WORKING PAPER

# A Monocentric Analysis of Housing Budget Restrictions, Including and Without Transportation

Nicolas Coulombel[a]

[a] Université Paris-Est, Laboratoire Ville Mobilité Transport, 19 rue Alfred Nobel, Cité Descartes – Champs sur Marne, F-77455 Marne la Vallée Cedex 2, FRANCE

Contact Information for the Corresponding Author:
E-mail: nicolas.coulombel@enpc.fr
Tel: (+33) 1 64 15 21 30
Fax : (+33) 1 64 15 21 40

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# ABSTRACT

Considering the last surge in fuel prices, the policy to limit the share of housing expenses in the households' budget, so as to secure their solvency, has been criticized. Supposedly, it induces people to get farther from the city center in search of cheaper housing prices, but with subsequent increased transportation costs that are often disregarded during the house search process. To address this issue, several researchers have advocated to set a constraint on the share of both housing and transportation expenditure.

The present paper analyzes and compares the effects of the two policies on the main features of the city and on the households' utility. The analysis is carried out within the classic monocentric model of urban economics. After a general analysis, an applied model is specified to capture the effects of each policy in straightforward formulae.

I find that constraining housing expenses may increase the well-being of households. Additionally, both policies prove to be effective in reducing urban sprawl and thereby energy consumption. Thus the choice of the optimal policy will depend on the local authorities' objectives.


**Keywords**: monocentric model, urban economics, housing expenses, transportation expenses, housing policy, location efficient mortgage

# INTRODUCTION

During the 2008 surge in oil prices, notable concerns rose about the "solvency" of households, which I define here as their ability to meet all their expenses[1]. This was especially the case in tight housing markets, where households have to face significant housing expenditures. And although the subsequent drop has somehow relieved the households' budgets, concerns remain over the long-term situation since oil prices are more than likely to be on the rise again. Under such circumstances, the relevance of capping housing expenditures at a given fraction of the household income, measure which had already been questioned, has become even more controversial[2]. Such practice is common in several countries in order to preserve the household solvency. In France it is enforced in two ways:

- Monthly payments for home loans are capped at one third of the household income (28% in the U.S. according to Duca and Rosenthal, 1994).
- When applying to rent a home, candidates must earn at least around three times the required rent[3].

While this policy does seem to secure the solvency of the households, it may spur them to settle far from the center of the agglomeration in search of moderate housing prices. Such is the case in the Greater Paris Region, whose central part desperately lacks affordable housing supply. This induces new homeowners to settle farther and farther in the suburbs, thereby contributing to urban sprawl (Polacchini and Orfeuil, 1999). Furthermore, because suburban households usually make the most extensive use of the car, we will see that they expose themselves to significant transport costs, which combined to the housing burden jeopardize the household budget. To prevent these undesirable collateral effects, several researchers (Hare, 1995, or Polacchini and Orfeuil, 1999) have advocated the equivalent of a joint budget constraint (housing plus transportation) for homebuyers instead of the current practices. Their aim is twofold:

- To increase public awareness of the extent of transportation costs implied by suburban and exurban lifestyles.
- Making near transit locations more affordable by increasing the size of the home loan for households willing to locate in such areas (based on future savings on transportation).

This idea was implemented in the U.S. under the name of "Location Efficient Mortgage"[4], but only in a limited number of housing markets.

Although there is abundant economic literature assessing land-use regulatory policies (e.g. Bertaud and Brueckner, 2005, or Brueckner, 2006), this is not the case for the specific

[1] This definition therefore encapsulates the usual notion of solvency as the ability of households to meet their financial obligations on time, and in particular mortgages.

[2] For the reminder of the text, I will equally use the terms "burden" or "expense ratio" to refer to the fraction of the household income dedicated to a budget item, e.g. housing or transportation. The housing expense ratio is also sometimes referred to as the front ratio.

[3] The ratio of one to three corresponds to a widespread practice in the Greater Paris Region, though few renters may even require up to four times the rent. In the remainder of France, income requirements may be less strict.

[4] See www.locationefficiency.com for more on the LEM initiative, which notably stemmed from the work achieved by Haas et alii (2006).

policies I have mentioned. I propose to remedy this gap by analyzing first the policy limiting the housing expense ratio (which I call the Constrained Housing Expense (CHE) policy), then the one capping the total share of transportation and housing expenses (the Constrained Housing+Transportation expenses (CH+T) policy). The analysis is carried out within the classic framework of urban economics, the monocentric model. The impacts of each policy on the main features of the city are brought to light and then compared; in particular I examine the issue of the well-being of the households, the city size and the related transport costs, and rent prices. As will be seen, both policies reduce urban sprawl (and thus could contribute to reduced energy consumption) while maintaining or even increasing the well-being of households.

I shall present this work as follows: the first section being the present introduction, section two describes the context and the scope of the study. Sections three and four respectively analyze at great length the CHE and CH+T policies. Section Five offers by way of conclusion a comparative analysis of the two measures, and policy recommendations.

# CONTEXT AND SCOPE OF THE STUDY

As mentioned in the introduction, no economic work has specifically tackled the issue of assessing the CHE and CH+T policies. Still three strands of works bring useful insights into the topic of this study. Along the presentation of the first strand, I start with providing indirect yet conclusive evidence for the significant influence exerted by the CHE policy. This is mainly achieved by examining housing and transportation burdens. It will also be shown that CHE and CH+T policies more specifically target lower-income households. Next, a survey of existing works on CHE and CH+T policies is carried out. Finally, I present the framework of analysis, namely the monocentric model, and specify the scope of the study.

## On housing and transportation burdens

A first set of empirical works gives a grasp of the size of the issue at stake, by focusing on the households' housing and transportation burdens. As a matter of fact, three questions are preliminary to the present study:

1. Does the CHE policy concern a significant number of households?
2. Is the impact on housing choices substantial?
3. Do spatial variations of transport costs really loom large in front of the housing burden?

By providing estimates of the housing and transportation burdens, Polacchini and Orfeuil (1999), Berri (2007) and Coulombel, Deschamps and Leurent (2007) bring first pieces of answer to question 1 and 3 for the Greater Paris Region (GPR). Despite differences in methodology or in the year of interest, all works draw similar conclusions regarding the housing and transportation expense ratios:

- The housing expense ratio is fairly stable over space, and is close to the maximum allowed by the CHE policy. Polacchini and Orfeuil (1999) find for the year 1991 an average front ratio of 32% for homebuyers and 26% for tenants of the private market. Coulombel, Deschamps and Leurent (2007) respectively find 28% and 39% for year 2001. Berri (2007) provides the lowest housing burdens, with 28% for homebuyers and 22% for tenants of the free sector in 1994.

- On the other hand, the transportation burden markedly increases with distance to the Central Business District (CBD), as a result of a greater car modal share in the suburbs as well as households making longer trips. Coulombel, Deschamps and Leurent (2007) have found that expense ratios range from 7% for inner Paris to 21% for the most remote parts of the GPR.

Given these two facts, all works bring to light an increasing spatial trend for the overall housing plus transportation burden.

Interestingly Haas et alii (2006) reach similar conclusions for the U.S. despite the notorious differences with Europe regarding general urban structure. Analyzing 28 metropolitan areas, they find the housing burden to be significantly less sensitive to location than the transportation burden, which strongly increases with distance to the nearest employment center. By example, the average housing burden of households with yearly income between 35,000 and 50,000$ fluctuates between 23 and 26% depending on the location within the metropolitan area, against 16 to 26% for transportation.

## Assessing the scope and significance of the CHE policy

Above findings naturally lead to the following statements:

- The relative constancy of the housing expense ratio over space (within a given metropolitan area), combined to its closeness to the theoretical upper bound, is most likely the effect of the CHE policy.
- Given this constancy, the increasing trend of the transportation share jeopardizes suburban and exurban households, who face a heavy joint housing & transportation burden (sometimes exceeding half their income).

Although the former statement might not come out as obvious at first, two elements corroborate it. First, housing burdens display volatility across households: thus an average housing burden close to the theoretical upper bound probably comes with a sizable number of constrained households. Secondly, exhibiting a front ratio lower than the upper bound does not automatically imply that the household was not constrained by the CHE policy at the time of its present housing choice[5]. In other words, the number of households concerned by the CHE policy probably surpasses the number of households with housing burdens at or above the theoretical upper bound[6].

To be thorough, one has to mitigate the first statement by underlining the key role of income in the previous analyses. As a matter of fact, all works accounting for income (e.g.

[5] Indeed, income usually rises along the household lifecycle, be it through inflation or job promotions. Since nominal mortgage payments are held constant over time, the housing burden decreases. Furthermore, when the household has successfully reimbursed its mortgage, its housing burden also drops. Because rent variations are regulated, similar phenomena often occur in the rental market. Therefore, one household usually sees its housing burden progressively decline until its next residential move. To mitigate this point, let us note that the housing burden may increase in case of an adverse event on the job market (e.g. unemployment spell), or if housing expenses increase due to specific conditions (renegotiation of the lease, flexible interest rate mortgage products...).

[6] Because the CHE policy is not enforced in a dynamic but in a static fashion (i.e. the capping is only checked once at the time of housing choice), households may bear housing burdens greater than the fixed bound.

Coulombel, Deschamps and Leurent, 2007 or Haas et alii, 2006) put forward the marked decrease of both the housing and transportation expense ratios with income[7]. Since households in the lower income bracket are more likely to face a heavy H+T burden than those in the upper one, they are also more prone to be effectively constrained by CHE and CH+T policies.

Additional figures provided by Gobillon and Le Blanc (2008) prove helpful in reaching conclusions on the scope and significance of the CHE policy,. Studying the effects of borrowing constraints, they estimate thanks to an econometric procedure that 53% of private sector tenants would be constrained were they to opt for homeownership[8]. The share of potentially constrained households is logically lower for homeowners (homebuyers and outright owners confounded) but still amounts to 20% of this category. Putting all the previous elements together, the significance of the CHE policy is clearly established in the case of new home buyers. Regarding tenancy, although previous studies have emphasized significant housing burdens that are likely to come along with a substantial fraction of constrained households, a more precise assessment of the phenomenon has yet to be made.

Coming back to the second above statement, this one may similarly seem peculiar at first. A sound economic reasoning would raise the fact that rational households with rational expectations freely and adequately choose their housing and transportation bundle. The high overall housing plus transportation burden endured by suburban households would be the result of an optimizing behavior and not a danger, even if this burden were to represent more than half the household income. Yet three arguments at least challenge this line of thinking:

- The housing market might not be perfectly competitive. In the presence of sticky prices[9], households already settled in the city center (the “insiders”) might stay to benefit from low transportation costs, pushing new households (the “outsiders”) towards the suburbs and the exurbs. These locations would enjoy housing prices which would be lower indeed, but still not compensating for the incurred surplus of transportation costs. In such a framework, stickiness of prices would slow the adjustment of housing prices in the central part of the agglomeration resulting from the strong associated demand, making insiders better-off than outsiders.
- Households might not be perfectly informed of transportation costs. In the case of car-owners, the coexistence of fixed and variable costs, the issue of maintenance, the cost of credit (when applicable), and the possibility of selling the car to get a new one, are all elements that hide the true cost of car ownership[10]. Besides, many households do not include fixed costs in the equation. They take the fact that they need one, two or three cars for granted, and thus compare

[7] Yet accounting for income leaves the spatial patterns of the housing and transportation expense ratios unaltered.

[8] To be more precise, the econometric model developed by the authors can estimate the number of households who given their current wealth and income would face borrowing constraints. This is achieved by predicting the value of the dwelling the household would be willing to purchase if it were to opt for homeownership now. The two main types of borrowing constraints are considered: the income-based one, which is at the core of our study, and the upfront payment constraint. The income constraint is found to prevail in most cases, corroborating the significance of the CHE policy.

[9] While not established here, the contents of numerous countries’ rent regulations strongly supports the assumption of sticky prices in the housing market.

[10] Based on informal interviews, many car owners have no clue of the order of magnitude of car-related fixed costs expressed in a yearly or in a per km basis.

the cost of transit to the variable cost of private transportation while they could save on one less car. Lastly, the volatility of fuel prices might be misunderstood or poorly taken into account.

- A last argument relying on moral hazard claims that households might not sufficiently consider the issue of bankruptcy (from a social welfare optimizing point of view) because of laws and public policies protecting financially distressed households.

## Overview of the existing literature on CHE and CH+T policies

While there is little work about the regulation of housing expenses on the rental market[11], the effects of borrowing constraints on the demand for housing have largely been documented by the economic literature. These works, described at great length in the survey carried out by Gobillon (2008), focus on the household decision to move and on the subsequent choice of tenure. Typically, households are assumed to choose the tenure and the quantity of housing stock (or housing service in the case of tenancy) that maximize their utility. Moving and/or transaction costs are usually introduced to induce punctual and not continuous housing adjustments (which occur under the form of a residential move[12]) based on a [s,S] rule. When a move occurs, household chooses its type of tenure according to the relative current and future prices of renting and owning and then opts for its quantity of housing consumption. Because borrowing constraints may prevent households from choosing their optimal value of housing stock, they have the double impact of making tenancy more attractive and hindering residential mobility. The latter effect would even prevail according to Zorn (1989) or Gobillon and Le Blanc (2008).

This strand of literature has shed significant light on the behavior of the household under borrowing constraints. It has also collected enough evidence to answer positively to above question 2.[13] Yet it displays two major shortcomings. Most works do not consider the housing supply side, and thus equilibrium mechanisms: in particular the impacts of borrowing constraints on housing prices are usually beyond scope. The omission of space is another significant weakness of most works on this topic, as well as of the few works which specifically deal with the issue of location efficient mortgages[14]. Since housing prices vary within the metropolitan area, borrowing constraints are likely to alter households' location choices. According to Hare (1995), what he calls "clunker mortgages" are even central in accounting for urban sprawl.

## Theoretical framework

Fully understanding the effects of CHE and CH+T policies implies considering the role of space as well as equilibrium mechanisms. This is where a third strand of the economic literature, which focuses on the analysis of land-use regulatory policies, proves useful. Various forms of regulation including restrictions on city size, lot size or density (or alternatively building-height with the introduction of maximal or minimal Floor Area Ratios) have been largely addressed based on the use of the classic framework of urban economics, namely the urban monocentric

[11] It is important to note that the regulation of housing expenses, which operates at the household level, differs from rent control, which is enforced at the dwelling level with rental price ceilings.

[12] The possibility of home improvements as a form of stock adjustment is seldom considered in this strand of literature.

[13] Once again I refer the reader to Gobillon (2008) for conclusive evidence on this issue.

[14] E.g. Blackman and Krupnick (2001)

model[15]. The ability of this model to represent both the demand and supply side of the housing market, and this within a spatial framework, makes it an adequate tool for the analysis of such policies. Recent contributions of Bertaud and Brueckner (2005) and Brueckner (2006) afford a good overview of this significant body of the urban economic literature.

Because the monocentric model has been shown to be particularly suitable to study housing or land use policies in a spatial equilibrium setting, I have chosen it for the present analysis and I will now outline its main characteristics. In the version of the model that I am going to use, households with income $Y$ maximize their utility $U(z,s)$ through a tradeoff between two goods under a budget constraint. The two goods are land ($s$ representing land consumption or lot size) and a composite good denoted by $z$ standing for all other goods. This economic behavior is represented by the following maximization problem:

$$\underset{z,s,r}{Max}\; U(z,s) \quad s.t. \quad R(r)s + z + T(r) = Y$$

While $R(r)$ stands for the relative land rent, $z$ is the numéraire good, and $T(r)$ represents transport costs. The variable $r$ represents location: since locating farther from the central business district (CBD) implies higher transport costs, households typically trade-off between accessibility and housing prices when choosing their location. The essence of this model lies in the endogeneity of housing prices, which vary according to the law of supply and demand. At equilibrium, prices reflect the “spatial advantage” of a given location.

## Scope of the study

The choice of this specific version of the monocentric model holds several assumptions, which I am now going to discuss. This will also provide me with the opportunity to specify the scope of the present study.

### *Transportation network*

Several assumptions are made about the transportation system:

(H1) The transportation network is assumed to be “unimodal” and dense.

(H2) Transportation costs solely include monetary costs.

(H3) These costs are isotropic and determined only by location.

(H4) They increase with distance.

Among the four assumptions (H2) is the most natural for two reasons: firstly, only monetary costs are considered in the CHE and CH+T constraints. Secondly, even if travel-time costs were to be included, Coulombel, Deschamps and Leurent (2007) have established for the Paris Metropolitan Area that neither location nor household income have a significant impact on travel time budgets[16]. (H4) is a usual assumption in a monocentric framework; it was verified for the Greater Paris Region by Coulombel, Deschamps and Leurent (2007).

[15] See Fujita (1989) for a very thorough analysis of this model.

[16] Yet even constant travel-time costs would obviously matter in case of distributed value of times (VOT).

Now let us consider (H1) and (H3). While transportation costs slightly increase with income, this feature is neglected for the sake of simplicity. Besides this point, the strongest assumption is to my view that of “unimodality”. It is important to note that this so-called “unimodality” hypothesis does not necessarily imply one single mode throughout the whole city (within the present stylized model). It rather corresponds to the fact that one location stands for one given amount of transportation costs. Such costs may correspond to either transit costs in the central part or car costs in the suburbs without affecting the validity of the model. Nevertheless, households may not choose between different modes at a given location. Thus, the “unimodality” assumption could be reformulated as the fact that practices of mobility are wholly determined by location. This is not far from being true, especially in the Greater Paris Region: walking and transit prevail in the dense and usually congested areas, while the car often represents the only sensible option for households living in the suburbs. Recent findings by Haas et alii (2006) corroborate this assumption: they establish that transportation costs are more driven by neighborhood characteristics than by household type or income.

[Under development...]

# CAPPING THE HOUSING EXPENSE RATIO

This section analyzes the impact of the CHE policy in terms of:

- Household utility
- Land use: city size, density
- Composition of the household budget

To do so, I first present the constrained housing expenses (CHE) model and solve the household maximization problem. Then I characterize the equilibrium city and proceed to comparative statics in the general case. Lastly, I study the different impacts of the CHE policy in the case of a linear city.

## The Constrained Housing Expense (CHE) model

Let us consider the general case of the monocentric model where $U(z,s)$ and $T(r)$ are assumed to comply with only the classic hypotheses :

- The utility function $U(z,s)$ is concave, strictly increasing with $z$ and $s$, and well-behaved[17].
- Transportation costs $T(r)$ increase with distance $r$ to the CBD.

[17] See definition provided in Fujita (1989) p.99

## *Presentation of the CHE model*

The CHE policy consists in capping expenditures at a given fraction of the household income. In order to study the effects of this policy, I amend the monocentric model with the corresponding constraint:

$$R(r)s \leq \alpha Y \qquad (E1)$$

where $\alpha \in [0,1]$.[18] The lower $\alpha$ is, the harsher the constraint is for households. Two special cases arise:

- $\alpha$=1 yields the original unconstrained model.
- $\alpha$=0 leads to a null housing expense (this special case will be discussed later).

Given the budget constraint of the household, (*E1*) is equivalent to the following constraint, which will prove easier to handle:

$$z \geq (1-\alpha)Y - T(r) \qquad (E2)$$

Consequently, the household maximization problem becomes:

$$\max_{z,s,r} U(z,s) \;\; s.t. \; \begin{cases} z + R(r)s + T(r) = Y \\ z \geq (1-\alpha)Y - T(r) \end{cases} \qquad (E3)$$

## *Notation*

I use the following notations throughout this section:

- A ~ superscript refers to the CHE model as opposed to the unconstrained one (for which no symbol is used).
- The parameter $\alpha$ may be included as an argument for the purpose of comparative statics.
- $S(z,u)$ and $Z(s,u)$ are the inverse functions of $U(z,s)$ relatively to either $s$ or $z$.
- $r_{max}$ is the farthest feasible location: $T(r_{\max}) = Y$ .

I also define two specific subsets of $\Re^+$:

- $E_A(u,\alpha) = \{r / z(r,u) < (1-\alpha)Y - T(r)\}$ is the strictly binding zone, defined as the set of locations $r$ where the Lagrange multiplier associated to (*E2*) is strictly positive.
- $E_I(u,\alpha) = \overline{E}_A(u,\alpha)$ is the nonbinding zone[19], and $\breve{E}_I(u,\alpha)$ its open subset.

where $z(r,u)$ is the solution of the bid-max program for the unconstrained model (see below).

## *The bid-max program*

**Bid rent function of the household** Bid rent functions are defined as usual:

[18] The monocentric is a single time period equilibrium model: savings and borrowings are not taken into consideration, and $\alpha$>1 is consequently irrelevant.

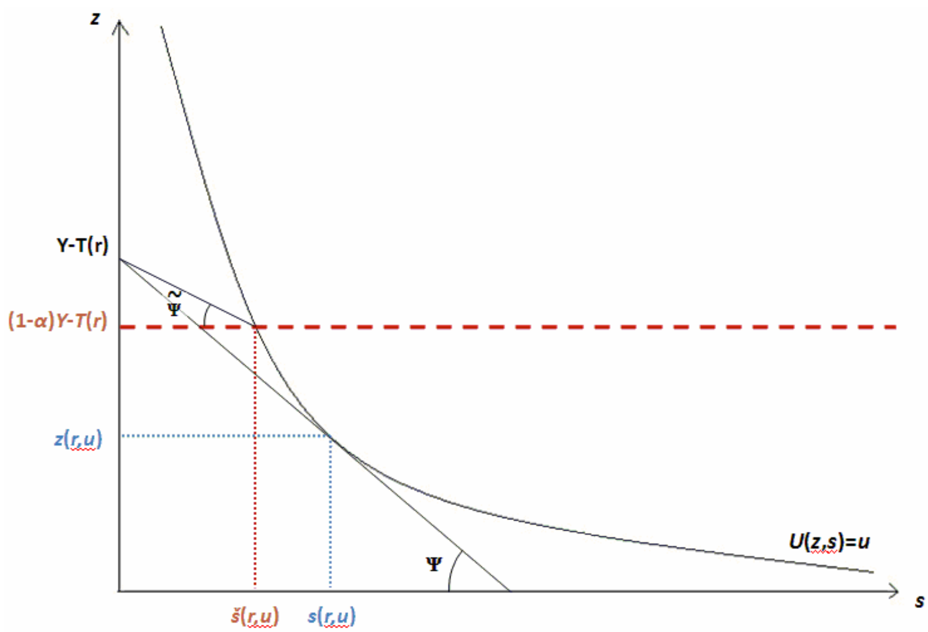
[19] Thus the complementary of $E_A(u,\alpha)$, which is also the zone where the constraint is Inactive.

$$\tilde{\Psi}(r,u)=\max_{z,s}\left\{\frac{Y-T(r)-z}{s}\middle| \begin{array}{l}U(z,s)=u\\ z\geq(1-\alpha)Y-T(r)\end{array}\right\}\qquad (CHE\ model)$$

$$\Psi(r,u)=\max_{z,s}\left\{\frac{Y-T(r)-z}{s}\middle| U(z,s)=u\right\}\qquad (unconstrained\ model)$$

Let us note the argmax of the maximization program $(z(r,u),s(r,u))$ in the unconstrained case, and $(\tilde{z}(r,u),\tilde{s}(r,u))$ in the constrained one. Figure 1 brings an enlightening insight into the economic interpretation of the solutions:

**FIGURE 1: GRAPHIC SOLVING OF THE BID-MAX PROGRAM**




*Source: Author*

The household starts with considering all consumption options providing it with its target utility. This comes back to restricting its choices to the isoutility curve *U*(*z*,*s*)=*u*. Now the household wishes to maximize its bidding power (i.e. its willingness to pay for a unit of land) while considering its remaining budget after accounting for transportation costs subsequent to its location choice (i.e. *Y*-*T*(*r*)). This bid-rent (*Y*-*T*(*r*)-*z*)/*s* is the slope of the "budget line"[20] passing by the two points (0,*Y*-*T*(*r*)) and (*s*,*z*).

In the unconstrained case, maximization of the bid-rent leads to the tangent budget line and the solution *Ψ*(*r*,*u*)= (*Y*-*T*(*r*)-*z*(*r*,*u*))/*s*(*r*,*u*).

In the constrained case, the set of feasible choices is restricted to the quarter-plane $z\geq(1-\alpha)Y-T(r)$ and *s*≥0, that is to say the area above the dashed horizontal line. If

[20] Let's recall that the bid rent program consists in looking for the maximal land rent that a household can afford given a target utility *u* and its income *Y*-*T*(*r*). Therefore the "price" is the output of this program, contrary to the classic economy framework where the price is set, which is why we use this precaution.

$z(r,u) < (1-\alpha)Y - T(r)$ then the unconstrained solution violates the constraint (which is the case in Figure 1). Constrained maximization within the intersection of the quarter-plane and the isoutility curve provides the sub-optimal budget line and $\tilde{\Psi}(r,u) < \Psi(r,u)$. It is also clear that in such a case $\tilde{z}(r,u) > z(r,u)$, and thereby $\tilde{s}(r,u) < s(r,u)$.

**Properties of the bid-max variables** Let us recall the main properties of $s(r,u)$, $z(r,u)$, and $\Psi(r,u)$:

- $s(r,u)$ increases with $r$ and $u$
- $\Psi(r,u)$ decreases with $r$ and $u$
- $z(r,u)$ decreases with $r$ (no specific result regarding the variation in $u$)

Considering that $\tilde{\Psi}(r,u)$ is obtained by adding (E2) to the unconstrained maximization program, we have the following property:

PROPERTY 1

$$\begin{cases} \tilde{z}(r,u) = \max[z(r,u),(1-\alpha)Y - T(r)] \\ \tilde{s}(r,u) = \min[s(r,u), S((1-\alpha)Y - T(r),u)] \\ \tilde{\Psi}(r,u) = \min[\Psi(r,u), \alpha Y / \tilde{s}(r,u)] \end{cases} \tag{E4}$$

While its rigorous proof is put back in the appendix, property 1 may be seen as a shortened and mathematical formulation of the above analysis of Figure 1.

A direct implication of this property is that $\forall(r,u)$, $\tilde{z}(r,u) \geq z(r,u)$, $\tilde{s}(r,u) \leq s(r,u)$ and $\tilde{\Psi}(r,u) \leq \Psi(r,u)$. In plain words, capping housing expenditures reduces[21]:

- the lot size which is bid for.
- the ability to pay for a unit of land.

Furthermore (*E4*) ensures that:

- $\tilde{s}(r,u,\alpha)$ increases with $r$, $u$ and $\alpha$
- $\tilde{\Psi}(r,u,\alpha)$ decreases with $r$, $u$ and increases with $\alpha$
- $\tilde{z}(r,u,\alpha)$ decreases with $r$ and $\alpha$

Conservation of the properties with respect to $r$ and $u$ will be central in demonstrating the existence and uniqueness of the equilibrium land use. Regarding the role of $\alpha$, relieving the constraint increases the maximum level of housing expenditures, which allows households to purchase bigger lots, increase their bid rent, and reduce their consumption of the $z$ good.

[21] This is for a given utility level. Because the CHE constraint alters the equilibrium utility of the city, we will see that implementing the CHE constraint may lead to larger lots than in the unconstrained city at equilibrium.

Additional results relating to the spatial variation of $\tilde{\Psi}(r,u)$ and the characterization of the binding zone $E_A(u,\alpha)$ were found. They are given in the appendix for a more complete analysis of the CHE model in the general case.

## The case of single household type

I investigate in this subsection the standard framework of a closed city with absentee landlords and inhabited by households of a given single type, with income $Y$ and utility function $U(z,s)$. After demonstrating the existence and uniqueness of the equilibrium in the CHE model, I perform comparative statics in order to compare the CHE equilibrium to the original equilibrium.

As is usual, I note as $N$ the number of households and I assume positive land supply $L(r)>0$ at all $r>0$.

### *Existence and uniqueness of the CHE equilibrium*

Demonstrating the existence and uniqueness of the equilibrium in the CHE model is equivalent to proving that there exists a single couple $(\tilde{u},\tilde{r}_f)$ that complies with the following system (following Fujita (1989)):

$$\begin{cases} \tilde{\Psi}(\tilde{r}_f,\tilde{u}) = R_A \\ \int_0^{\tilde{r}_f} \dfrac{L(r)}{\tilde{s}(r,\tilde{u})} dr = N \end{cases} \qquad (E5)$$

The first equality is the boundary condition that determines the edge $\tilde{r}_f$ of the city: at $\tilde{r}_f$ bid rent equates the opportunity cost of land, $R_A$. The second equality corresponds to the population constraint: integration of the density function within the city gives $N$, the total number of households. Note that density $n(r)$ is given by the available land supply divided by the land consumption per household, i.e. $n(r) = L(r) \div \tilde{s}(r,\tilde{u})$

PROPOSITION 1

The CHE monocentric model with single household type admits a unique equilibrium.

PROOF OF PROPOSITION 1

Similarly to Fujita (1989), we consider the outer boundary function $\tilde{b}(u)$ characterized by $\int_0^{\tilde{b}(u)} \dfrac{L(r)}{\tilde{s}(r,u)} dr = N$. $\tilde{b}(u)$ determines the city size for a given target utility $u$. Since $\tilde{s}(r,u)$ exhibits the same required features as $s(r,u)$, that is to say $\tilde{s}(r,u)$ is decreasing in $u$, tends toward $+\infty$ when $u \to +\infty$ and tends toward 0 when $u \to -\infty$, we could proceed similarly to Fujita and show that $\tilde{b}(u)$ is well-defined on an interval $]-\infty,a[$, where possibly $a=+\infty$. Besides, $\tilde{b}(u)$ strictly increases with $u$ and ranges from 0 to $+\infty$ when $u$ ranges from $-\infty$ to $a$.

Then we consider $\tilde{R}_{Bound}(x) = \tilde{\Psi}(x, \tilde{U}(x))$ where $\tilde{U}(x) = \tilde{b}^{-1}(x)$ for $x \in [0, r_{max}[$. $\tilde{R}_{Bound}(x)$ is the land rent at the edge $x$ of a city, the utility of which has been chosen so as to procure the required size $x$. Since $\tilde{b}(u)$ increases strictly with $u$, $\tilde{U}(x)$ also increases strictly with $x$, implying that $\tilde{R}_{Bound}(x)$ is strictly decreasing in $x$ (remember that $\tilde{\Psi}(r,u)$ is decreasing in both $r$ and $u$). Since $\tilde{R}_{Bound}(r_{\max}) = 0$ and $\tilde{R}_{Bound}(x) \underset{x \to 0}{\to} +\infty$, the equation $\tilde{R}_{Bound}(x) = R_A$ admits one and only one solution $\tilde{r}_f$. Eventually, by taking $\tilde{u} = \tilde{U}(\tilde{r}_f)$, it is trivial to check that $(\tilde{u}, \tilde{r}_f)$ satisfies system (*E6*).

## *Comparative statics in the general case*

I determine here the influence of the constraint parameter $\alpha$ on the equilibrium city.

**City Size** Quite intuitively, the CHE policy reduces the city size:

PROPOSITION 2

For any set $(N, Y, R_A)$ the size $\tilde{r}_f(\alpha)$ of the CHE city increases with $\alpha$

PROOF OF PROPOSITION 2

Let us first show that the constrained boundary rent curve $\tilde{R}_{Bound}(x, \alpha_1)$ is below the second one, i.e.: $\tilde{R}_{Bound}(x, \alpha_1) \le \tilde{R}_{Bound}(x, \alpha_2)$

As $\forall(r,u)$, $\tilde{s}(r,u,\alpha_1) \le \tilde{s}(r,u,\alpha_2)$ then $\int_0^x \frac{L(r)}{\tilde{s}(r,u,\alpha_1)} dr \ge \int_0^x \frac{L(r)}{\tilde{s}(r,u,\alpha_2)} dr$.

Since $\int_0^{\tilde{b}(u,\alpha_1)} \frac{L(r)}{\tilde{s}(r,u,\alpha_1)} dr = \int_0^{\tilde{b}(u,\alpha_2)} \frac{L(r)}{\tilde{s}(r,u,\alpha_2)} dr = N$, this implies $\tilde{b}(u,\alpha_1) \le \tilde{b}(u,\alpha_2)$, which in turn implies that the inverse functions are in reversed order, that is to say $\tilde{U}(x,\alpha_1) \ge \tilde{U}(x,\alpha_2)$.

Using the inequality $\forall(r,u)$, $\tilde{\Psi}(r,u,\alpha_1) \le \tilde{\Psi}(r,u,\alpha_2)$, we have:

$$\tilde{\Psi}(x, \tilde{U}(x,\alpha_1), \alpha_1) \le \tilde{\Psi}(x, \tilde{U}(x,\alpha_2), \alpha_1) \le \tilde{\Psi}(x, \tilde{U}(x,\alpha_2), \alpha_2)$$

$$\Rightarrow \quad \tilde{R}_{Bound}(x,\alpha_1) \le \tilde{R}_{Bound}(x,\alpha_2)$$

which is the claimed property. Considering this, demonstration of proposition 2 is straightforward since $\tilde{R}_{Bound}(\tilde{r}_f(\alpha_1), \alpha_1) = \tilde{R}_{Bound}(\tilde{r}_f(\alpha_2), \alpha_2) = R_A$.

Because $\alpha$=1 yields the original model, proposition 2 unveils that the CHE city is smaller than the original one.

**Equilibrium utility** The constraint bearing on housing expenses induces two effects that alter the equilibrium utility level:

- Being constrained in their choices, households achieve a lower utility at a given location and land rent price
- But capping housing expenses has a depressing effect on bid prices, hence on land rents, which tends to increase the utility of the households (which may be seen as an income effect).

Depending on the relative magnitude of these two effects, the resulting utility level of the HE city is higher or lower than that of the original city. We will see in the application to come that both cases are possible: setting a moderate constraint leads to a higher utility level at equilibrium ($\tilde{u}(\alpha) \geq u_{eq}$), but $\tilde{u}(\alpha) < u_{eq}$ arises when the constraint puts an excessive burden on the households.

Despite this difficulty, the following proposition casts some light on this issue:

PROPOSITION 3

For any couple $\alpha_1<\alpha_2$, if the household located at the edge of the $\alpha_2$ city spends less than $\alpha_1 Y$ on housing (i.e. $\tilde{r}_f(\alpha_2) \notin E_A(\tilde{u}(\alpha_2),\alpha_1)$), then the equilibrium utility $\tilde{u}(\alpha_1)$ of the $\alpha_1$ city is superior to the equilibrium utility $\tilde{u}(\alpha_2)$ of the $\alpha_2$ city.

PROOF

For a household located at $\tilde{r}_f(\alpha_2)$, we have the following relations:

- $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_1) = \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_2)$ from $\tilde{r}_f(\alpha_2) \in \overline{E}_A(\tilde{u}(\alpha_2),\alpha_1)$
- $\tilde{\Psi}(\tilde{r}_f(\alpha_1),\tilde{u}(\alpha_1),\alpha_1) = R_A = \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_2)$ (boundary conditions)
- $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_1),\alpha_1) \leq \tilde{\Psi}(\tilde{r}_f(\alpha_1),\tilde{u}(\alpha_1),\alpha_1)$ due to $\tilde{r}_f(\alpha_1) \leq \tilde{r}_f(\alpha_2)$ (proposition 2)

By combining these relations, we have $\tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_1),\alpha_1) \leq \tilde{\Psi}(\tilde{r}_f(\alpha_2),\tilde{u}(\alpha_2),\alpha_1)$, which implies $\tilde{u}(\alpha_1) \geq \tilde{u}(\alpha_2)$.

Proposition 3 gives specific conditions under which the equilibrium utility of the CHE city decreases with $\alpha$. When setting $\alpha_2$=1, it gives a sufficient but not necessary criterion for the CHE city to display a higher equilibrium utility than the unconstrained city (with equilibrium utility $u_{eq}$).

### Housing expenses

Determining the influence of $\alpha$ on housing expenses proves not trivial, because tightening the HE constraint may result in a lower utility level, which in turn may increase the housing expenses of unconstrained households[22]. Nonetheless, when the equilibrium utility rises, it is possible to

[22] While we do mention this possibility, it would be very rare and would involve atypical utility functions. As a matter of fact, we can show thanks to proposition B in Appendix that as long as the binding zone is always a disk (centered on the CBD) whatever the target utility, the amount of housing expenses increases with $\alpha$. This is the case for the family of log-linear utility functions for instance. There is also strong support (but no definite proof at this stage) for the fact that the total differential rate would also increase with $\alpha$ under the previous conditions.

show that tightening the constraint always diminishes the total land rent distributed to the landlords. This also holds when the constraint is binding for the whole city[23].

## Application to a linear city

Considering the limitations of the general case, I now provide a special case consisting in a linear city $L(r)=1$, with linear transport costs $T(r)=ar$, and that is inhabited by $N$ households with income $Y$ and a log-linear utility function $U(z,s)=1/2\log z+1/2\log s$.

The aim of this specification choice is twofold:

- It allows for the analytical derivation of the equilibrium land use while maintaining several flexible parameters ($\alpha$, $a$ for transportation costs, etc.). This in turn allows for an illustration of my previous results and a deeper analysis of the CHE.
- It also aims at developing a reference framework for a further comparison of the CHE and CHT policies, which would not be feasible in the general case.

Consequently, this subsection is divided into two parts: first a derivation of the equilibrium city, then the performance of comparative statics.

I did not choose a disk-shaped city (i.e. $L(r)=2\pi r$) since calculations prove more complex, especially for deriving analytical results.

### *Derivation of the equilibrium city*

After establishing the binding zone, I derive the different variables of interest, that is to say bid-max variables, utility level and city size. They will be used in the next subsection to analyze the equilibrium outcomes.

#### Determination of the binding zone

The log-linear form of the utility function proves particularly convenient to handle thanks to its property of allocating fixed fractions of the disposable income for each expense item[24]. Here $z(r,u)=1/2(Y-ar)$, and the housing expenditure constraint is strictly binding when:

$$r<r_{bind}(\alpha)=\frac{(1-2\alpha)Y}{a} \qquad (E6)$$

Thus:

- if $\alpha\geq 1/2$ the HE constraint is never binding. The CHE model is equivalent to the unconstrained model.

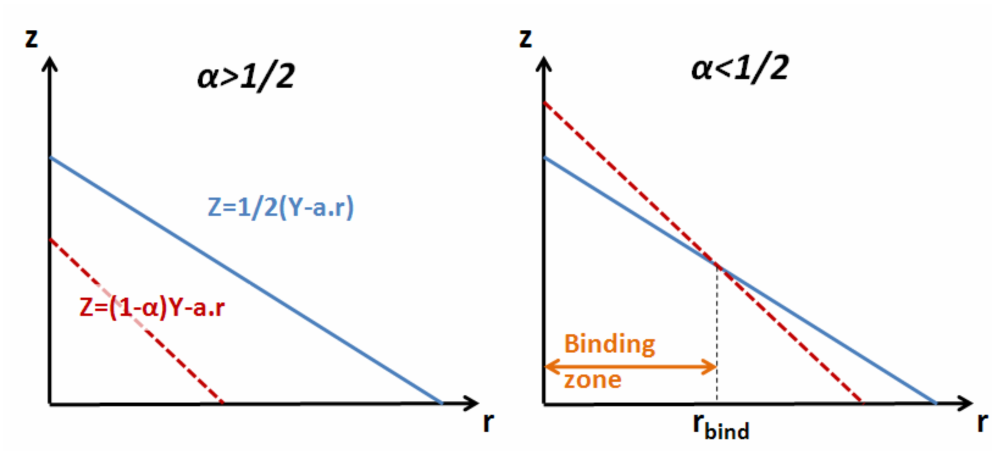
[23] I refer the reader to the Appendix for a demonstration of these results.

[24] For reminder, in the case of a log-linear utility function $U(z,s)=\beta\log z+(1-\beta)\log s$, the household allocates a fraction $\beta$ of its disposable income *Y-T(r)* to the composite good *z*, and the remaining fraction *1-β* to the housing good.

- if $\alpha < 1/2$, only households located closer than $r_{bind}(\alpha)$ are effectively submitted to the HE constraint.

These two cases are illustrated in Figure 2, which depicts the variable *z(r,u)* and the HE constraint for two values of α, one below 1/2 and one above:

**FIGURE 2 : ILLUSTRATION OF THE BINDING ZONE**




*Source: Author*

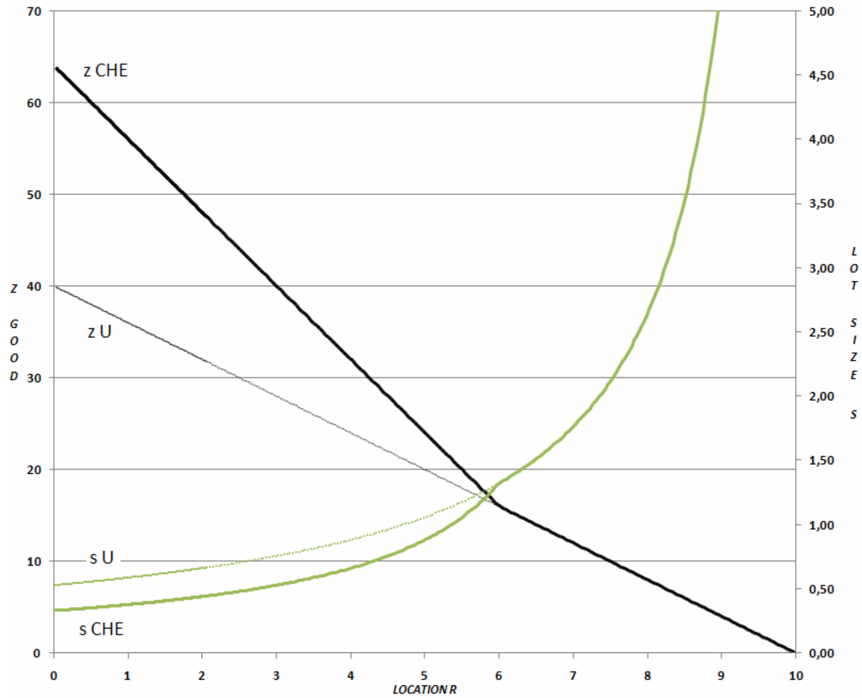
Based on its preferences specified by our linear model, the household chooses to spend half its disposable income $Y$-$T(r)$ on housing expenses. Therefore when $\alpha \geq 1/2$ the constraint line is always below the $z(r,u)$ line as is illustrated on Figure 2, meaning that the constraint is never violated. On the other hand, if $\alpha < 1/2$, the constraint is binding for locations close to the CBD. Because the constraint bears on total income and not on disposable income, it becomes less and less binding as transport costs grow until location $r_{bind}(\alpha)$ is reached, where the constraint is not binding anymore.

**Characterization of the equilibrium** Resolution of the bid-max program brings about the following formulae:

$$r \leq r_{bind}(\alpha) \begin{cases} \tilde{z}(r,u) = (1-\alpha)Y - ar \\ \tilde{s}(r,u) = e^{2u} / \tilde{z}(r,u) \\ \tilde{\Psi}(r,u) = e^{-2u} \alpha Y \{(1-\alpha)Y - ar\} \end{cases} \qquad r \geq r_{bind}(\alpha) \begin{cases} \tilde{z}(r,u) = (Y - ar)/2 \\ \tilde{s}(r,u) = e^{2u} / \tilde{z}(r,u) \\ \tilde{\Psi}(r,u) = e^{-2u}(Y - ar)^2 / 4 \end{cases} \qquad (E8)$$

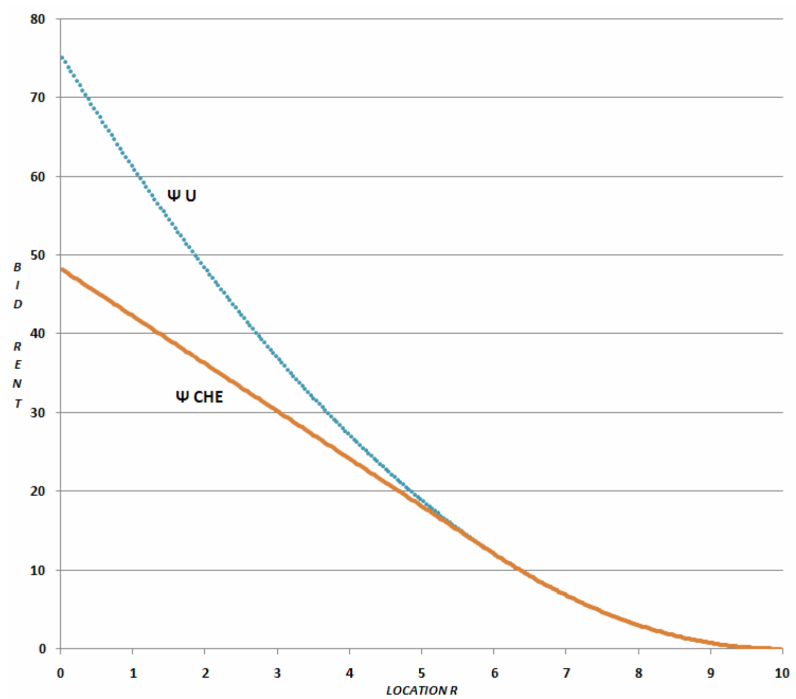
Figure 3 illustrates these solutions for the following settings (which will constitute the reference model): $N$=10, $Y$=80, $a$=8 and $R_A$=20. In addition to that I choose $\alpha$=0.20 and $u$=21.21 (which corresponds to the equilibrium utility of the CHE model for the chosen settings). For these settings $r_{max}$=10 and $r_{bind}$=6.

**FIGURE 3A : LOT SIZE AND Z GOOD CONSUMPTION IN THE UNCONSTRAINED (U) AND CHE MODELS**




*Source: Author's calculations*

**FIGURE 3B : BID RENT FUNCTIONS**




*Source: Author's calculations*

As previously observed, for a given utility the HE constraint reduces both the lot size and the bid rent inside the binding zone, and increases the consumption of the composite good. Outside the binding zone, we find the same solutions for the CHE and unconstrained models.

We are now ready to characterize the equilibriums.

PROPOSITION 4

In the applied case, the equilibrium is characterized as follows:

| | $\alpha \le \alpha_{cr}$ | $\alpha \in [\alpha_{cr}, 1/2]$ | $\alpha \ge 1/2$ |
|---|---|---|---|
| $e^{2\tilde{u}}$ | $\dfrac{\alpha^2 Y^2}{R_A^2}\left(\sqrt{a^2N^2+\left(\dfrac{1-\alpha}{\alpha}\right)^2 R_A^2}-aN\right)$ | $\dfrac{Y^2\left(1-2\alpha+2\alpha^2\right)}{2(aN+R_A)}$ | $\dfrac{Y^2}{4(aN+R_A)}$ |
| $\tilde{r}_f$ | $\dfrac{Y}{a}\left(1-\alpha\left(1+\sqrt{\left(\dfrac{aN}{R_A}\right)^2+\left(\dfrac{1-\alpha}{\alpha}\right)^2}-\dfrac{aN}{R_A}\right)\right)$ | $\dfrac{Y}{a}\left(1-\sqrt{\dfrac{R_A}{aN+R_A}\left(2-4\alpha+4\alpha^2\right)}\right)$ | $\dfrac{Y}{a}\left(1-\sqrt{\dfrac{R_A}{aN+R_A}}\right)$ |

where $\alpha_{cr}=\left(1+\sqrt{1+\dfrac{2aN}{R_A}}\right)^{-1}$

Calculations are based on the distinction of 3 cases:

- $\alpha \ge 1/2$ yields the unconstrained model
- If $\alpha \in [\alpha_{cr}, 1/2]$, the edge of the city is beyond $r_{bind}(\alpha)$
- If $\alpha \le \alpha_{cr}$, the HE constraint is active for the whole city

## *Comparative statics for the applied model*

After computing the different equilibriums, we can proceed to a more precise analysis of the role of α.

**Utility level** In the applied model, while an appropriate choice of $\alpha$ increases the households' utility compared to the unconstrained city, setting $\alpha$ to too low a value usually decreases it.

PROPERTY 2

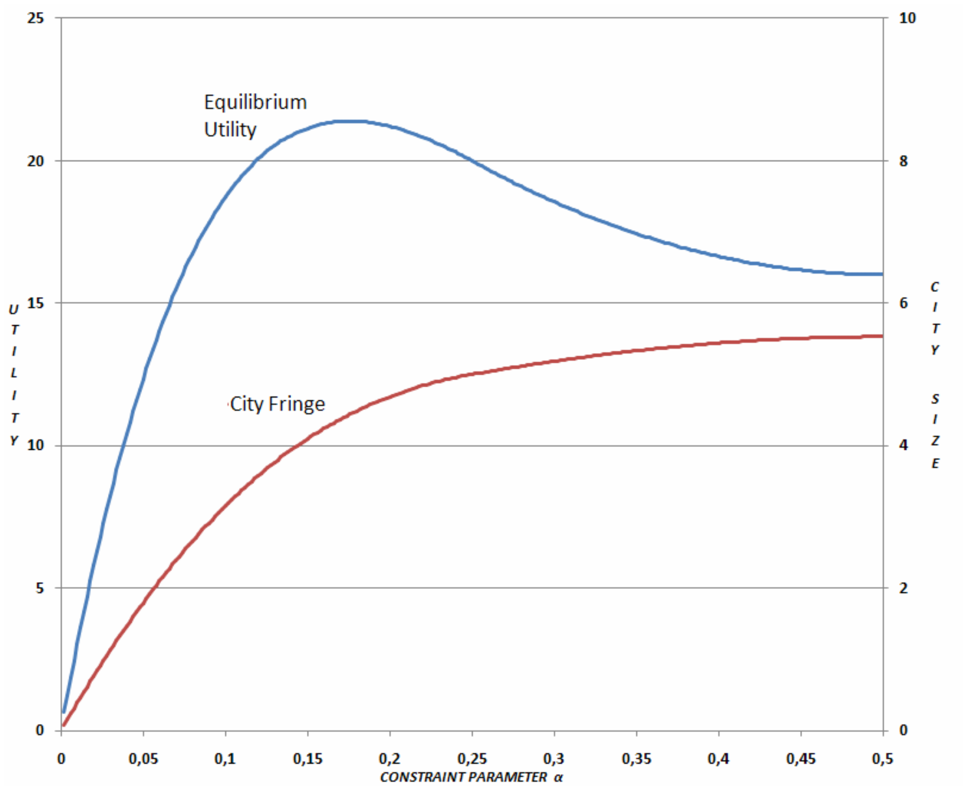
For any given set of parameters ($N,Y,R_A>0,a$), the equilibrium utility $\tilde{u}(\alpha)$ of the CHE city strictly decreases on $[\alpha_{cr},1/2]$ with $\tilde{u}(1/2)=u_{eq}$. It is maximal for $\alpha_{max}<\alpha_{cr}$, with $\tilde{u}(\alpha_{\max})>u_{eq}$. Furthermore, $\tilde{u}(\alpha)\underset{\alpha\to 0}{\to}-\infty$

If $R_A=0$, $\tilde{u}(\alpha)$ strictly decreases on $]0,1/2]$ and therefore is maximal when $\alpha$ tends toward 0.

Demonstration (proof omitted) is carried out by using proposition 4.

Figure 4 depicts the variations of $e^{2\tilde{u}(\alpha)}$ for the reference model (corresponding to *N*=10, *Y*=80, *a*=8 and $R_A$=20); for these settings $\alpha_{cr}$=0.25.

**FIGURE 4 : UTILITY LEVEL AND SIZE OF THE CHE CITY**




*Source: Author's calculations*

One can check that 0.176=$\alpha_{max}$<$\alpha_{cr}$ =0.25, which corroborates property 2.

Property 2 confirms proposition 3: whenever the city fringe is beyond the binding zone (i.e. $\tilde{r}_f(\alpha) \geq r_{bind}(\alpha)$ which is equivalent to $\alpha \geq \alpha_{cr}$ ), the CHE city displays a higher utility level than the unconstrained city. On the other hand, if the city is entirely constrained, reducing $\alpha$ proves worthwhile at first but quickly utility dwindles.

In fact, when the outside competition for land (the agricultural sector) is mild, the constraints put on households' choices are more than compensated for by the drop in prices that results from less fierce competition for land within the binding zone. This increases the utility of all households. Conversely, if the competitiveness of the households is too weak relative to the agricultural sector, the reduction of the city size is exacerbated and leads to declining utility.

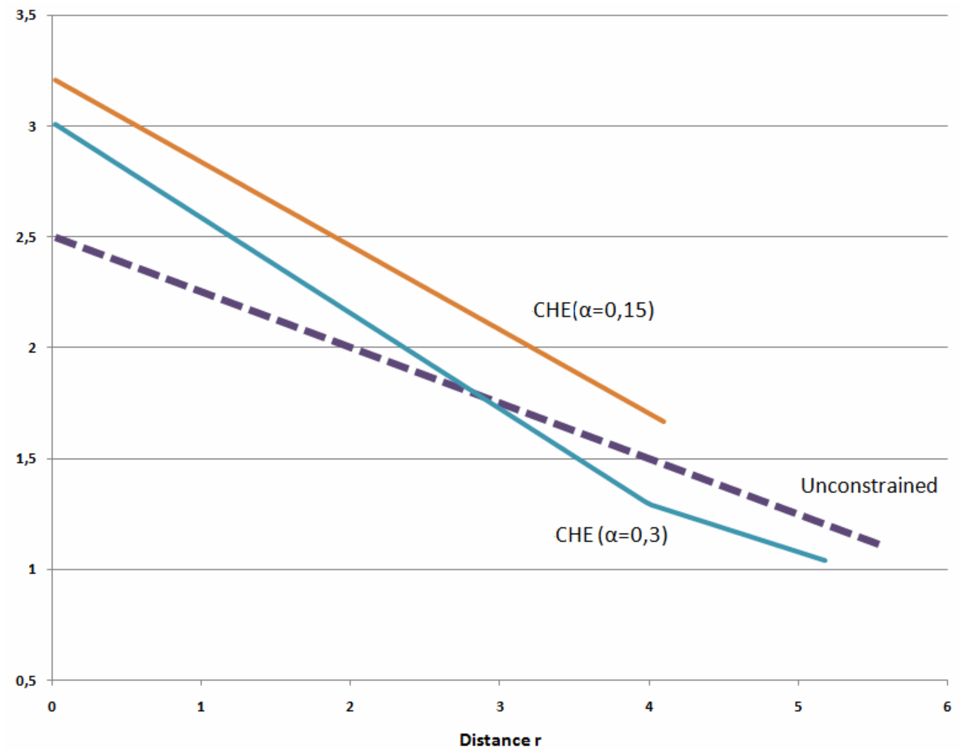
**City Size and Density** Unlike the utility level, tightening the housing budget constraint always reduces city size (see proposition 2) as shown on Figure 2. When $\alpha$ is decreased from 0.5 to 0, city size shrinks, and this phenomenon is accentuated when $\alpha$<$\alpha_{max}$, i.e. when the HE constraint becomes too significant relatively to the need to compete with the agricultural sector for land.

Reduction of the city size is achieved in different ways according to the value of $\alpha$:

- When the utility level increases, owing to higher densities near the CBD that overweigh lower densities in the suburban area
- When the utility level decreases, density uniformly rises throughout the city

Figure 5 illustrates the equilibrium densities for the original city, and two CHE cities with $\alpha$=0.3 and $\alpha$=0.15:

**FIGURE 5 : INFLUENCE OF $\alpha$ ON DENSITY IN THE REFERENCE MODEL**




*Source: Author's calculations*

When $\alpha$ is chosen within [$\alpha_{max}$,1/2], one observes as predicted higher densities near the CBD, but lower densities in the suburbs. When $\alpha$ is chosen within [0,$\alpha_{max}$], density rises throughout the whole town.

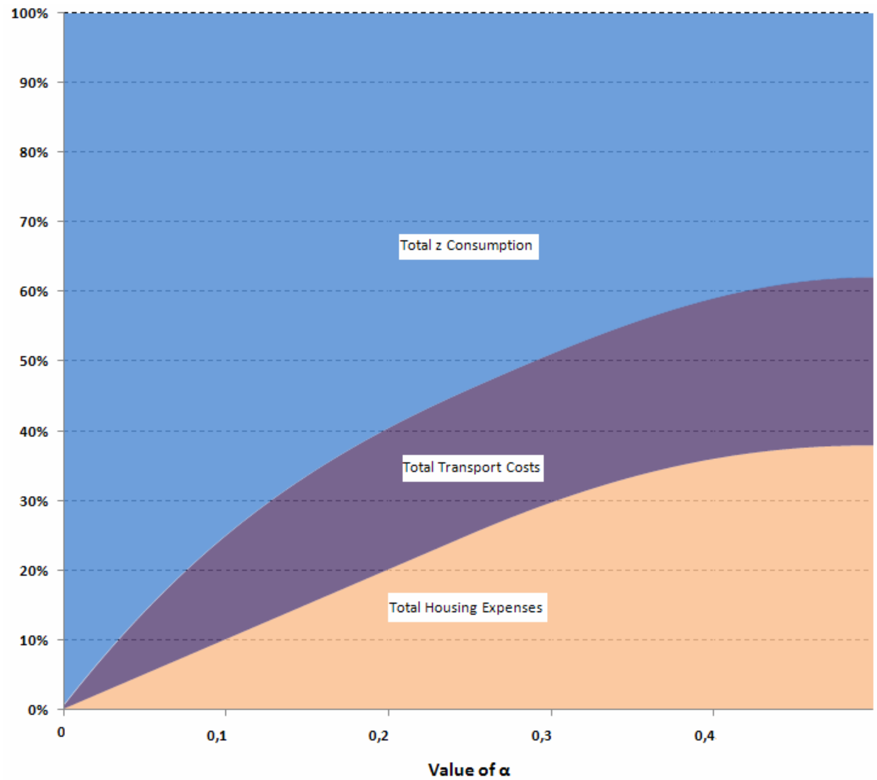
**Average composition of household budgets** Since the HE policy was designed to cap housing expenses so as to ensure the solvency of the households, one key issue is the average composition of the household budget at equilibrium land use (proof omitted):

PROPERTY 3

For any given set ($N$,$Y$,$R_A$,$a$), the average expenditures for both housing and transportation are rising with $\alpha$, inducing a declining consumption of the $z$ good

Figure 6 exemplifies Property 3 for the reference model. For high values of $\alpha$ (between 0.4 and 0.5), decreasing $\alpha$ only slightly reduces the housing and transportation budget shares, because a limited number of households is affected by the constraint. If $\alpha$ further decreases (approximately until $\alpha_{max}$=0.176), housing expenses decrease more sharply while transport costs are moderately affected. On this interval, decreasing $\alpha$ has a more significant depressing effect on prices than on lot sizes. Below $\alpha_{max}$, the constraint weighs more on the households' choices of lot size, resulting in smaller cities and lower transportation and housing expenditures.

**FIGURE 6 : INFLUENCE OF $\alpha$ ON THE AVERAGE COMPOSITION OF THE HOUSEHOLD'S BUDGET**




*Source: Author's calculations*

**Concluding remarks for the CHE model** In sum, capping housing expenditures has the twofold effect of distorting households' residential choices (regarding lot size) and reducing equilibrium prices of the housing market. At first, the latter effect overweighs the former, leading to an increase in the utility level while the global structure of the city (size, use of transportation) remains relatively unchanged. Nevertheless decreasing $\alpha$ further eventually drastically reduces lot sizes, resulting in a drop in both utility level and city size.

Of course, the increase in utility generated by ad hoc values of $\alpha$ has a cost: the total housing expenses distributed to landlords (more precisely, the adequate notion would be the total differential land rent presented in (*9*) but to maintain the simplicity of the argument I refer to total housing expenses). By enforcing reduced prices, the CHE policy proceeds to a form of redistribution from the landlords to the households similar to the public ownership case described in Fujita (*9*), where rents are redistributed to the households. This redistribution is at the origin of the higher utility than in the unconstrained city with absentee landlords.

Given the analysis of the Herbert-Stevens model (*9*), we know that utility of the closed-city model is maximized in the case of public ownership. No other configuration of the city, and in particular the CHE city, can outperform this one in utility grounds. Yet, the CHE policy is widely enforced and accepted, while such is not the case for the public ownership of land. Thus it is an interesting policy that can improve the solvency of the households and increase their utility at the same time, though being detrimental to landlords.

# CONSTRAINING THE SHARE OF BOTH HOUSING AND TRANSPORTATION

Let us now turn our attention to an alternative policy, consisting in capping the total share of housing and transportation expenditures. As previously, I examine the impacts of such a policy on the equilibrium city, in particular the influence of the constraint parameter $\mu$.

Considering the similarities between the CH+T and CHE policies, I first present the main results, omitting the proofs, and then focus on the application to the linear city.

## The Constrained Housing+Transportation (CH+T) model

### *Overview of the CH+T model*

The CH+T model is a monocentric model amended with the following additional constraint:

$$R(r)s + T(r) \leq \mu Y \qquad (E9)$$

The sum of housing and transportation expenditures is capped to within a fraction $\mu$ of the household's income $Y$. The case $\mu \geq 1$ is consequently tantamount to the classic unconstrained model.

Enforcement of such a policy yields the same effects as the CHE policy:

- constraining lot size choices of the households (actually it sets a *de facto* a minimal density)
- lowering prices

Yet this time it can be shown that the constraint concerns above all the households in the suburban area (starting from the edge of the city). The tighter it becomes, the more households it affects until covering the whole city.

### *Equilibrium features in the general case*

The CH+T land use equilibrium exists and is unique. The only specific property of the equilibrium in the general case is that city size increases with $\mu$, which is the result of the minimal density enforcement. The HT constraint induces the two same economic forces that influence the equilibrium utility level:

- By obliging the households to make sub-optimal choices, the latter achieve a lower utility level
- But capping HT expenses generates a “discount” on housing prices, which is beneficial to the households

Nevertheless, unlike the HE policy, there is no obvious case where one can predict the outcome. The same goes for housing expenses.

## Application to a linear city

In order to compare the CHE and CH+T policies, let us come back to the application where: $U(z,s) = 1/2\log z + 1/2\log s$, $T(r) = ar$ and $L(r) = 1$.

### *Derivation of the equilibrium city*

**Determining the binding zone** The HT constraint is strictly binding when:

$$r > r_{bind}(\mu) = \frac{(2\mu - 1)Y}{a} \qquad (E10)$$

Hence the following cases:

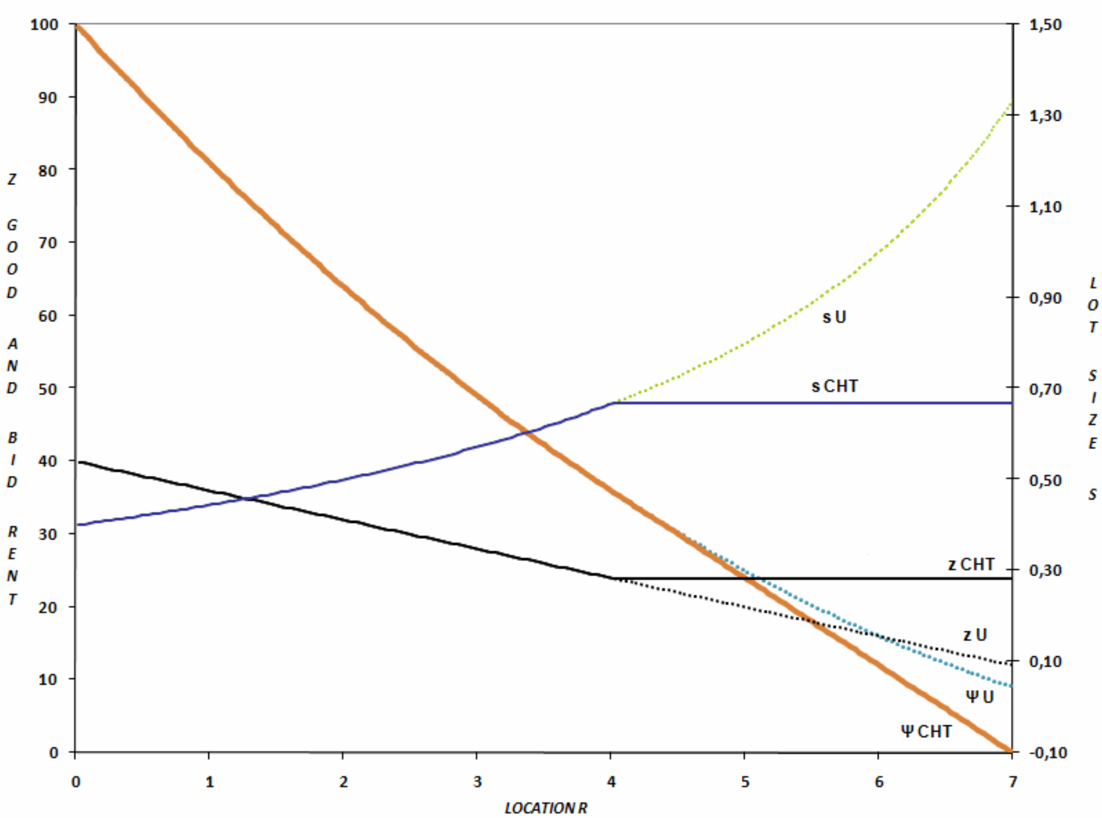
- If $\mu$<1/2 the HT constraint is always binding.
- If $\mu \geq 1/2$, households located beyond $r_{bind}(\mu)$ are bound by the HT constraint.

**Characterization of the equilibrium** Resolution of the bid-max program brings about the following system of equations:

$$r \leq r_{bind}(\mu) \begin{cases} \hat{z}(r,u) = (Y - ar)/2 \\ \hat{s}(r,u) = e^{2u} / \hat{z}(r,u) \\ \hat{\Psi}(r,u) = e^{-2u}(Y - ar)^2 / 4 \end{cases} \qquad r \geq r_{bind}(\mu) \begin{cases} \hat{z}(r,u) = (1-\mu)Y \\ \hat{s}(r,u) = e^{2u} / \hat{z}(r,u) \\ \hat{\Psi}(r,u) = e^{-2u}(1-\mu)Y(\mu Y - ar) \end{cases} \qquad (E11)$$

Figure 7 illustrates (*E11*) for the settings of the reference model ($N$=10, $Y$=80, $a$=8, $R_A$=20). Moreover I choose $\mu$=0.70 and $u$=16 (corresponding to the equilibrium utility of the CH+T reference model for the selected value of $\mu$), which yields $r_{max}$=7 and $r_{bind}$=4.

**FIGURE 7 : BID RENT, LOT SIZES AND Z GOOD CONSUMPTIONS IN THE UNCONSTRAINED (U) AND CH+T MODELS**




*Source: Author's calculations*

As previously stated, for $r \leq r_{bind}$ the HT constraint is ineffective, leading to the same solution that in the original case. For $r \geq r_{bind}$, the constraint becomes active, leading to constant values for lot size and $z$ good consumption.

From (*E11*), we can derive the equilibrium utility and city size of the CH+T city:

PROPOSITION 5

In the applied case, the equilibrium is characterized as follows:

| | $\mu \leq 1/2$ | $\mu \in [1/2, \mu_{cr}]$ | $\mu \geq \mu_{cr}$ |
|---|---|---|---|
| $e^{2\hat{u}}$ | $\dfrac{\mu(1-\mu)Y^2}{aN + R_A}$ | $\dfrac{Y^2}{4(aN + R_A)}$ | $\dfrac{Y^2}{4(aN + R_A)}$ |
| $\hat{r}_f$ | $\dfrac{\mu Y}{a}\dfrac{aN}{aN + R_A}$ | $\dfrac{Y}{a}\left\{\mu - \dfrac{1}{4(1-\mu)}\dfrac{R_A}{(aN + R_A)}\right\}$ | $\dfrac{Y}{a}\left(1 - \sqrt{\dfrac{R_A}{aN + R_A}}\right)$ |

where $\mu_{cr} = 1 - \frac{1}{2}\sqrt{\frac{R_A}{aN + R_A}}$

Calculations are based on three distinct cases:

- $\mu \geq \mu_{cr}$ yields the unconstrained model.
- If $\mu \in [1/2, \mu_{cr}]$, $r_{bind}(\mu) \geq 0$, thus households living in the central area of the city are unconstrained.
- If $\mu \leq 1/2$, $r_{bind}(\mu) \leq 0$. The HT constraint is active for the whole city.

## *Comparative statics for the applied model*

**Utility level** Starting from $\mu$=1, while decreasing $\mu$ has no impact at first on the utility level of the households (compared to the unconstrained city), for $\mu \leq 1/2$ it decreases the utility level.

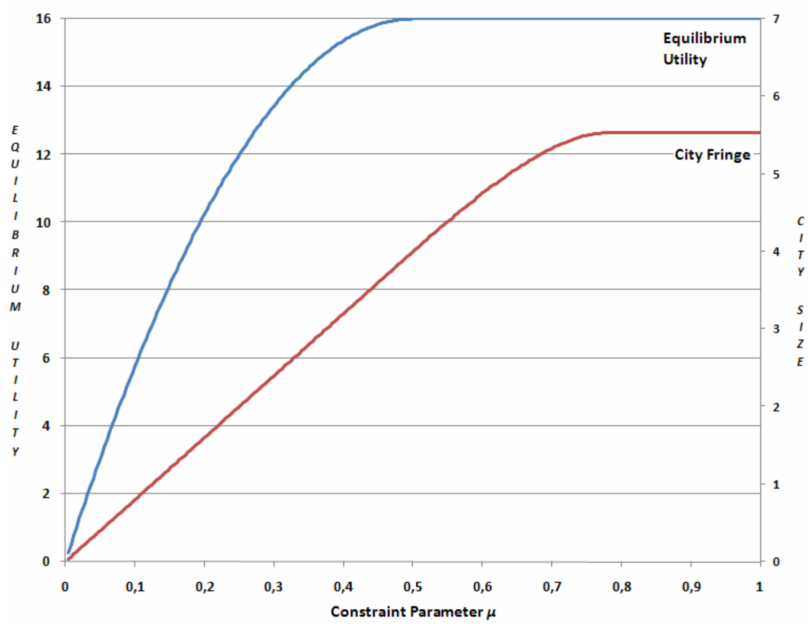
PROPERTY 4

For any given set of parameters $(N, Y, R_A, a)$, the equilibrium utility $\hat{u}(\mu)$ of the CH+T city strictly increases with $\mu$ on [0,1/2] and is constant for $\mu \geq 1/2$.

Considering Proposition 5, Property 4 is straightforward. Yet, this property proves enlightening for it states that, for $\mu \in [1/2, \mu_{cr}]$, the capped lot sizes perfectly compensate for the "discount" on housing prices given to the households. If $\mu \leq 1/2$, the constraint becomes too strong, inducing a drop in the utility level.

Figure 8 depicts the variations of $e^{2\hat{u}(\mu)}$ for the reference model, where $\mu_{cr}$=0.776:

**FIGURE 8 : UTILITY LEVEL AND SIZE OF THE CH+T CITY**

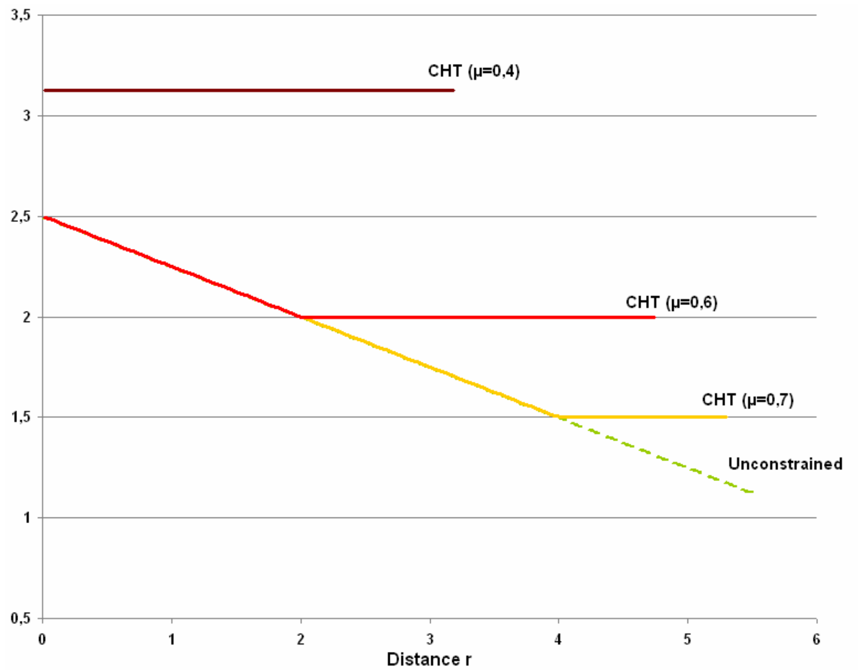



*Source: Author's calculations*

**City Size and Density** As previously stated for any given set of parameters ($N,Y,R_A,a$), the city size increases with $\mu$, which is illustrated for the reference model in Figure 6. On $[1/2,\mu_{cr}]$ the city size is fairly well approximated by a linear function, which demonstrates the efficiency of this policy in reducing the city size (relatively to the CHE policy).

Similarly to the CHE policy, the CH+T policy alters the spatial distribution of density, but this time it sets a minimum density level that affects either the most remote part of the city ($\mu \in [1/2, \mu_{cr}]$), or the whole city ($\mu \leq 1/2$). This phenomenon is illustrated in Figure 9:

**FIGURE 9 : INFLUENCE OF μ ON EQUILIBRIUM DENSITY IN THE REFERENCE MODEL**




*Source: Author's calculations*

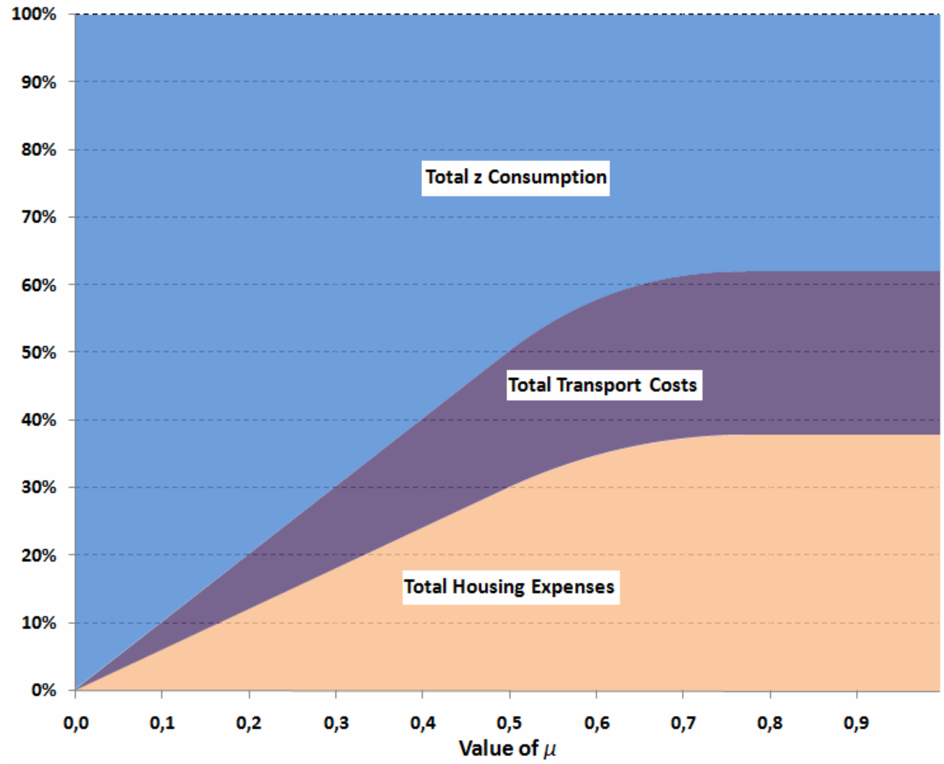
**Average composition of the household budgets** Similarly to the HE policy, the HT policy brings about lower housing and transportation expenditures for the households, as illustrated in Figure 10 (proof of following property omitted):

PROPERTY 5

For any given set ($N,Y,R_A,a$), the average expenditures for both housing and transportation increase with $\mu$, while the average consumption of the composite good decreases with $\mu$.

When the HT policy becomes active (starting from $\mu_{cr}$), increasing the constraint results in decreasing transport costs and housing expenses. Unlike the HE policy, the two items decrease simultaneously in similar proportions, which results from capping housing and transportation expenses instead of only housing expenditures. When $\mu$ falls below 1/2, the decrease steepens.

**FIGURE 10: INFLUENCE OF $\mu$ ON THE AVERAGE COMPOSITION OF THE HOUSEHOLD'S BUDGET**




*Source: Author's calculations*

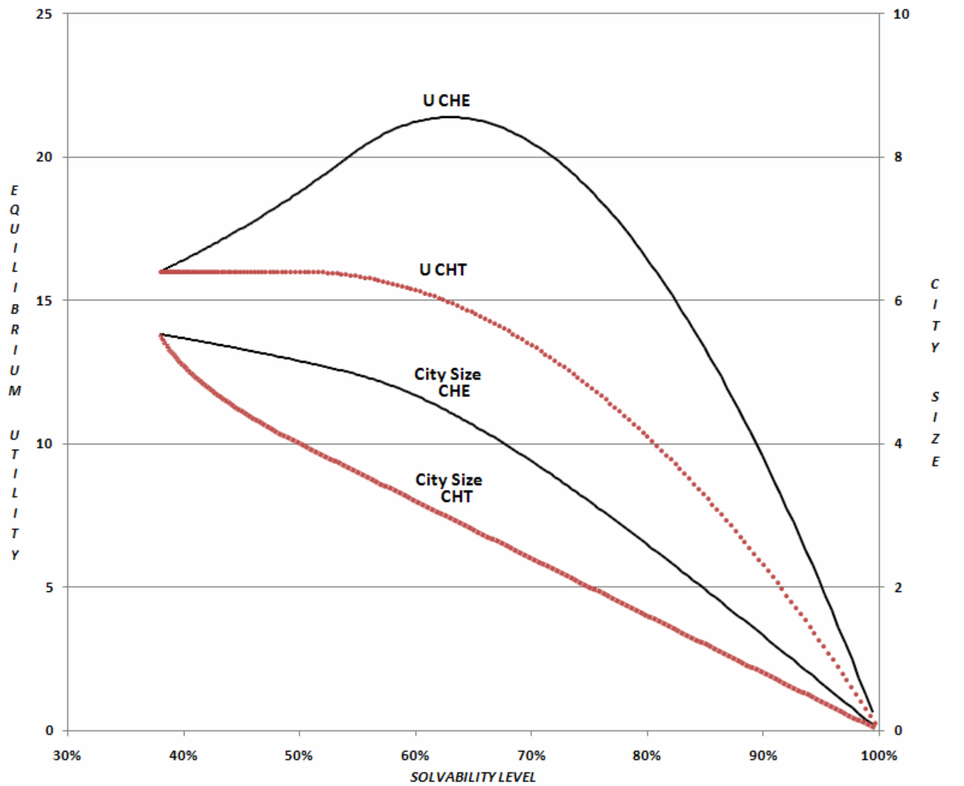
# CONCLUSIONS

Let us compare the main results concerning a linear city implementing either the CHE or the CH+T policy. I shall present the equilibrium utility level and city size for the reference applied model with a target solvency level, defined as the fraction of income remaining after paying the housing and transportation costs (Figure 11).

In both models, increasing the solvency of households is done by tightening the corresponding constraint, until reaching the maximal solvency level of 100% for a value of the constraint parameter equal to zero. Since a constraint parameter of one yields the unconstrained model in both cases, each pair of curves starts at the same point.

Figure 11 reveals that the CHE policy provides a greater utility for any target level of solvency, but at the cost of a greater city size. Regarding land use, while both policies induce shrinkage of the city, the CHE policy steepens the density curve when the utility rises, while the CH+T policy always flattens the density curve. Moreover, the CH+T policy is more efficient in reducing the city size, and consequently transportation costs and energy consumption.

**FIGURE 11 : UTILITY LEVEL AND SIZE OF THE CHE AND CH+T CITIES ACCORDING TO TARGET SOLVENCY LEVEL**




*Source: Author*

Consequently, the linear model suggests that while the CHE policy is beneficial to households on utility grounds, and does improve their solvency while simultaneously reducing city size and transport expenses, the CH+T policy makes a better tool to struggle against urban sprawl and transportation costs. Because the model includes neither several externalities such as pollution or congestion, nor the scarcity of energy, the CH+T policy might prove a better choice than the CHE policy depending on the objectives of the local authorities, and this despite utility considerations. In all cases, both policies can be used to secure a target level of solvency for the households.

While the model developed in the present paper was helpful in understanding the CHE and CH+T policies, several improvements are planned to assess the effects of these policies in more realistic settings:

- Considering the case of a disk-shaped city, which will complicate the calculations.
- Calibrating the utility functions and the parameters against existing metropolitan areas.
- Considering the policy impacts in terms of car ownership decision and modal choice, especially for the CH+T policy.

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# APPENDIX

## Proofs

PROOF OF PROPERTY 1

Demonstration of property 1 is straightforward:

- If $z(r,u) \geq (1-\alpha)Y - T(r)$ the CHE model and the unconstrained model yield the same solutions for the bid-max program, and $\tilde{z}(r,u) = z(r,u)$
- If $z(r,u) < (1-\alpha)Y - T(r)$, the constraint (E2) is binding in the CHE bid-max program and $\tilde{z}(r,u) = (1-\alpha)Y - T(r)$.

Expressing these two alternatives into a single formulation yields:

$$\tilde{z}(r,u) = \max[z(r,u),(1-\alpha)Y - T(r)]$$

$\tilde{s}(r,u) = \min[s(r,u), S((1-\alpha)Y - T(r),u)]$ and $\tilde{\Psi}(r,u) = \min[\Psi(r,u), \alpha Y / \tilde{s}(r,u)]$ are straightforward considering that either we are faced with the unconstrained case, or (E2) is binding.

## Additional results for the CHE model

Additional results relating to the spatial variation of $\tilde{\Psi}(r,u)$ and the characterization of the binding zone $E_A(u,\alpha)$ are presented here and complete the analysis of the CHE model in the general case.

### *Spatial variation of the bid rent function*

Property A gives us the partial derivative of $\tilde{\Psi}(r,u)$ with respect to *r*:

PROPERTY A

$$\tilde{\Psi}_r(r,u) = -\frac{T'(r)}{\tilde{s}(r,u)} \frac{\tilde{\Psi}(r,u)}{\Psi^*(r,u)}$$

where $\Psi^*(r,u) = -\frac{\partial Z}{\partial s}(\tilde{s}(r,u),u) = \left\{-\frac{\partial S(\tilde{z}(r,u),u)}{\partial z}\right\}^{-1}$

PROOF

Proof of property A is based on the distinction of three cases: $r \in \breve{E}_I(u)$, $r \in E_A(u)$ and $r \in E_I(u) - \breve{E}_I(u)$.

If $r \in \breve{E}_I(u)$ we are faced with the unconstrained case. Application of the envelop theorem gives (Fujita (1989) equation (2.27)):

$$\tilde{\Psi}_r(r,u) = -\frac{T'(r)}{\tilde{s}(r,u)} = -\frac{T'(r)}{\tilde{s}(r,u)} \frac{\tilde{\Psi}(r,u)}{\Psi^*(r,u)}$$

since $\tilde{\Psi}(r,u)=\Psi(r,u)=\Psi*(r,u)$ on $E_I(u)$.

If on the contrary $r\in E_A(u)$, the constraint ($E2$) is binding. Partial derivation of $\tilde{\Psi}(r,u)=\alpha Y/\tilde{s}(r,u)$ relatively to $r$ gives:

$$\tilde{\Psi}_r(r,u)=-\frac{\alpha Y}{\tilde{s}(r,u)^2}\frac{\partial\tilde{s}(r,u)}{\partial r}=-\frac{\alpha Y}{\tilde{s}(r,u)^2}\frac{\partial\tilde{z}(r,u)}{\partial r}\frac{\partial S(\tilde{z}(r,u),u)}{\partial z}=\frac{\alpha Y}{\tilde{s}(r,u)^2}T'(r)\left(\frac{-1}{\Psi*(r,u)}\right)$$

By substituting $\alpha Y/\tilde{s}(r,u)$ by $\tilde{\Psi}(r,u)$, we obtain the desired relationship:

$$\tilde{\Psi}_r(r,u)=-\frac{T'(r)}{\tilde{s}(r,u)}\frac{\tilde{\Psi}(r,u)}{\Psi*(r,u)}$$

where $\Psi*(r,u)$ is the land rent that rationalizes the bundle ($\tilde{z}(r,u),\tilde{s}(r,u)$) in the unconstrained household bid-max program under target utility $u$. When ($E2$) is binding, bid rents are ranked in the following order: $\tilde{\Psi}(r,u)<\Psi(r,u)<\Psi*(r,u)$.[25] A graphic interpretation of $\Psi*(r,u)$ can be given thanks to Figure 1: $\Psi*(r,u)$ is the slope (in absolute terms) of the isoutility curve at the point ($\tilde{s}(r,u),\tilde{z}(r,u)$).

The untreated case $r\in E_I(u)-\breve{E}_I(u)$ ($E_I(u)$ deprived of its open component) can be treated using continuity considerations.

The standard equation $\Psi_r(r,u)=-T'(r)/s(r,u)$ is amended by a distortion factor $\tilde{\Psi}(r,u)/\Psi*(r,u)$ varying between 0 and 1, 1 corresponding to the unconstrained case[26]. Considering the discussion on $\Psi*(r,u)$ provided in the preceding proof, the more binding the housing expense constraint, the lower the distortion factor.

## *Determination of the binding zone*

The analysis of the CHE model naturally raises the question of the specification of the binding zone, i.e. the set of locations where households effectively cap their housing expenditures at the maximum allowed $\alpha Y$. Is this zone near the center of the city? Or near the edge? And is this zone even a convex set?

Unfortunately, the standard monocentric model exposed in the introduction provides no answer to this matter in the general case. It is even more than plausible that one may encounter any kind of situation depending on the specification of the utility function. As a matter of fact, two economic forces counterbalance each other: when a household gets farther from the city center, its housing consumption $s(r,u)$ increases while equilibrium land rent $R(r)$ decreases. Thus nothing general can be said about the variation of the housing expenditure $R(r)s(r,u)$. Depending on the preferences of the household, the zone where $R(r)s(r,u)\geq\alpha Y$ can take many forms from a segment near the centre or near the edge, to a collection of segments dispersed over space.

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[25] $\Psi(r,u)<\Psi*(r,u)$ comes from the fact that $-\frac{\partial Z}{\partial s}(s,u)$ is decreasing in $s$ thanks to the concavity of the utility function.

[26] Since $\tilde{\Psi}(r,u)=\Psi(r,u)=\Psi*(r,u)$ on $E_I(u)$.

Despite this difficulty, a few things can be said about the binding zone. First, $E_A(u)$ is an open set of $\Re^+$: because $E_A(u)$ can also be defined as the set $\{r / z(r,u)-(1-\alpha)Y - T(r) < 0\}$, the continuity of $z(r,u)-(1-\alpha)Y - T(r)$ ensures this result. Similarly, $E_I(u)$ is a closed set of $\Re^+$. Secondly, beyond a certain location (such that $T(r)=(1-\alpha)Y$ ), the constraint can never be binding since the remaining disposable income is too low. Lastly, proposition A gives a utility-based condition under which the binding zone has the shape of a disk (logically centered on the CBD).

PROPOSITION A

$\forall s > 0, \dfrac{-Z_{ss}(s,u).s}{Z_s(s,u)} > 1 \Rightarrow$ $E_A(u)$ is either an empty set or an interval of the form $[0, r_{bind}(u)[$.

PROOF

To demonstrate this proposition, let us first derive the housing expenditure of the unconstrained equilibrium relatively to $r$:

$$\frac{\partial(R(r)s(r,u))}{\partial r} = -T'(r) + R(r)\frac{\partial s(r,u)}{\partial r} = -T'(r) + (-f_u'(s(r,u))).\left(-\frac{\partial s_{Hicks}}{\partial R}\frac{T'(r)}{s(r,u)}\right)$$

where $s_{Hicks}$ stands for the Hicksian (compensated) demand for land[27]. Using the relation $\dfrac{\partial s_{Hicks}(R(r),u)}{\partial R} = -\dfrac{1}{Z_{ss}(s(r,u),u)}$, we finally obtain:

$$\frac{\partial(R(r)s(r,u))}{\partial r} = -T'(r) - T'(r)\frac{Z_s(s(r,u),u)}{Z_{ss}(s(r,u),u)s(r,u)} = -T'(r)\left\{1 + \frac{Z_s(s(r,u),u)}{Z_{ss}(s(r,u),u)s(r,u)}\right\}$$

Then $\dfrac{-Z_{ss}(s,u)s}{Z_s(s,u)} > 1 \Rightarrow 1 + \dfrac{Z_s(s(r,u),u)}{Z_{ss}(s(r,u),u)s(r,u)} > 0$, which implies that $\dfrac{\partial(R(r)s(r,u))}{\partial r} < 0$.

Since $R(r)s(r,u)$ is strictly decreasing in $r$, then proposition A is clear.

In the case of a log-linear utility function $U(z,s) = \gamma \log z + \beta \log s$, we have $-Z_{ss}(s,u)s / Z_s(s,u) = (\gamma + \beta)/\gamma > 1$, which means that proposition A applies.

All these elements tend to indicate the prevalence of disk-shaped binding zones put aside the case of peculiar utility functions.

## *Comparative statics: the case of housing expenses*

Due to the different possible outcomes of the CHE policy, analysis of the influence of $\alpha$ proves not trivial. This is because tightening the constraint on housing expenses may result in a lower utility level, which may in turn increase the housing expenses of unconstrained households.

[27] Which is defined as the argmax $\tilde{s}(R,u)$ of $\min_{s,z} z + Rs \;\; s.t. \;\; U(z,s) = u$

Nonetheless, when the equilibrium utility rises, tightening the constraint bearing on the households' housing expenditures diminishes the total land rent distributed to the landlords. The same goes for the total differential land rent (TDR).

For reminder, the TDR is the sum of the housing expenses made by the households, to which has been substracted the opportunity cost of land:

$$TDR(\alpha) = \int_0^{\tilde{r}_f(\alpha)} L(r)\left(\tilde{\Psi}(r,\alpha) - R_A\right)dr \qquad \text{(II.9)}$$

These points are synthesized in the following proposition

PROPOSITION B

For any couple $\alpha_1 < \alpha_2$ checking $\tilde{u}(\alpha_1) \geq \tilde{u}(\alpha_2)$, the amount $H(\alpha_1)$ of housing expenses of the $\alpha_1$ city is inferior to the $\alpha_2$ city's one. Similarly, $TDR(\alpha_1) \leq TDR(\alpha_2)$.
If both the $\alpha_1$ and $\alpha_2$ city are fully constrained ($\left[0, \tilde{r}_f(\alpha_i)\right] \subset E_A(\tilde{u}(\alpha_i), \alpha_i)$ for $i$=(1,2)), then we also have $H(\alpha_1) < H(\alpha_2)$

PROOF

Let us first consider the case $\tilde{u}(\alpha_1) \geq \tilde{u}(\alpha_2)$. This implies $\tilde{\Psi}(r, \tilde{u}(\alpha_1), \alpha_1) \leq \tilde{\Psi}(r, \tilde{u}(\alpha_2), \alpha_2)$ for any location $r$ ($\tilde{\Psi}(r,u,\alpha)$ decreases with $u$ and increases with $\alpha$). Since proposition 2 also implies $\tilde{r}_f(\alpha_1) \leq \tilde{r}_f(\alpha_2)$, this shows:

$$H(\alpha_1) = \int_0^{\tilde{r}_f(\alpha_1)} L(r)\tilde{\Psi}(r,\alpha_1)dr \leq \int_0^{\tilde{r}_f(\alpha_2)} L(r)\tilde{\Psi}(r,\alpha_2)dr = H(\alpha_2)$$

Since $\tilde{\Psi}(r,\alpha_1) - R_A \leq \tilde{\Psi}(r,\alpha_2) - R_A$, we also have $TDR(\alpha_1) \leq TDR(\alpha_2)$.
In the second case, if $\left[0, \tilde{r}_f(\alpha_i)\right] \subset E_A(\tilde{u}(\alpha_i), \alpha_i)$, all households are constrained. Then $H(\alpha_i) = N\alpha_i Y$ for $i$=(1,2) and obviously $H(\alpha_1) < H(\alpha_2)$.