects of such information. One implication of such analysis is that information availability *per se* need not lead to market efficiency. Instead, we point out that information *accessibility* -which focuses on whether information is presented in a way to ease its interpretation- should matter as well. In this sense, the quest for market stability may require considering issues of simplicity of information, or even of information overload, rather than just increasing the amount of potentially available information.

Along the line of our analysis, one could even argue that some news may have a destabilizing effect, as they may lead partially sophisticated investors to get excessively excited, thereby feeding the bubble phenomenon. This is in a sense what happens within our model when unexpected increases in the price lead partially sophisticated investors to overestimate the duration of the bubble and stay invested for too long. If these investors ignored the news, and in particular the realized price, they would stay invested less long and so leave less room for bubbles (as shown in the Example above). Information accessibility and news-driven euphoria may be useful starting points also for exploring the role of media in stimulating or undermining a speculative phenomenon.<sup>28</sup>

## 5.3 Rational investors and market efficiency

A classic proposition views market efficiency as the result of rational arbitrageurs' strategies. However, several models, apart from the present one, show that rational agents need not have the incentive to immediately stabilize the market. These include Abreu and Brunnermeier (2003) and De Long et al. (1990b). These two models differ in their predictions. In Abreu and Brunnermeier (2003), increasing the share of rational agents reduces the maximal bubble as it reduces the buying capacity of irrational agents. Conversely, in

---

<sup>28</sup>The strong relation between media coverage and abnormal returns has been recently documented e.g. by Dyck and Zingales (2003) and Veldkamp (2006).

De Long et al. (1990b), increasing the share of rational agents increases the size of the bubble as it further distorts irrational agents' expectations.

By contrast, in our model, the relation between the maximal bubble and the share of rational investors can go both ways, and it depends crucially on investors' attitudes towards uncertainty. If investors disregard uncertainty, increasing the share of rational investors makes irrational traders less optimistic and induces rational investors to exit earlier, which reduces the maximal bubble (see Proposition 4). Conversely, in a setting with uncertainty averse agents, speculative investments increase with the share of rational investors, who face less uncertainty, and such an increase makes irrational traders more optimistic and it may then allow for longer bubbles (see Section 4.3).

Hence, in our setting, rational investors are not necessarily a stabilizing force. Instead, market efficiency would be achieved by increasing the fraction of people who admit that their predictions can be imprecise, and that apparently strange observations may not be the result of chance, but rather of a wrong model.

## 5.4 Extensions and future research

While we have described a world with only two cognitive types, a natural extension would be to enrich the range of cognitive types. There are many ways this could be done and we will review only a few ideas here.

First, we could consider the case of investors who distinguish a bit between the various phases of the bubble, thereby further differentiating investors by how many phases they consider. Apart from generalizing our results, this exercise might generate additional predictions, for example by revealing that the order of exit from the speculative market need not be monotonic in the degree of sophistication (how many phases are distinguished). Perhaps also, some agents may decide to re-enter the speculative market after having exited, creating a richer (and possibly more complicated) set of trading behaviors.<sup>29</sup>

---

<sup>29</sup>In fact, the threshold property that characterizes optimal strategies in our main analy-

Second, we could introduce agents who consider different aggregate statistics from the data. Instead of average investment strategies, investors could consider for example average price changes along the bubble, average prices at the peak of the bubble, average durations of a bubble. For some of these aggregate statistics, bubbles are less likely to arise.<sup>30</sup> We view a more systematic exploration on which kind of aggregate statistics is likely to give rise to speculative phenomena as an important direction for future research.

---

sis (see Lemma 1 and Section 7.1) would hold within each phase that agents distinguish, not necessarily for the entire duration of the market.

<sup>30</sup>For example, if investors only knew the average  $T$  in the past bubbles, the bubble would not arise by standard backward induction arguments. In fact, since our model is completely deterministic, knowing the duration of past bubbles would be enough to perfectly predict the duration of the current bubble. In a less literal (and more realistic) interpretation of our model, however, each bubble may be somewhat different from the previous ones, as described for example by a different realization of a stochastic element ( $K$ ,  $r$  and  $z$ ). In such world, having a correct understanding of other investors' strategies, as opposed to simply knowing the realizations of  $T$ , would provide a much more useful information about the dynamics of the current bubble.

## References

- Abreu, Dilip and Markus K. Brunnermeier**, “Bubbles and Crashes,” *Econometrica*, 2003, *71* (1), 173–204.
- Allen, Franklin, Stephen Morris, and Andrew Postlewaite**, “Finite Bubbles with Short Sale Constraints and Asymmetric Information,” *Journal of Economic Theory*, 1993, *61* (2), 206–229.
- Bianchi, Milo**, “Speculative Bubbles and Rationality - A review of the theory,” Mimeo: Stockholm School of Economics April 2007.
- and **Philippe Jehiel**, “Bubbles and Crashes with Partially Sophisticated Investors,” Paris School of Economics Working Paper No. 2008-62 2008.
- Brunnermeier, Markus K. and Stefan Nagel**, “Hedge Funds and the Technology Bubble,” *Journal of Finance*, October 2004, *59* (5), 2013–2040.
- Chen, Zengjing and Larry Epstein**, “Ambiguity, Risk, and Asset Returns in Continuous Time,” *Econometrica*, July 2002, *70* (2), 1403–1443.
- Cochrane, John H.**, “Stocks as Money: Convenience Yield and the Tech-Stock Bubble,” NBER Working Paper No. 8987 June 2002.
- Cutler, David M., James M. Poterba, and Lawrence H. Summers**, “What Moves Stock Prices?,” *Journal of Portfolio Management*, Spring 1989, *15* (3), 4–12.
- , — , and — , “Speculative Dynamics and the Role of Feedback Traders,” *American Economic Review*, May 1990, *80* (2), 63–68.
- De Long, J. Bradford, Andrei Shleifer, Lawrence H. Summers, and Robert J. Waldmann**, “Noise Trader Risk in Financial Markets,” *Journal of Political Economy*, 1990, *98* (4), 703–38.

- , — , — , and — , “Positive Feedback Investment Strategies and Destabilizing Rational Speculation,” *Journal of Finance*, June 1990, *45* (2), 379–395.
- Dekel, Eddie, Drew Fudenberg, and David K. Levine**, “Learning to play Bayesian games,” *Games and Economic Behavior*, February 2004, *46* (2), 282–303.
- Dyck, Alexander and Luigi Zingales**, “The Media and Asset Prices,” Mimeo: Harvard Business School and University of Chicago 2003.
- Eyster, Erik and Matthew Rabin**, “Cursed Equilibrium,” *Econometrica*, 09 2005, *73* (5), 1623–1672.
- Garber, Peter M.**, “Famous First Bubbles,” *Journal of Economic Perspectives*, 1990, *4* (2), 35–54.
- Genotte, Gerard and Hayne Leland**, “Market Liquidity, Hedging, and Crashes,” *American Economic Review*, December 1990, *80* (5), 999–1021.
- Gilboa, Itzhac and David Schmeidler**, “Maxmin Expected Utility with Nonunique Prior,” *Journal of Mathematical Economics*, 1989, *18*, 141–153.
- Haruvy, Ernan, Yaron Lahav, and Charles N. Noussair**, “Traders’ Expectations in Asset Markets: Experimental Evidence,” *American Economic Review*, December 2007, *97* (5), 1901–1920.
- Higgins, E. Tory**, “Knowledge activation: Accessibility, applicability and salience.” in E. Tory Higgins and Arie E. Kruglanski, eds., *Social psychology: Handbook of basic principles*, New York: Guilford., 1996, pp. 133–168.
- Hirshleifer, David**, “Investor Psychology and Asset Pricing,” *Journal of Finance*, 2001, *56* (4), 1533–1597.

- Jehiel, Philippe**, “Limited horizon forecast in repeated alternate games,” *Journal of Economic Theory*, 1995, *67*, 497–519.
- , “Analogy-based expectation equilibrium,” *Journal of Economic Theory*, 2005, *123* (2), 81–104.
- and **Dov Samet**, “Valuation Equilibria,” *Theoretical Economics*, 2007, *2*, 163–185.
- Kahneman, Daniel**, “Maps of Bounded Rationality: Psychology for Behavioral Economics,” *American Economic Review*, December 2003, *93* (5), 1449–1475.
- Kindleberger, Charles**, *Manias, panics, and crashes: a history of financial crises*, Wiley and Sons, 2005.
- Klibanoff, Peter, Massimo Marinacci, and Sujoy Mukerji**, “A Smooth Model of Decision Making under Ambiguity,” *Econometrica*, November 2005, *73* (6), 1849–1892.
- Milgrom, Paul and Nancy Stokey**, “Information, trade and common knowledge,” *Journal of Economic Theory*, February 1982, *26* (1), 17–27.
- Porter, David P. and Vernon L. Smith**, “Stock Market Bubbles in the Laboratory,” *Journal of Behavioral Finance*, 2003, *4* (1), 7–20.
- Rubinstein, Ariel**, *Modeling Bounded Rationality*, The MIT Press, 1998.
- Sargent, Thomas**, *Bounded Rationality in Macroeconomics*, Oxford University Press, 1993.
- Scheinkman, Jose A. and Wei Xiong**, “Overconfidence and Speculative Bubbles,” *Journal of Political Economy*, 2003, *111* (6), 1183–1219.
- Shiller, Robert J.**, *Market Volatility*, Cambridge MA: MIT Press, 1989.
- , “Measuring Bubble Expectations and Investor Confidence,” *Journal of Psychology and Financial Markets*, 2000, *1* (1), 49–60.

- Shleifer, Andrei**, *Inefficient Markets*, Oxford University Press, 2000.
- Stiglitz, Joseph E.**, “Symposium on Bubbles,” *Journal of Economic Perspectives*, 1990, 4 (2), 13–18.
- Temin, Peter and Hans-Joachim Voth**, “Riding the South Sea Bubble,” *American Economic Review*, December 2004, 94 (5), 1654–1668.
- Tirole, Jean**, “On the Possibility of Speculation Under Rational Expectations,” *Econometrica*, 1982, 50 (5), 1163–1182.
- Veldkamp, Laura**, “Slow boom, sudden crash,” *Journal of Economic Theory*, October 2005, 124 (2), 230–257.
- , “Media Frenzies in Markets for Financial Information,” *American Economic Review*, June 2006, 96 (3), 577–601.

## 6 Omitted proofs

### 6.1 Proof of Lemma 1

Notice first that if agent  $i \in \theta$  exits from the market at  $t$ , then he must expect the price to drop at  $t + 1$ , i.e.  $p_{t+1}^{\theta,t} < p_t^{\theta,t}$ , otherwise he would rather exit at  $t + 1$ . By equation (5), and the discussion thereafter, if  $p_{t+1}^{\theta,t} < p_t^{\theta,t}$  then  $K_{t+2}^{\theta,t} = 0$ , so this agent expects the market to close at  $t + 2$ .  $R$ -agents' expectations are correct, so the price indeed drops at  $t + 1$  and the market closes at  $t + 2$ . Hence, these agents will not re-enter at  $t + 1$ . Now consider  $I$ -agents. By equation (2),  $p_{t+1}^{I,t} < p_t^{I,t}$  is equivalent to  $B_{t+1}^{I,t} < S_{t+1}^{I,t}$ , which, given equation (11), writes as  $\bar{\beta}K_{t+1}^{I,t} < \bar{\sigma}(1 - K)$ . By equation (5), we have  $K_{t+1}^{I,t} = \bar{\beta}K_t - \min\{\bar{\beta}K_t, \bar{\sigma}(1 - K)\}$ , so  $p_{t+1}^{I,t} < p_t^{I,t}$  is equivalent to

$$K_t < \frac{\bar{\sigma}(1 - K)(1 + \bar{\beta})}{\bar{\beta}^2}.$$

Now, since  $K_t$  cannot increase over time, it must be that

$$K_s < \frac{\bar{\sigma}(1 - K)(1 + \bar{\beta})}{\bar{\beta}^2} \text{ for every } s \geq t.$$

This implies that, at any  $s \geq t$ ,  $I$ -agents expect the price to drop at  $s + 1$  and the market to close at  $s + 2$ . Hence, such agents will never enter again.

### 6.2 Proof of Proposition 1

If agent  $i \in \theta$  expects  $p_{t+1}^{\theta,t} \geq p_t^{\theta,t}$  he will buy at  $t$  since the strategy of buying at  $t$  and selling at  $t + 1$  gives a positive expected profit. Notice that, for this reason, the proposed strategy is optimal even though the agent may be hit by a liquidity shock which forces him to sell at  $t + 1$ . Conversely, if  $p_{t+1}^{\theta,t} < p_t^{\theta,t}$ , then  $B_{t+1}^{\theta,t} < S_{t+1}^{\theta,t}$ . By equation (5), this implies  $K_{t+2}^{\theta,t} = 0$ , so the agent expects the market to close at  $t + 2$ . Hence, given that the agent expects that selling at  $t + 1$  would be unprofitable and selling after  $t + 2$  would be impossible, he does not buy at  $t$ . Hence,  $p_{t+1}^{\theta,t} \geq p_t^{\theta,t}$  is also necessary for  $i \in \theta$  to buy/stay in the market at  $t$ . Finally, as already noted in (2), equations



(12) and (13) are equivalent.

### 6.3 Proof of Corollary 1

According to equation (11), condition (13) can be written as

$$\bar{\beta}K_{t+1}^{I,t} \geq \bar{\sigma}(1-K), \quad (25)$$

where by (5) we have  $K_{t+1}^{I,t} = \bar{\beta}K_t - \min\{\bar{\beta}K_t, \bar{\sigma}(1-K)\}$ . If  $\bar{\beta}K_t < \bar{\sigma}(1-K)$ , the agent would expect the price to drop at  $t$  and he would exit. This corresponds to condition (14), since  $\bar{\sigma}(1-K)/\bar{\beta} < W$  and so  $\bar{\beta}K_t < \bar{\sigma}(1-K)$  implies  $K_t < W$ . If instead  $\bar{\beta}K_t \geq \bar{\sigma}(1-K)$ , then  $K_{t+1}^{I,t} = \bar{\beta}K_t - \bar{\sigma}(1-K)$ . Substituting into (25) gives the result.

### 6.4 The conditions defining $T$

In order to express conditions (20), (21) and (22) in terms of our exogenous parameters, notice first that, iterating equation (5), the amount of potential buyers in period  $s$  can be written as the difference between the initial amount of potential buyers  $K$  and the accumulated amount of exits up to period  $s-1$ , that is

$$K_s = K - \sum_{t=1}^{t=s-1} [V_t + (1-\beta_t)K_t]. \quad (26)$$

The volumes of trade induced by the bubble equilibrium are

$$V_t = \begin{cases} z(1-K) & \text{for } t \leq T-1, \\ z(1-K)(1-r) + (1-K)r & \text{for } t = T, \\ 0 & \text{for } t = T+1. \end{cases} \quad (27)$$

Hence, using equations (16), (26), (27) and rearranging terms, we get

$$K_T = K - z(1-K)(T-1),$$

and

$$K_{T+1} = (1-r)[K - z(1-K)T] - r(1-K).$$

Condition (20) requires  $\beta_T K_T \geq \sigma_T(1 - K)$ . With simple algebra, it can be written as

$$T \leq \frac{K - r}{z(1 - K)(1 - r)} \equiv T_1.$$

We then turn to conditions (21) and (22). To see their structure, we first define the functions

$$F(T) \equiv K - z(1 - K)(T - 1),$$

and

$$G(T) \equiv (1 - r)[K - z(1 - K)T] - r(1 - K),$$

where by construction  $F(T) = K_T$  and  $G(T) = K_{T+1}$ . Notice that these functions are decreasing in  $T$  and they both tend to minus infinity as  $T$  goes to infinity. Furthermore, with simple algebra, one can show that the function  $W(T)$ , as defined in equation (15) and in which  $\bar{\beta}$  and  $\bar{\sigma}$  are given by (18) and (19), is decreasing and convex in  $T$ , and that it tends to  $2z(1 - K)$  as  $T$  goes to infinity.<sup>31</sup> Hence, both  $F(T)$  and  $G(T)$  can intersect  $W(T)$  at most twice in  $\mathbb{R}_+$ .

Suppose indeed that both  $F(T)$  and  $G(T)$  intersect  $W(T)$  twice. Let  $T_5$  and  $T_2$  be the roots solving  $F(T_5) = W(T_5)$  and  $F(T_2) = W(T_2)$ , with  $T_5 < T_2$ ; and similarly let  $T_4$  and  $T_3$  be the roots solving  $G(T_4) = W(T_4)$  and  $G(T_3) = W(T_3)$ , with  $T_4 < T_3$ . Since  $G(T) < F(T)$  for every  $T$ , we then have that  $T_2 > T_3 > T_4 > T_5$ . In this case, the bubble equilibrium writes as  $T \in [T_5, T_4] \cup (T_3, T_{\max}]$ , where  $T_{\max} \equiv \min\{T_1, T_2\}$ .

The possibility of two disjoint intervals defining the bubble equilibrium depends on the fact that, in our model, both the amount of potential buyers at  $T$  and  $I$ -agents' expectations depend on  $T$ , as expressed by the functions  $F(T)$ ,  $G(T)$  and  $W(T)$ . If  $F(T)$  and  $G(T)$  were constant (i.e. if  $z$  were zero), then we would only have equilibria of the type  $[T_5, T_4]$ . According to condition (21), we would need  $T \geq T_5$  in order to make  $I$ -agents' expectations sufficiently optimistic and induce them to buy (recall that  $W(T)$  is decreasing, i.e.  $I$ -agents' optimism increases in  $T$ ). On the other hand, con-

---

<sup>31</sup>Being only algebra, the proof is omitted.

dition (22) would require  $T < T_4$  :  $R$ -agents could not sell too late otherwise  $I$ -agents' expectations would be too optimistic and they would never sell, so the crash would not occur.

Conversely, if  $W(T)$  were constant, we would only have equilibria of the type  $(T_3, T_{\max}]$ . Condition (21) would require that  $T \leq T_2$ . If  $R$ -agents sell too late,  $I$ -agents would not buy since the amount of cash observed at that stage would be too low. On the other hand, condition (22) requires  $T > T_3$ . If  $R$ -agents sell too early,  $I$ -agents would not exit at  $T + 1$ , so the crash would not occur. Hence, it would be optimal to stay in the market rather than selling at  $T$ .

As one expects, equilibria of the type  $[T_5, T_4)$  occur when  $F(T)$  and  $G(T)$  are very high, so the binding constraint is the evolution of  $I$ -agents' expectations; while equilibria of the type  $(T_3, T_{\max}]$  occur when  $F(T)$  and  $G(T)$  are very low, so the binding constraint is the evolution the amount of cash observed in the economy. Indeed, for  $K$  sufficiently high, equilibria of the type  $[T_5, T_4)$  do not exist, since we have  $T_4 < 1$  (as in Example 3.2.1). More generally, depending on the value of  $K$ ,  $r$  and  $z$ , such  $T_2, T_3, T_4, T_5$  may not exist or their value may be less than one. This means that the constraints defined above may or may not bind.

Rather than providing a full treatment of such  $T_2, T_3, T_4, T_5$ , our analysis was mainly interested in defining conditions for the existence of equilibrium bubble (as expressed in Proposition 3 and in Section 4.1) and in characterizing the comparative statics on the maximal equilibrium bubble  $T_{\max}$  (as expressed in Proposition 4 and in Section 4.3).

## 6.5 Proof of Proposition 3

Notice first that, for every  $K$ ,  $z$  and  $r$ , we have  $T_3 < T_{\max} \equiv \min\{T_1, T_2\}$ . In fact, since  $G(T) < F(T)$  for every  $T$ , we have that  $T_3 < T_2$ . Moreover, by definition,  $G(T_1) = 0$ , so condition (22) holds for sure at  $T_1$  and then  $T_3 < T_1$ . Given the shape of the function  $W(T)$  described in Section 6.4, the bubble equilibrium exists if and only if  $W(T)$  and  $F(T)$  intersect at least once, i.e. if there exists a  $T_2 \geq 1$  such that  $F(T_2) = W(T_2)$ . In fact, when this is the case,

$T_{\max}$  can always be sustained as equilibrium. Hence, a *sufficient* condition for the existence of a bubble equilibrium is that  $W(T)$  and  $F(T)$  intersect once and only once, that is the case when  $K \geq W(1)$ . With some algebra, one writes

$$K \geq W(1) \iff K \geq \frac{[z(1-K)(1-r) + (1-K)(1+r)](3-r)}{(1-r)^2}. \quad (28)$$

Condition (28) can be rearranged to define a  $K^*$  such that if  $K \geq K^*$  then  $K \geq W(1)$ , and so a bubble equilibrium exists. Moreover, one can see that such  $K^*$  is always smaller than one, and it increases in  $r$  and  $z$ .<sup>32</sup>

## 6.6 The conditions defining $T_{\max}$

We now turn to the analysis of the conditions under which  $T_1$  or  $T_2$  defines  $T_{\max} \equiv \min\{T_1, T_2\}$ . Notice first that  $T_1 < T_2$  if and only if  $W(T_1) < F(T_1)$ . By definition of  $T_1$ ,  $F(T_1)(1-r) = S_T$  and  $S_T = z(1-K)(1-r) + r(1-K)$ , so  $W(T_1) < F(T_1)$  writes

$$\frac{z(1-K)(1-r) + r(1-K)}{1-r} > W(T_1). \quad (29)$$

In Section 3.3, we claimed that  $T_{\max} = T_2$  when  $r$  is small, and  $T_{\max} = T_1$  when  $z$  or  $K$  are small. We now show that this is indeed the case. Consider the first claim. Rearranging condition (29), we can define a threshold  $\bar{r}$  such that  $T_1 < T_2$  if and only if  $r > \bar{r}$ . Such threshold is implicitly defined by  $\bar{r} = P(\bar{r})$ , where

$$P(r) \equiv \frac{W(T_1) - z(1-K)}{W(T_1) + (1-z)(1-K)}. \quad (30)$$

In fact,  $P(r)$  is increasing in  $W(T_1)$ , and  $W(T_1)$  is increasing in  $r$ . Moreover,  $P(0) > 0$  and  $P(1) < 1$ . Hence  $r > P(r)$  holds for  $r > \bar{r}$ , where  $\bar{r}$  is uniquely defined by  $\bar{r} = P(\bar{r})$ .

Now consider the case of  $z \rightarrow 0$ , i.e. the probability of liquidity shocks is very small. Both  $T_1$  and  $T_2$  tend to infinity as  $z$  tends to zero, but  $T_2$  exceeds

<sup>32</sup>Equation (28) can alternatively be rearranged to define a  $r^*$  and a  $z^*$  such that if  $r \leq r^*$  or if  $z \leq z^*$ , then a bubble equilibrium exists.

$T_1$ . In fact if  $z \rightarrow 0$ , then  $z(1 - K) \rightarrow 0$ ,  $T_1 \rightarrow \infty$  and  $W(T_1) \rightarrow 0$ . Hence,  $P(r) \rightarrow 0$ , so  $r$  always exceeds  $P(r)$  and  $T_{\max} = T_1$ .

Finally, consider the conditions on  $K$ . Condition (29) can be rearranged as

$$K < \frac{r + (1 - r)[z - W(T_1)]}{z(1 - r) + r} \equiv Q(K).$$

Notice first that if  $K = 1$ , then  $W(T_1) = 0$  and so  $Q(1) = 1$ . That is, if  $K = 1$ , then  $T_1 = T_2$ . For  $K = 0$ , no bubble equilibrium exists, so we only have to consider  $K \geq K_{\min}$ , where  $K_{\min}$  corresponds to the case  $T_1 = 1$  and it writes as

$$K_{\min} \equiv \frac{r + z(1 - r)}{1 + z(1 - r)}.$$

Now, it can be shown (with simple algebra) that  $Q(K_{\min}) > K_{\min}$ , which means that  $T_1 < T_2$  for  $K = K_{\min}$ .

## 6.7 Proof of Proposition 4

By differentiating equation  $T_1 = (K - r)/[z(1 - K)(1 - r)]$ , one sees that  $T_1$  increases in  $K$  and decreases with  $z$  and  $r$ . To see the effects on  $T_2$ , define the function  $L(T) \equiv F(T) - W(T)$ . By definition,  $L(T_2) \equiv 0$ . Differentiating the function  $L(T)$ , one can see that it decreases in  $T_2$ ,  $z$  and  $r$  and it increases in  $K$ . Hence, by the implicit function theorem,  $T_2$  increases in  $K$  and decreases with  $z$  and  $r$ . The second part of the Proposition can be shown by noticing that if  $K \rightarrow 1$  or  $z \rightarrow 0$ , then  $z(1 - K) \rightarrow 0$ . Both  $T_1$  and  $T_2$  tend to infinity as  $z(1 - K) \rightarrow 0$ .

## 6.8 Proof of Proposition 6

Condition (24) writes  $\bar{\beta}K_{T+1} < \bar{\sigma}(1 - K)$ . Recall that condition (21) requires  $\bar{\beta}K_{T+1}^{I,T} \geq \bar{\sigma}(1 - K)$ . Hence, conditions (24) and (21) jointly require  $K_{T+1} < K_{T+1}^{I,T}$ . Recall that  $K_{T+1} = \beta_T K_T - S_T$ , and  $K_{T+1}^{I,T} = \bar{\beta}K_T - \bar{\sigma}(1 - K)$ . Hence,  $K_{T+1} < K_{T+1}^{I,T}$  if and only if

$$(\beta_T - \bar{\beta})K_T + [\bar{\sigma}(1 - K) - S_T] < 0. \quad (31)$$

Consider the first term in (31). Recall that  $\beta_T = (1-r)$  and  $\bar{\beta} = (T-r)/(T+1)$ , so  $\beta_T < \bar{\beta}$  requires  $(T+1)(1-r) < (T-r)$ , that is  $rT > 1$ . Now consider the second term in equation (31). Recall that  $\bar{\sigma} = ((T-r)z + 1 + r)/(T+1)$  and  $S_T = r(1-K) + z(1-K)(1-r)$ . Hence,  $\bar{\sigma}(1-K) < S_T$  requires  $r(1-K) - z(1-K)Tr > 1 - K - z(1-K)$ , that is  $rT > 1$ . Hence, condition (31) is satisfied if and only if  $rT > 1$ . In particular, recall that we must have  $T \leq T_1$ , where  $T_1 = (K-r)/[z(1-K)(1-r)]$ , so condition (31) requires  $r > [z(1-K)(1-r)]/(K-r)$ . Doing the algebra, the last inequality is satisfied for  $r \in (r_1, r_2)$ , where  $r_1 > 0$ . Hence, there exists a  $r_{\min} > r_1 > 0$  such that if  $r \leq r_{\min}$  condition (31) cannot hold.

## 6.9 Proof of Proposition 7

Notice first that, as in equation (21),  $L$ -investors buy/stay in at  $t$  if and only if  $K_t \geq W$ , while  $H$ -investors buy/stay in at  $t$  if and only if  $K_t \geq W(\varepsilon)$ , where

$$W(\varepsilon) \equiv \frac{(1-K)(\bar{\sigma} + \varepsilon)(1 + \bar{\beta} - \varepsilon)}{(\bar{\beta} - \varepsilon)^2}.$$

Since  $W(\varepsilon)$  increases in  $\varepsilon$ ,  $H$ -investors always sell before  $L$ -investors.<sup>33</sup> We then define an equilibrium in which for  $t < \tilde{T}$  no investor wants to exit (apart from the liquidity traders);  $H$ -investors leave the market at  $\tilde{T}$ , selling to rational agents  $R$  and to low ambiguity averse agents  $L$ .<sup>34</sup> At  $T > \tilde{T}$ , rational investors sell to  $L$ -agents. At  $T+1$ ,  $L$ -agents realize that the crash is about to occur and they want to sell, while no one is willing to buy. The

<sup>33</sup> $W(\varepsilon)$  is simply obtained from (21) by replacing  $\bar{\beta}$  with  $\bar{\beta} - \varepsilon$ , and  $\bar{\sigma}$  with  $\bar{\sigma} + \varepsilon$ .

<sup>34</sup>Since the distribution of reservation prices, cognitive type and probability of liquidity shock are all independent, the proportion of  $R$  and  $L$  in the speculative market remains constant until  $T$ . Hence, for  $t \in (\tilde{T}, T)$ , the proportion of  $R$ -investors in the market is

$$\frac{r}{r + (1-h)(1-r)},$$

and, similarly, the proportion of  $L$ -agents is

$$\frac{(1-r)(1-h)}{r + (1-h)(1-r)}.$$

crash occurs and the market closes.

We are interested in how  $T_{\max}$  varies with  $r$ , for a given proportion of ambiguity averse agents  $h$ .<sup>35</sup> Consider the effects on  $T_1$ , recalling that  $T_1$  is defined by  $B_T = S_T$ . In our equilibrium,  $S_T$  includes all  $R$ -investors with stocks at  $T$  and the exogenous sales  $z(1-K)$ , while  $B_T$  includes all  $L$ -investors with cash at  $T$ . That is,  $S_T = \sigma_t(1-K)$ , where

$$\sigma_t = z + (1-z)\frac{r}{1-h(1-r)},$$

and  $B_T = \beta_t K_T$ , where

$$\beta_t = \frac{(1-r)(1-h)}{1-h(1-r)},$$

and

$$K_T = K - z(1-K)(T-1) - (1-r)h[(1-z)(1-K) + K - z(1-K)(\tilde{T}-1)].$$

In order to simplify the exposition, from now on we assume that  $H$ -investors perceive enough uncertainty to be induced to sell immediately, i.e. that  $\varepsilon$  is large enough to have  $\tilde{T} = 1$ .<sup>36</sup> Hence,  $B_T \geq S_T$  defines the condition  $T \leq T_1$ , where

$$T_1 \equiv \frac{1}{z(1-K)} \left[ K + z(1-K) - (1-z+zK)(1-r)h - \frac{z(1-K) - hz(1-K)(1-r) - (1-z)(1-K)r}{(1-r)(1-h)} \right]. \quad (32)$$

After simple algebra, we can see that

$$\frac{\partial T_1}{\partial r} = \frac{1}{z(1-K)} \left[ h(1-z+zK) - \frac{1-K}{(1-r)^2(1-h)} \right], \quad (33)$$

<sup>35</sup>The effect of  $h$  is clear:  $T_{\max}$  decreases with  $h$ , that is the fraction of people leaving the market early.

<sup>36</sup>For example  $H$ -investors may think that  $\beta_t$  and  $\sigma_t$  are respectively drawn by distributions with mean  $\bar{\beta}$  and  $\bar{\sigma}$  and support on  $[0, 1]$ . As they are extremely ambiguity adverse, they assume  $\beta_t = 0$  and  $\sigma_t = 1$  for all  $t$ , so they exit as soon as possible. In other words, given that there is a one-to-one mapping between  $\tilde{T}$  and  $\varepsilon$ , we now consider  $\tilde{T}$  as an exogenous parameter of the model.

which is positive when

$$r \leq 1 - \sqrt{\frac{1 - K}{h(1 - h)(1 - z + zK)}}. \quad (34)$$

Proposition 7 claims that if  $z \rightarrow 0$ , then  $T_{\max}$  increases in  $r$  for every  $r \leq \hat{r}$ , where

$$\hat{r} \equiv 1 - \sqrt{\frac{1 - K}{h(1 - h)}}.$$

To see that, notice first that  $T_{\max} = T_1$  when  $z \rightarrow 0$ . In fact, one can replicate the analysis of Section 6.6 in the setting with ambiguity aversion and write that  $T_1 < T_2$  if and only if  $r$  exceeds a threshold implicitly defined by

$$r > \frac{(1 - h)[W(T_1) - z(1 - K)]}{(1 - h)[W(T_1) - z(1 - K)] + 1 - K}.$$

If  $z \rightarrow 0$  the right hand side of the last equation tends to zero, and so  $T_1 < T_2$ . Substituting  $z = 0$  into equation (34) gives the result.