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Non-smooth Dynamics and Multiple Equilibria in a Cournot-Ramsey Model with Endogenous Markups.

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Abstract

We develop a simple Ramsey model with numerous Cournotian industries where entry generates an endogenous markup. The model produces two different regimes: a monopoly and an oligopoly one. We provide a rigorous study of non-smooth dynamics and we also analyse the global dynamics of the model, demonstrating the model exhibits robust heteroclinic orbits, either of the smooth or the non-smooth type. Similar economies may be in any of these regimes and they may change regime along its convergence path. Fixed

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costs and elasticities of demand, play a crucial role and changing their values may alter the dynamics in a radical way, either by inducing a discontinuous transition or a discontinuous hysteresis.
KEYWORDS: endogenous mark-ups, non-smooth dynamics, discontinuous induced bifurcations, heteroclinic orbits.
JEL CLASSIFICATION: C62, D43, E32,

1 Introduction

In this paper we allow for a process of free entry in the manner of Cournot which results in an endogenous markup of price over marginal cost. We embed this entry process into an otherwise fairly standard intertemporal representative-household macromodel. The Ramsey household consumes and accumulates capital. Free entry drives profits in each instant to zero, leading to an endogenous markup and hence wedge between the marginal product of capital and the marginal revenue product. The typical industry in the economy can be in one of two states: monopoly, where there is one firm only producing the output and charging the monopoly markup, or oligopoly where there is more than one firm and the markup is below the monopoly markup. These two states result in two dynamic regimes for economy: for low levels of capital the dynamics is in the monopoly regime, for high levels it is in the oligopoly regime, and in-between there is a *switching* boundary and resultant non-smooth dynamics.

We find that there can be one, two or three steady-state equilibria in this economy. There are two types of stable equilibria (*saddles*): one is a low-output high-markup monopoly; the other is a high-output low-markup oligopoly. All other types of equilibria are unstable. This leads to a "Rostovian" threshold effect. Unless the economy starts off with a high enough capital stock, the economy will be trapped in the low-output high-markup monopoly. If, however, the capital stock is high enough, the economy will be attracted to the high-output low-markup equilibrium. The dividing line is (for a range of parameters) a totally unstable equilibrium, i.e. an unstable focus. The implication of this threshold effect is that an economy may be stuck in a monopoly with a high markup. The reason for this is that the markup reduces the marginal revenue product of capital below its marginal product, and discourages saving so that only a low steady-state capital emerges. This would make a good argument for the government to intervene in some way to enable a *great leap forward* to achieve the critical capital stock so that it can then leave the outcome to the market. This intervention could take the form of regulation (reducing the gap between the marginal product of capital and the return to savings) to encourage savings and the accumulation of capital. More radical alternatives would be forced saving or the nationalization of the means of production in the initial stages of development.

If we take the two types of stable equilibria, monopoly and oligopoly, we find that only the oligopoly equilibrium has an markup which varies. In this case, it is counter-cyclical: higher output leads to more firms and a lower markup with Cournot competition. The empirical literature has shown there is strong evidence of a mildly countercyclical markup - see, *inter alia*, Martins and Scarpetta (2002). This pattern is consistent with a model of the markup with frequent demand shocks and relatively rare supply shocks. Additionally, the procyclical business creation/destruction pattern observed in reality is also consistent with this type of models.

Existing models with Cournot competition and entry have tended to be in a discrete-time overlapping-generations framework (e.g. Chatterjee et al. (1993), D'Aspremont et al. (1995), dos Santos Ferreira and Lloyd-Braga (2005), or Kaas and Madden (2005)), or discrete-time Real Business Cycle (RBC) models (e.g. dos Santos Ferreira and Dufour (2006) ,Portier (1995), Costa (2001) and Costa (2006)). Continuous-time models where Cournot competition is the mechanism generating endogenous markups are not abundant. Zilibotti (1994) use Cournot competition and entry in the intermediation sector in a growth model. Galí and Zilibotti (1995) present an endogenous-growth model with Cournot competition and free entry, where the marginal product of capital is constant as in the AK model of Rebelo (1991). Costa (2004) also uses this framework in a Ramsey model with endogenous labour, but restricts its analysis to study steady-state fiscal policy.

Our contribution is two-fold. First, on the methodological front, we provide a rigorous study of non-smooth dynamics, in particular of discontinuity-induced bifurcations. We also analyse the global dynamics of the model, which is non-trivial in a multiple-equilibrium environment, demonstrating the model exhibits robust heteroclinic orbits, i.e. orbits which connect the different equilibria together, and we do it in both the smooth and non-smooth cases. Second, on the economic relevance of the model, we show that two fundamentally identical economies may behave very differently, as they may be in two different regimes with distinct dynamic behaviour, especially in terms of markups. Even for the same economy, there is the possibility of regime change along the convergence to a stable long-run equilibrium. The "deep" parameters associated with the dominant market structure, i.e. fixed costs and elasticities of demand, play a crucial role in this model and a change in their values may alter the dynamics in a radical way, either by inducing a discontinuous transition or a discontinuous hysteresis. Thus, the interaction between industrial policy and macroeconomic stability emerges as a likely outcome to explore.

In section 2 of the paper, we extend the standard continuous-time Ramsey model to include the free entry of firms in the context of Cournot competition. In section 3 we characterise the dynamics taking into account the switching boundary between monopoly and oligopoly. In section 4 we characterise the steady-states: we partition the parameter space to determine the number and type of steady-states (including possible bifurcations). In section 5 we characterise the local dynamics of any steady-state equilibria which exist. In section 6 we derive the global dynamics of the economy. In section 7 we discuss the relationship to the existing literature and section 8 concludes.

2 A Ramsey Model with Endogenous Mark-ups

2.1 Households

We assume there is a single infinitely living household that consumes a basket of goods and supplies one unit of labour and K units of capital to firms. Total population is constant and has been normalised to unity and there is no technical progress, i.e. we assume that the rate of technical progress is zero[1]. Thus, quantity variables may be interpreted as expressed in units of efficient labour. The household is assumed to maximise an intertemporal utility function in the absence of uncertainty:

$$\max_{C(t),L(t)} U = \int_0^\infty e^{-\rho t} \ln \left[C(t) \right] dt, \tag{1}$$

where $\rho > 0$ represents the rate of time preference, C stands for consumption. There is an exogenous labour supply that we set at unity. For sake of simplicity we assume a logarithmic felicity function, but the results hold with a general isoelastic function.

The household sells human and physical capital services to firms obtaining labour and non-labour income in exchange. The final good can be used either for consumption or for capital accumulation. The price of the final good P is normalised to unity, i.e. the final good is used as *numéraire*. Therefore, the instantaneous budget constraint is given by

$$\dot{K}(t) = w(t) + R(t) K(t) + \Pi(t) - C(t) - \delta K(t), \tag{2}$$

where w is the wage rate, R stands for the rental price of capital, Π represents real pure profits, and $\delta > 0$ is capital depreciation.

Optimal consumption and labour supply paths verify the arbitrage and the transversality conditions:

$$\frac{\dot{C}(t)}{C(t)} = R(t) - (\rho + \delta), \tag{3}$$

$$\lim_{t \to \infty} e^{-\rho t} \frac{K(t)}{C(t)} = 0. \tag{4}$$

[1]This is for simplicity. Exogenous population growth or exogenous technical change do not change the main message of the model, but it complicates notation substantially.

2.2 The final-good sector

The final good, Y, is produced in a competitive retail sector using a CES technology that transforms a continuum of intermediate goods, with mass equal to unity, into a final homogeneous good. The technology exhibits constant returns to specialisation:

$$Y(t) = \left[\int_0^1 y(v,t)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}}, \tag{5}$$

where $\sigma > 0$ represents the elasticity of substitution between inputs and $y(v,t)$ stands for intermediate consumption of variety $v \in [0,1]$ at the moment t.

The maximisation problem can be solved in two steps: (i) determining demand functions for each input that minimises total cost for a given level of final output; (ii) determining the optimal level of output for the representative firm. The first step gives us the following intratemporal demand function for each input:

$$y(v,t) = \left[\frac{p(v,t)}{P(t)} \right]^{-\sigma} Y(t), \tag{6}$$

where $p(v,t)$ stands for the price of good v and P is the appropriate cost-of-producing index form this firm given by

$$P(t) = \left[\int_0^1 p(v,t)^{1-\sigma} dv \right]^{\frac{1}{1-\sigma}}. \tag{7}$$

The cost function can be written as $P(t)Y(t)$. Therefore, the second step in the maximisation program equals the price of the final good to its marginal cost:

$$1 = P(t). \tag{8}$$

2.3 The intermediate goods sector

Industry v (V) is composed of $n(v,t) \geq 1$ producers at moment t[2] and each firm $i = 1, ..., n(v,t)$ has the following technology:

$$y_i(v,t) = \max\left\{ A(t) K_i(v,t)^{\alpha} L_i(v,t)^{1-\alpha} - \phi, 0 \right\}, \tag{9}$$

where y_i represents the output of firm i, K_i and L_i represent its capital and labour inputs, $A(t) > 0$ stands for total factor productivity (TFP), $0 < \alpha < 1$, and $\phi > 0$

[2] Of course this number is an integer. However, we will treat it as a real number, for simplicity. We can think about it as the average number of firms in each industry.

induces increasing returns to scale. The industry can be in two states: *monopoly* when only the initial firm is operative and *oligopoly* when more than one is operative.

Using the terminology in D'Aspremont et al. (1997), we assume Monopolistic Competition (CMC), i.e. firms compete over quantities within the same industry, and they compete over prices across industries. Therefore, firm i faces the following residual demand for its variety, given the outputs of the other firms in the same industry ($k \neq i \in V$) and given the prices of firms producing goods that are an imperfect substitute to good v,

$$y_i(v,t) = \left[\frac{p(v,t)}{P(t)}\right]^{-\sigma} D(t) - \sum_{k \neq i \in V} y_k(v,t), \tag{10}$$

where $D = C + I$ represents total demand for the final good in the economy, and I stands for gross investment defined as

$$I(t) = \dot{K}(t) + \delta K(t). \tag{11}$$

The representative firm maximises its real profits given by

$$\begin{aligned} \max_{L_i(v,t),K_i(v,t)} \Pi_i(v,t) &= \frac{p(v,t)}{P(t)} y_i(v,t) - \frac{w(t)}{P(t)} L_i(v,t) - \frac{R(t)}{P(t)} K_i(v,t), \quad (12) \\ p(v,t) &= \left[\frac{y_i(v,t) + \sum_{k \neq i \in V} y_k(v,t)}{D(t)}\right]^{-\sigma} P(t), \\ y_i(v,t) + \phi &= A(t) K_i(v,t)^{\alpha} L(v,t)^{1-\alpha}. \end{aligned}$$

Notice this is a static problem, as the firm does not accumulate capital. The first-order conditions are given by

$$[1 - \mu_i(v,t)] f_{L,i}(v,t) = \frac{w(t)}{p(v,t)}, \tag{13}$$

$$[1 - \mu_i(v,t)] f_{K,i}(v,t) = \frac{R(t)}{p(v,t)}, \tag{14}$$

where $\mu_i(v,t) = [p(v,t) - MC_i(v,t)]/p(v,t) \in (0,1)$ is the Lerner index for firm i in industry V, $MC_i(v,t) = w(t)/f_{L,i}(v,t) = R(t)/f_{K,i}(v,t)$ represents the marginal cost of production, and $f_{L,i}(v,t) = (1-\alpha) A(t) [K_i(v,t)/L_i(v,t)]^{-\alpha}$ and $f_{K,i}(v,t) = \alpha A(t) [K_i(v,t)/L_i(v,t)]^{\alpha-1}$ stand for marginal products.

2.4 From micro to macro

Let us now assume an intra-industrial symmetric equilibrium exists for all industries. In this case, we have $\mu_i(v,t) = \mu(v,t) = 1/[\sigma n(v,t)]$. Notice that for an equilibrium to exist we must have $\sigma n(v,t) > 1$. Considering our Cobb-Douglas production function is homogeneous of degree one, its partial derivatives are homogeneous of degree zero. Thus, we can rewrite equations (13) and (14) to represent the entire industry:

$$[1 - \mu(v,t)] \left\{ (1-\alpha) A \left[\frac{K(v,t)}{L(v,t)} \right]^{-\alpha} \right\} = \frac{w(t)}{p(v,t)}, \tag{15}$$

$$[1 - \mu(v,t)] \left\{ (1-\alpha) A \left[\frac{K(v,t)}{L(v,t)} \right]^{\alpha-1} \right\} = \frac{R(t)}{p(v,t)}, \tag{16}$$

where $L(v,t) = \sum_{i \in V} L_i(v,t) = n(v,t) L_i(v,t)$ and $K(v,t) = \sum_{i \in V} K_i(v,t) = n(v,t) K_i(v,t)$ represent labour and capital demand in industry V. Given the previous equations, we can obtain total profits in industry V:

$$\Pi(v,t) = \frac{p(v,t)}{P(t)} \left\{ \mu(v,t) \left[A(t) K(v,t)^{\alpha} L(v,t)^{1-\alpha} \right] - n(v,t) \phi \right\}. \tag{17}$$

If all industries are identical, i.e. if there is an inter-industrial symmetric equilibrium, we have $n(v,t) = n(t)$ (consequently $\mu(v,t) = \mu(t)$), and $p(v,t) = P(t)$. Also, when the final output market is in equilibrium we have $Y(t) = D(t)$. Therefore we can write aggregate output as

$$Y(t) = A(t) K(t)^{\alpha} L(t)^{1-\alpha} - n(t) \phi,$$

where $L(t) = \int_0^1 L(v,t)\, dv$ and $K(t) = \int_0^1 K(v,t)\, dv$ represent total labour and capital demand, and market demand for labour and capital can be written as[3]

$$(1-\mu) f_L(K,L) \equiv (1-\mu) \left[(1-\alpha) A \left(\frac{K}{L} \right)^{-\alpha} \right] = \frac{w}{P}, \tag{18}$$

$$(1-\mu) f_K(K,L) \equiv (1-\mu) \left[\alpha A \frac{K}{L}^{\alpha-1} \right] = \frac{R}{P}. \tag{19}$$

The market-clearing condition for the labour market is given by $L = 1$. Therefore, we can derive an aggregate production function for final output given by

$$Y = F(K) - n\phi, \tag{20}$$

where $F(K) = AK^{\alpha}$ is a reduced-form production function ignoring the fixed cost.

[3]In general, we will ignore time indices from this point onwards, for sake of simplicity. We will reintroduce them only wherever they are needed.

2.5 Entry

Profit income is obtained by aggregating profits across all industries, i.e. $\Pi(t) = \int_0^1 \Pi(v,t)\,dv = Y(t) - w(t) - R(t)K(t)$. Considering the equilibrium factor prices and the aggregate production function, total profits can be expressed as

$$\Pi = \mu F(K) - n\phi. \tag{21}$$

Assuming instantaneous free entry, the number of firms in each industry adjusts in order to keep pure profits of all firms except the initial firm equal to zero. Although we treat n as a continuous variable, we still want to impose a lower bound of $n \geq 1$, so that the mark-up cannot take a value greater than σ^{-1}. Taking into account that $n = 1/(\sigma\mu)$, we obtain the rule governing the endogenous mark-up:

$$\mu = \mu(K) = \min\left\{m(K), \frac{1}{\sigma}\right\}, \tag{22}$$

$$m(K) = \sqrt{\frac{\phi}{\sigma F(K)}}. \tag{23}$$

The actual mark-up μ is the smaller of the monopoly mark-up and the zero-profit markup $m(K)$.

3 The Switching Boundary

The economy can be in one of two states monopolistic competition (MC) or Cournotian monopolistic competition (CMC), depending on the level of the capital stock. Equations (20) and (22) allow us to define two particular values of K:

$$\underline{K} = \left(\frac{\phi}{A}\right)^{1/\alpha} = \{\ K : F(K) = \phi\}, \tag{24}$$

which is the minimum level of capital required to produce a non-negative output in a MC state, and

$$\tilde{K} = \left(\frac{\phi\sigma}{A}\right)^{1/\alpha} = \{\ K : m(K) = \frac{1}{\sigma}\}. \tag{25}$$

is the level of capital at which free-entry reduces the number of firms to exactly 1.

As $\sigma > 1$ implies $\tilde{K} > \underline{K} > 0$ then there is a range of the capital stock for each the economy operates in a MC state. If $K > \tilde{K}$ then $m(K) > 1/\sigma$ and the economy operates in a CMC state.

Therefore, when the stock of capital crosses to value $\tilde{K}$ we have an endogenous change in the state of the economy. We say that there is a *switching* boundary if $K = \tilde{K}$ or the economy undergoes a regime switch.

However, for $\underline{K} < K < \widetilde{K}$ the economy can only be in one of two states: i) $n = 1$ is a monopolistic competition state and ii) $n = 0$ means a complete shut down. We are interested in the case where monopolistic competition can exist and it can be compared with CMC. Thus, we introduce a simple assumption that makes the two cases comparable: we assume there is a public or private coordination machanism that allows households to sustain a monopolistic-competition equilibrium if it is globally better for them than the alternative of complete shut down.

One example of a simple mechanism in this class is a lump-sum tax equal at least to $\frac{1}{\sigma}F(K) - \phi$ and levied on households and transfered to firms. In this case, losses from firms[4] are offset by lump-sum transfers. Thus, there is no incentive for households to shut down firms due to a coordination failure and they are better off.

Nonetheless, the qualitative results of the model are still the same under an alternative formulation where entry is still free in $\underline{K} < K < \widetilde{K}$, as it can be seen in the appendix.

The functions for the return on capital and output change qualitatively when there is a regime switch. In a MC regime the rate of return of capital and the output are given by $R_1(K) \equiv (1 - 1/\sigma)F^{'}(K)$ and by $Y_1(K) \equiv F(K) - \phi$, and in a CMC regime are given by $R_2(K) \equiv (1 - m(K))F^{'}(K)$ and $Y_2(K) \equiv (1 - m(K))F(K)$, respectively.

Then, the return to capital, defined on its whole domain, is formally given by the continuous and piecewise smooth function

$$R(K) = \begin{cases} R_1(K) & \text{if } \underline{K} \leq K < \tilde{K} \\ R_2(K) & \text{if } \tilde{K} \leq K. \end{cases} \tag{26}$$

Next we will prove that function $R(K)$ can take one of the two shapes we call $R^a(K)$ and $R^b(K)$ (see Figure 1). Let us introduce a new value for σ

$$\bar{\sigma} = \bar{\sigma}(\alpha) = \frac{2 - \alpha}{2(1 - \alpha)} > 1, \tag{27}$$

and the associate level of the capital stock associated to a maximum return for a CMC regime

$$\overline{K} \equiv \left\{ K : \, m(K) = \frac{1}{\bar{\sigma}} \right\} = \left[\frac{\phi \bar{\sigma}^2}{\sigma A} \right]^{1/\alpha}. \tag{28}$$

The next result relates value $\overline{K}$ with the switching boundary:

[4]Notice that for $\underline{K} < K < \widetilde{K}$ all industries have negative profits.

Lemma 1

1. *If* $1 < \sigma \leq \bar{\sigma}$ *then* $\underline{K} < \tilde{K} \leq \overline{K}$.
2. *If* $\sigma \geq \bar{\sigma}$ *then* $\tilde{K} \geq \overline{K}$.

Proof See the appendix.

This means that in the first case a maximum value for the rate of return exists within the CMC regime and is absent in the second case. This allows us to

The properties of $R(K)$ are given by the following proposition:

Proposition 1 *: Properties of* $R(K)$

Consider function $R(K)$ *given in equation (26) in the domain* $K \geq \underline{K}$. *It has the following properties:*

1. $R(K)$ *is positive (for finite* K*), is continuous and is piecewise smooth.*
2. *In addition:*
   (a) *if* $1 < \sigma < \bar{\sigma}$ *then* $R(K) = R^a(K)$ *is non-monotonous;*
   (b) *if* $\sigma \geq \bar{\sigma}$ *then* $R(K) = R^b(K)$ *is monotonous,*

   where

   (a) $R^a(K)$ *is decreasing along branch* $R_1(K)$, *if* $K \leq \tilde{K}$, *changes slope at* $K = \tilde{K}$, *becomes increasing along branch* $R_2(K)$, *if* $\tilde{K} < K < \overline{K}$, *reaches a local maximum at* $K = \overline{K}$ *and then becomes decreasing for* $K > \overline{K}$:

$$R^{a'}(K) \begin{cases} < 0, & \text{if } K \in [\underline{K}, \tilde{K}) \cup (\overline{K}, \infty), \\ > 0, & \text{if } K \in (\tilde{K}, \overline{K}), \\ = 0, & \text{if } K = \overline{K}) \end{cases}$$

   (b) $R^b(K)$ *is decreasing in all its domain, but the slope jumps discontinuously at the switching point* $K = \tilde{K}$, *where* $R_1'(\tilde{K}) < R_2'(\tilde{K}) \leq 0$.

Proof See the appendix.

At the switching point $K = \tilde{K}$ the function $R(K)$ is non-smooth: the left and right derivatives differ. However, we can determine a generalized derivative as the convex hull of the left and right (classical) derivatives,

$$\partial R(K)|_{K=\tilde{K}} = \{R_q'(K) : \ 0 \leq q \leq 1, \ K = \tilde{K}\}$$

where:

$$\left. R_q^{'}(K) \right|_{K=\tilde{K}} = (1-q)R_1^{'}(\tilde{K}) + qR_2^{'}(\tilde{K})$$
$$= -\frac{(1-\alpha)}{\sigma}\left[\,(1-q)(\sigma-1)+q(\sigma-\tilde{\sigma})\right]\frac{F^{'}(\tilde{K})}{\tilde{K}}.$$

Though it is non-smooth in both cases the return function is convex-concave if $R(K) = R^a(K)$ and is globally concave if $R(K) = R^b(K)$. In the first case we offer an microeconomic foundation for function as Skiba (1978). In the second case the return function is closer to the Ramsey case. However, in both case there are two different market regimes and a switching boundary in which the return function is non-smooth.

`Figure 1 around here`

We can also define the output function on its whole domain as a continuous and non-smooth function

$$Y(K) = \begin{cases} Y_1(K) & \text{if } \underline{K} \le K < \tilde{K} \\ Y_2(K) & \text{if } \tilde{K} \le K. \end{cases} \tag{29}$$

Proposition 2 *: Properties of $Y(K)$*

1. *$Y(K)$ is positive, continuous and piecewise smooth.*
2. *For any values of the parameter σ, at the switching point $K = \tilde{K}$, we have $Y_1(\tilde{K}) = Y_2(\tilde{K})$ and $Y_1^{'}(\tilde{K}) > Y_2^{'}(\tilde{K})$.*

Also, we can determine a generalized derivative at $K = \tilde{K}$,

$$\partial Y(K)|_{K=\tilde{K}} = \{Y_q^{'}(K):\ 0 \le q \le 1,\ K = \tilde{K}\}$$

where

$$\left. Y_q^{'}(K) \right|_{K=\tilde{K}} = (1-q)Y_1^{'}(\tilde{K}) + qY_2^{'}(\tilde{K}) = F^{'}(\tilde{K})\left(1-\frac{q}{2\sigma}\right).$$

Proof See the appendix.

The $Y(K)$ function only changes qualitatively with the supply side of the market structure.

4 General Equilibrium

4.1 The two regimes

From now on, let us consider only the domain the following partition over the domain of (C, K):

$$\begin{aligned} S_1 &= \{\ (C,K) \in R_+^2 :\ C \geq 0,\ \underline{K} < K < \tilde{K}\}, \\ \Sigma &= \{\ (C,K) \in R_+^2\ :\ C \geq 0,\ K = \tilde{K}\}, \\ S_2 &= \{\ (C,K) \in R_+^2\ :\ C \geq 0,\ K > \tilde{K}\}. \end{aligned}$$

In words, S_1 corresponds to a regime in which there is monopolistic competition and S_2 in which there is CMC and there is a switching boundary between regimes in Σ.

The general equilibrium (GE) of our model is defined by the flow $\{(C^*(t), K^*(t)) : t \in R_+\}$ which is generated by the planar piecewise-smooth ordinary differential equation

$$\dot{C} = (R_j(K) - (\rho + \delta))\, C,\ j = 1, 2, \Sigma, \tag{30}$$

$$\dot{K} = Y_j(K) - C - \delta K,\ j = 1, 2, \Sigma. \tag{31}$$

where $R_\Sigma = R(\tilde{K})$ and $Y_\Sigma = Y(\tilde{K})$, together with the initial condition $K(0) = K_0$ and the transversality condition

$$\lim_{t\to\infty} \frac{K(t)}{C(t)} e^{-\rho t} = 0. \tag{32}$$

In order to study the GE dynamics, we first determine the conditions over the parameters related to the number of the steady state equilibria. We find that the economy can display five different types of structurally stable phase diagrams. In the following sections we study the dynamics.

4.2 Steady-state equilibria

Stationary equilibria are defined as

$$(C^*, K^*) = \{(C, K) \in S_1 \cup \Sigma \cup S_2 :\ \dot{C} = \dot{K} = 0,\ C > 0\},$$

where the condition $C > 0$ ensures that the transversality condition holds. Let

$$C_j(K^*) = Y_j(K^*) - \delta K^*,\ j = 1, 2, \Sigma.$$

Then, we have the equivalent conditions for an admissible [5] equilibrium

$$\rho + \delta = \begin{cases} R_1(K^*) & \text{if } \underline{K} \leq K^* < \tilde{K} \\ R_2(K^*) & \text{if } K^* \geq \tilde{K} \end{cases}, \tag{33}$$

and

$$C^* = \begin{cases} C_1(K^*), & \text{if } \underline{K} \leq K^* < \tilde{K} \\ C_2(K^*), & \text{if } K^* \geq \tilde{K} \end{cases}. \tag{34}$$

We say that we have a boundary equilibrium $(C^*, K^*) = (\tilde{C}, \tilde{K})$ if

$$\begin{aligned} R_1(K^*) &= R_2(K^*) = R_\Sigma(K^*) = R(\tilde{K}) = \rho + \delta, \\ C_1(K^*) &= C_2(K^*) = C_\Sigma(K^*) = C(\tilde{K}). \end{aligned}$$

The equilibrium determination is recursive: equation (33) determines the equilibria (isolated or multiple) for K and equation (34) determines the associated equilibria for C.

In proposition 1 we found that $R(K)$ is non-monotonous in case $R^a(K)$ and is monotonous in case $R^b(K)$. In proposition 2 we proved that $Y(K)$ is always monotonous. Then we may have multiplicity of equilibria if $R(K) = R^a(K)$, and uniqueness if $R(K) = R^b(K)$.

The next proposition presents all the possible cases. As we may have bifurcations with a maximum co-dimension of order two, two parameters unfold all the cases. A first natural choice is σ, the second is overhead ϕ.

We first derive some critical values for ϕ as functions of the other parameters, σ, α, A, ρ and δ.

The two first critical values are related to the MC equilibrium, $\underline{\phi}$ and $\underline{\phi}_c$: one is related to equilibrium with zero production and the other to equilibrium with zero consumption.

The critical value for ϕ such that there is a steady state MC equilibrium for $K = \underline{K}$, is determined from $F(K) = \phi$, and the equilibrium condition in the subset

[5] See di Bernardo et al. (2008) A virtual equilibrium is defined as the values of (C, K) such that

$$\rho + \delta = \begin{cases} R_1(K) & \text{if } K > \tilde{K} \\ R_2(K) & \text{if } \underline{K} < K < \tilde{K} \end{cases},$$

and

$$C = \begin{cases} C_1(K) = Y_1(K) - \delta K & \text{if } K > \tilde{K} \\ C_2(K) = Y_2(K) - \delta K & \text{if } \underline{K} < K < \tilde{K}. \end{cases}.$$

S_1, $R_1(K) = \rho + \delta$, that is $\underline{\phi} \equiv \{ \phi : R_1(\underline{K}) = \rho + \delta\}$. It is uniquely given by

$$\underline{\phi} = \underline{\phi}(\sigma) \equiv \left[A \left(\frac{\alpha}{\rho + \delta} \left(1 - \frac{1}{\sigma} \right) \right)^{\alpha} \right]^{1/(1-\alpha)} \quad \text{if } \sigma > 1. \tag{35}$$

The critical value for ϕ such that there is a steady state MC equilibrium for $C^* = C(K^*) = 0$, is the resultant of $R_1(K) = \rho + \delta$ and $C_1(K) = Y_1(K) - \delta K = 0$, $\underline{\phi}_c = \{ \phi : \ R_1(K) = \rho + \delta, : C_1(K) = 0\}$

$$\underline{\phi}_c = \underline{\phi} - \delta K^* = \underline{\phi} - \delta \left(\frac{\alpha A}{\rho + \delta} \left(1 - \frac{1}{\bar{\sigma}} \right) \right)^{1/(1-\alpha)}. \tag{36}$$

The third critical value for ϕ is related to the existence of a boundary steady state equilibrium, that is to an equilibrium in the switching region Σ. It is defined as $\tilde{\phi} = \{\phi : R_1(K) = R_2(K) = R_\Sigma(K) = \rho + \delta\}$ and is

$$\tilde{\phi} = \frac{\underline{\phi}(\sigma)}{\sigma}. \tag{37}$$

A fourth critical value related to the existence of a steady state CMC equilibrium for $K = \overline{K}$, that is, at the local maximum return to capital in the CMC regime. It is determined from the relation between $R_2'(K) = 0$, and the equilibrium condition in the subset S_2, $R_2(K) = \rho + \delta$, $\overline{\phi} \equiv \{ \phi : R_2(\overline{K}) = \rho + \delta\}$. It is uniquely given by

$$\overline{\phi} = \overline{\phi}(\sigma) \equiv \frac{\sigma}{\bar{\sigma}^2} \left[A \left(\frac{\alpha}{\rho + \delta} \left(1 - \frac{1}{\bar{\sigma}} \right) \right)^{\alpha} \right]^{1/(1-\alpha)} \quad \text{if } 1 < \sigma \leq \bar{\sigma}, \tag{38}$$

where $\bar{\sigma}$ is given in equation (27).

Then the following relationships hold:

Lemma 2 $\underline{\phi}_c(\sigma) < \underline{\phi}(\sigma)$ for all $\sigma > 1$.

Lemma 3 $\underline{\phi}(\sigma) > \tilde{\phi}(\sigma)$ for all $\sigma > 1$.

It is useful to define two further critical values for ϕ, which are independent from σ $\phi_a = \underline{\phi}_c(\sigma_a) = \overline{\phi}(\sigma_a)$ where

$$\sigma_a = \{\sigma : \ \underline{\phi}_c(\sigma) = \overline{\phi}(\sigma) \ \}, \tag{39}$$

and

$$\phi_b = \frac{1}{\bar{\sigma}} \left[A \left(\frac{\alpha}{\rho + \delta} \left(1 - \frac{1}{\bar{\sigma}} \right) \right)^{\alpha} \right]^{1/(1-\alpha)}. \tag{40}$$

Lemma 4 If $1 < \sigma < \bar{\sigma}$ then $\overline{\phi}\ (\sigma) > \tilde{\phi}(\sigma)$. If $\sigma \bar{=} \bar{\sigma}$ then $\overline{\phi}(\bar{\sigma}) = \tilde{\phi}(\bar{\sigma}) = \phi_b$.

The previous critical values allow us to define a partition over the domain of (ϕ, σ), that is $\mathbf{R}^2_+$ (see Figure 2):

$$\mathcal{A} = \{\ (\sigma, \phi) :\ \sigma > 1, \phi > \max\{\overline{\phi}, \underline{\phi}_c\ \}, \tag{41}$$

$$\mathcal{B} = \{\ (\sigma, \phi) :\ \sigma > \sigma_a,\ \max\{\overline{\phi}, \tilde{\phi}\ \} < \phi < \underline{\phi}_c\ \}, \tag{42}$$

$$\mathcal{C} = \{\ (\sigma, \phi) :\ 1 < \sigma < \sigma_a,\ \overline{\phi} < \phi < \underline{\phi}_c\}, \tag{43}$$

$$\mathcal{D} = \{\ (\sigma, \phi) :\ 1 < \sigma < \bar{\sigma},\ \min\{\overline{\phi}, \underline{\phi}_c\ \} < \phi < \tilde{\phi}\}, \tag{44}$$

$$\mathcal{E} = \{\ (\sigma, \phi) :\ \sigma > 1,\ 0 < \phi < \tilde{\phi}\}. \tag{45}$$

`Figure 2 around here`

Now, we present our main classification result concerning the existence, uniqueness and multiplicity of stationary equilibria, in the interior subspaces, S_1 and S_2 and in the switching boundary Σ.

Proposition 3 *: Stationary equilibria: existence and multiplicity*

1. *If $(\sigma, \phi) \in \mathcal{B}$ then there is a single stationary equilibrium in which there is monopolistic competition.*
2. *If $(\sigma, \phi) \in \mathcal{C}$ then there are two stationary equilibria in which there both with CMC.*
3. *If $(\sigma, \phi) \in \mathcal{D}$ then there are three stationary equilibria, one in which there is monopolistic competition and two in which there is CMC.*
4. *If $(\sigma, \phi) \in \mathcal{E}$ then there is an unique stationary equilibrium in which there is CMC.*
5. *If $(\sigma, \phi) \in \mathcal{A}$ then there is no stationary equilibrium.*

Proof See the appendix.

Now consider a further partition on $\mathcal{B}$:

$$\mathcal{D}^s = \{\ (\sigma, \phi) :\ 1 < \sigma < \bar{\sigma},\ \phi = \overline{\phi}\}, \tag{46}$$

$$\mathcal{D}^n = \{\ (\sigma, \phi) :\ 1 < \sigma < \bar{\sigma},\ \phi = \tilde{\phi}\}, \tag{47}$$

(48)

Corollary 1 *If $(\sigma, \phi) \in \mathcal{D}^s \cup \mathcal{D}^n$ then there are two equilibria.*

5 Local Dynamics

Before we are able to characterize completely the out-of-steady state GE dynamics, and the possible phase diagrams for (C, K), we need to address the local dynamic properties in the three subsets of the state space: S_1, S_2 and Σ, and, in particular the existence of local bifurcations. In this section we study local dynamics and in the next section we study the global dynamics scenarios.

Given the non-smoothness of the dynamic system generated by equations (30)-(31) , we have to resort to a specific bifurcation analysis, in order to offer a characterization of the dynamics. In principle, we may have both smooth and non-smooth (local and global) bifurcations. The dynamic systems theory for non-smooth differential equations is a relatively new topic in the dynamic systems literature, and there is not yet a complete taxonomy of the non-smooth bifurcations (see Leine (2006) and di Bernardo et al. (2008) [6]).

5.1 Local dynamics for stationary states outside the switching boundary

Let us first consider non-boundary stationary states belonging to any branch of the state space, $K_j^* \neq \tilde{K} \ \in S_j$ for $j = 1, 2$. The local dynamics in the neighbourhood of the stationary points belonging to each branch of the state space can be characterised by the eigenvalues of the associated Jacobians.

The Jacobian evaluated at $K_j^* \in S_j$ is

$$J_j(K_j^*) = \begin{pmatrix} 0 & C_j(K_j^*)R_j^{'}(K_j^*) \\ -1 & C_j^{'}(K_j^*) \end{pmatrix}, K_j^* \in S_j, \ j = 1, 2.$$

It has trace and determinant given by

$$\text{tr}(J_j(K_j^*)) = C_j^{'}(K_j^*), K_j^* \in S_j, j = 1, 2, \tag{49}$$

$$\det(J_j(K_j^*)) = C_j(K_j^*)R_j^{'}(K_j^*), K_j^* \in S_j, j = 1, 2. \tag{50}$$

Then the associated eigenvalues for an admissible equilibria point, $K^* \neq \tilde{K}$ are

$$\lambda_j^- = \frac{C_j^{'}(K_j^*)}{2} - \Delta(J_j(K_j^*))^{1/2}, \tag{51}$$

$$\lambda_j^+ = \frac{C_j^{'}(K_j^*)}{2} + \Delta(J_j(K_j^*))^{1/2}, \tag{52}$$

[6] According to these authors, it may even not be possible given the large number of different types of non-smooth dynamical systems that exist.

where the discriminant is

$$\Delta(J_j) = \left(\frac{C_j'(K_j^*)}{2}\right)^2 - C_j(K^*)R_j'(K^*). \tag{53}$$

Lemma 5 *Assume that the condition for the existence of an equilibrium belonging to branch S_1 holds ($\tilde{\phi} < \phi < \underline{\phi}_c$). Then, the equilibrium point (C_1^*, K_1^*) is saddle-point stable.*

Proof See the appendix.

Lemma 6 *Assume that (C_2^*, K_2^*) is a stationary equilibrium belonging to branch S_2. If $R_2'(K_2^*) < 0$ then the equilibrium is saddle point stable. If $R_2'(K_2^*) = 0$ then the equilibrium is a continuous bifurcation point. If $R_2'(K_2^*) > 0$ then the equilibrium is an unstable node.*

Proof See the appendix.

In all cases, the eigenvalues are continuous functions of the parameters, and, in particular of ϕ and σ. If an eigenvalue associated to a non-boundary equilibrium passes through the imaginary axis in a continuous way, as a result of a continuous change in a parameter, then we say that the equilibrium point (C_i^*, K_i^*) undergoes a *continuous bifurcation* (see Leine (2006)).

5.2 Local dynamics at a switching boundary stationary state

Boundary equilibria, that is equilibria such that $(C_1^*, K_1^*) = (C_2^*, K_2^*) = (\tilde{C}, \tilde{K})$ analogously to admissible equilibria, may or may not be (local) bifurcation points. In the second case we say that we have a boundary-equilibrium bifurcation (see (di Bernardo et al., 2008, p. 220)) or a discontinuous bifurcation (see Leine (2006)).

Eigenvalues evaluated at boundary equilibria are discontinuous functions of the parameters. If the eigenvalues associated to the two branches evaluated at the boundary equilibrium shift discontinuously but do not change sign when they cross the boundary equilibrium, we say that the system undergoes a *transition*, or displays persistence (see (di Bernardo et al., 2008, p. 220)). If the eigenvalues shift discontinuously, we have to determine generalized eigenvalues. If the eigenvalues, evaluated at the boundary equilibrium, display an unique or multiple crossings to the imaginary axis then we say that the system undergoes a *discontinuous bifurcation*. In this case the number of equilibria change when the bifurcation point is crossed, and the number of equilibria which coexist or disappear when the bifurcation point is crossed allows for a characterization of the type of the bifurcation.

Lemma 7 *Let $K^* = \tilde{K}$ be a boundary equilibrium point. Then*

$$0 < \mathrm{tr}(J_2(\tilde{K})) < \mathrm{tr}(J_1(\tilde{K})),$$

and

$$\det(J_2(\tilde{K})) = \det(J_1(\tilde{K})) \left(\frac{\sigma - \bar{\sigma}}{\sigma - 1} \right).$$

Proof See the appendix.

Now consider the generalized Jacobian evaluated at the boundary equilibrium

$$\partial J(\tilde{K}) = \{\ J_q(K) : 0 \leq q \leq 1,\ K = \tilde{K}\},$$

where

$$J_q(\tilde{K}) = \begin{pmatrix} 0 & (1-q)C_1(\tilde{K})R_1'(\tilde{K}) + qC_2(\tilde{K})R_2'(\tilde{K}) \\ -1 & (1-q)C_1'(\tilde{K}) + qC_2'(\tilde{K}) \end{pmatrix}, \tag{54}$$

and the generalized eigenvalues

$$\Lambda^{\mp}(\tilde{K}) = \{\lambda_q^{\mp} :\ \ 0 \leq q \leq 1,\ K = \tilde{K}\},$$

where

$$\lambda_q^{\mp} = \frac{\mathrm{tr}(J_q)}{2} \mp \Delta(J_q)^{1/2}.$$

Lemma 8 Consider the generalized Jacobian (54). Then:

1. $\mathrm{tr}(J_q(\tilde{K})) > 0$ *for all* $q \in [0,1]$.
2. *If* $1 < \sigma \leq \bar{\sigma}$ *then there is a value for* $q \in [0,1]$ *such that* $\det J_q(\tilde{K}) = 0$.
3. *If* $\sigma > \bar{\sigma}$ *then* $\det J_q(\tilde{K}) > 0$ *for all* $q \in [0,1]$.

Proof See the appendix.

Then, at the boundary equilibrium there is a "jump" in the eigenvalues, $\lambda_1^-(\tilde{K}) \neq \lambda_2^-(\tilde{K})$ and $\lambda_1^+(\tilde{K}) \neq \lambda_2^+(\tilde{K})$, for any values of the parameters. More importantly, the generalized eigenvalue associated to stability, Λ^- has an infinite number of values containing or not the zero. In the last lemma we found that $0 \notin \Lambda^-$ if $\sigma > \bar{\sigma}$ and, that $0 \in \Lambda^-$ if $1 < \sigma \leq \bar{\sigma}$. In the last case we say that there is a *discountinous bifurcation*. In the case in which $\sigma = \bar{\sigma}$ we would have $\lambda_2^-(\tilde{K}) = 0$ only if $q = 1$. This means that a discontinuous bifurcation would coincide with a local continuous bifurcation. Observe that all those kinds of bifurcations are specific to non-smooth ordinary differential equations (ODE) and do not occur in smooth ODE. This is why they are also called *discontinuity induced bifurcations*.

The next proposition presents the a description of the local dynamics in the presence of a boundary or switching stationary equilibrium.

Proposition 4 *: Local dynamics for boundary equilibria*

Let $K^* = \tilde{K}$ and $\phi = \tilde{\phi}$. Then:

1. *There are no limit cycles in a neighborhood of the equilibrium point.*
2. *If $\sigma > \bar{\sigma}$ there is persistence of the steady state and a non-smooth saddle-saddle transition.*
3. *If $\sigma < \bar{\sigma}$ there is coexistence of equilibria for $\phi > \tilde{\phi}$, where $K^* < \tilde{K}$ is a saddle point and $K^* > \tilde{K}$ is an unstable focus.*
4. *If $\sigma = \bar{\sigma}$ there is a local bifurcation of the fold type.*

Proof See the appendix.

Figure 3 illustrates the occurrence of a local discontinuous bifurcation in the last case when ϕ crosses the bifurcation value $\tilde{\phi}$. We also see, that there is a single crossing at the imaginary axis.

`Figure 3 around here`

5.3 Summing up

`Figure 4 around here`

The next proposition gathers all the results in this section as regards both the number of equilibria and the local dynamics, for all parameters values such that a stationary equilibrium exists.

Proposition 5 *: Local dynamics*

1. *If $(\sigma, \phi) \in \mathcal{B} \cup \mathcal{E}$ then there is an unique saddle-point stable stationary state.*
2. *If $(\sigma, \phi) \in \mathcal{D}$ then there are three stationary states: saddle-unstable focus-saddle.*
3. *If $(\sigma, \phi) \in \mathcal{C}$then there are two stationary states: unstable focus-saddle.*

Proposition 6 *: Local bifurcations*

1. *If $(\sigma, \phi) \in \mathcal{D}^s$ then there is a continuous subcritical pitchfork bifurcation if $\phi = \overline{\phi} \neq \tilde{\phi}$*

2. *If* $(\sigma, \phi) \in \mathcal{D}^n$ *then there is a discontinuous supercritical pitchfork bifurcation if* $\phi = \tilde{\phi} \neq \overline{\phi}$.
3. *If* $\sigma = \overline{\sigma}$ *and* $\phi = \overline{\phi} = \tilde{\phi}$, *that is, if* $(\sigma, \phi) \in \mathcal{D}^s \cap \mathcal{D}^n$ *then, there is a discontinuous fold bifurcation.*

Figure 4 presents a bifurcation diagram in the space (K, ϕ) for case $1 < \sigma < \bar{\sigma}$ and for case $\sigma \geq \bar{\sigma}$: in the first case we have a discontinuous saddle-saddle transition and in the second a non-smooth hysteresis. If we consider it together with figure 2, we can have a graphical description of the number and stability properties of the stationary equilibria.

The left subfigure of figure 4 corresponds to the case in which $R(K) = R^a(K)$ and to regions $\mathcal{B}_1$, $\mathcal{D}$ and $\mathcal{E}_1$, and $\mathcal{D}^s$ and $\mathcal{D}^n$ in the two boundaries. For high values of $\phi > \overline{\phi}$, we have successively a unique MC equilibrium which is saddle point stable, that we call (C^*_M, K^*_M) from now on. For lower values of $\overline{\phi} < \phi\tilde{\phi}$ we have three equilibria, one MC , (C^*_M, K^*_M), and two CMC equilibra. the first is unstable and is associated to a high markup, (C^*_H, K^*_H), and the second saddle point stable and associated with a low markup, (C^*_L, K^*_L). If $\phi < \tilde{\phi}$ we have again a unique equilibrium which is CMC, it is saddle point stable and is associated with a low markup. In the transitions, if ϕ decreases, we pass through bifurcation values of the parameters. In the first case, we have a continuous biburcation, if $\phi = \overline{\phi}$, such that the MC equilibria continuous to exist but the two CMC equilibria arise. In the second case, we have a discontinuity induced bifurcation, if $\phi = \tilde{\phi}$, such that a MC and a CMC equilibria disappear. The bifurcation diagram is similar to the hysteresis for smooth dynamic systems.

The right subfigure of figure 4 corresponds to the case $\sigma > \overline{\sigma}$ which implies $R(K) = R^b(K)$ and displays regions $\mathcal{B}_2$, if $\phi > \tilde{\phi}$, and $\mathcal{E}_2$, if $\phi < \tilde{\phi}$ [7]. We have only one equilibrium for any values of the parameters, corresponding to a MC, (C^*_M, K^*_M), in the upper part of the figure, and to a CMC, (C^*_L, K^*_L), in the lower part. In the boundary we still have a unique equilibrium $(\tilde{C}, \tilde{K})$. All the equilibria are saddle point stable, including the boundary equilibrium. In this case we said that the passage through the discontinuity involves a transition: the qualitative dynamics does not change, but the left are right transitional adjustments are different, threr will be a MC dynamics in the left and a CMC dynamics in the right.

When the parameters are in region $\mathcal{C}$, this corresponds to the case in the left subfigure in figure 4 when the curve $C(K) = 0$ intersects the equilibrium condition $R(K) = \rho + \delta$ in the positively slopped part.

[7]In fact we have the case $\sigma = \overline{\sigma}$. The boundary, in this case verifies $\tilde{\phi} = \overline{\phi}$, and we have the case 3 in Proposition 6.

6 Global General-Equilibrium dynamics

This section deals with the GE dynamics and in particular to the behavior of the economy when it is far away from a steady, and in particular, when the initial structure of the economy differs from the asymptotic one. We do this by offering a (almost) complete characterization of the phase diagrams. In some cases, the phase diagrams can only be built if we consider global dynamics.

As we saw in the last section, our model displays two types of structurally stable dynamics, both involving the two MC and CMC regimes, one in which the return function is $R^a(K)$ and the other when it is $R^b(K)$. We may call them two alternative GE scenarios.

Let us represent the GE flows as $\Phi(t) = (\Phi_C(t), \Phi_K(t))$, where $\Phi_C(t, K(0), \sigma, \phi) = C^*(t)$ and $\Phi_K(t, K(0), \sigma, \phi) = K^*(t)$, generated by equations (30), (31), and (32), for any initial state of the economy, and for different industrial parameters $(K(0), \sigma, \phi)$.

Proposition 7 *: Assume that* $(\sigma, \phi) \in \mathcal{B}$ *and that* $K(0) > \underline{K}$ *. Then there is an unique monopolistic-competition stationary equilibrium* (C^*_M, K^*_M)*, of the saddle-point type. If* $K(0) > \underline{K}$*, independently of the regime at time* $t = 0$*,then the GE flow* $\{\Phi(t, K(0)) : \ t \geq 0\}$*, will converge asymptotically to a steady state. In particular, if* $K(0) > \tilde{K}$ *then both capital and consumption will adjust downwards, and the convergence will be piecewise smooth, with a "jump" in the rate of convergence, at time* $\tau > 0$ *when* $\Phi_K(\tau, K(0)) = \tilde{K}$*.*

Proof See the appendix.

Figure 5 depicts the complete phase diagram associated to Proposition 7. If the initial capital stock verifies $\underline{K} < K(0) < \tilde{K}$ then both the initial and the stationary state of the economy will have a MC regime and the markup will be constant and independent of the transitional dynamics. The dynamics will be smooth and will depend on the relative position of $K(0)$ and K^*_M. If $K(0)$ if it is greater (smaller) then K^*_M then there will be a downward (upward) adjustment. If the initial capital stock verifies $K(0) > \tilde{K}$ then the economy will start from a CMC regime and will have a regime shift to MC along the transition path at time $t = \tau$. In the beginning of the adjustment the markup will be endogenous and anti-cyclic: both output and consumption will go down while the markup is adjusting upwards. At time τ there will be only one firm per industry, as the economy changes from a CMC to a MC regime. At this point, the markup will become exogenous and constant while the size of the monopolist will shrink continuously along the convergence to the stationary equilibrium (C^*_M, K^*_M). The GE equilibrium paths will lie along the stable manifold associated with MC equilibrium (C^*_M, K^*_M), $W^s_M = W^s(C^*_M, K^*_M)$, which belongs to both branches S_1 and S_2, and is therefore non-smooth.

There are two conditions for the existence of a GE equilibrium: first, there should exist at least one stationary, or steady state, equilibrium, and, second, the initial level of the stock of capital should be admissible. The first condition involves restrictions over the parameters ϕ and σ such that case $\mathcal{A}$ in proposition 5 is ruled out. The second condition, is, that $K(0) > \underline{K}$.

`Figure 5 around here`

Proposition 8 : *Assume that $(\sigma, \phi) \in \mathcal{E}$ and that $K(0) > \underline{K}$. Then there is an unique CMC steady-state equilibrium associated to the low markup (C_L^*, K_L^*), of the saddle-point type. Irrespective of the regime associated to $K(0)$, at time $t = 0$, the GE flow $\{\Phi(t, K(0)) : \; t \geq 0\}$ will converge asymptotically to the stationary equilibrium . If $K(0) < \tilde{K}$ then both capital and consumption will adjust upwards, and the convergence will be piecewise smooth, with a "jump" in the rate of convergence when $\Phi_K(\tau, K(0)) = \tilde{K}$ for $\tau > 0$.*

Proof See the appendix.

Figure 6 depicts the complete phase diagram associated to Proposition 8. It has a similar interpretation as figure 5. The GE equilibrium paths will lie along the stable manifold associated with the low markup CMC equilibrium (C_L^*, K_L^*), $W_L^s = W^s(C_L^*, K_L^*)$, which also belongs to both branches S_1 and S_2, and is non-smooth. In this case, if the initial state of the economy is MC and the initial capital stock verifies $\underline{K} < K(0) < \tilde{K}$ then the capital stock will increase in the transition and there will be a change in regime along the way. The markup wil be at the start uncorrelated with activity and there will be entry. When entry begins competition will drive the markup anti-cyclicaly down.

`Figure 6 around here`

The other phase diagrams involve multiple equilibria. In order to build them we need some global dynamics results. we call Γ_{HL} to the heteroclinic orbit joining the two CMC stationary equilibria (C_H^*, K_H^*) and (C_L^*, K_L^*),

$$\Gamma_{HL} = W_H^u \cap W_L^s.$$

The heteroclinic orbit coincides with the intersection between the unstable manifold associated to (C_H^*, K_H^*), W_H^u, and the stable manifold associated to (C_L^*, K_L^*), W_L^s.

Lemma 9 *Assume that $1 < \sigma < \bar{\sigma}$ and $\overline{\phi} < \phi < \tilde{\phi}$. Then there is a smooth heteroclinic orbit, Γ_{HL}. If $K_H^* < K(0) < K_L^*$ then Γ_{HL} is the equilibrium trajectory.*

Proof See the appendix.

Therefore, the heteroclinic orbit joining the two equilibria is directed from (K^*_H, C^*_H) to (K^*_L, C^*_L) has a positive slope, it is tangent to the direction defined by the eigenvector associated to the eigenvalue with smaller absolute value in the neighborhood of (K^*_H, C^*_H), $E^{u,-}_H$, and by the negative eigenvector, E^s_L, in the neighborhood of (K^*_L, C^*_L).

We also call Γ_{HM} to a second heteroclinic orbit joining the CMC equilibrium (C^*_H, K^*_H) to the MC equilibrium (C^*_M, K^*_M),

$$\Gamma_{HM} = W^u_H \cap W^s_M.$$

Again, the heteroclinic orbit is defined as the intersection between the unstable manifold associated to (C^*_H, K^*_H), W^u_H, and the stable manifold associated to (C^*_M, K^*_M), W^s_M. As we prove next it is continuous and piecewise-smooth.

Lemma 10 *Assume that* $1 < \sigma < \bar{\sigma}$ *and* $\overline{\phi}_c < \phi < \tilde{\phi}$. *Then there is a piecewise-smooth heteroclinic orbit,* $\Gamma_{H,M}$, *connecting* (C^*_H, K^*_H) *and* (C^*_M, K^*_M). *If* $K^*_M < K(0) < K^*_H$ *then* Γ_{HM} *is the equilibrium trajectory.*

Proof See the appendix.

Proposition 9 *: Assume that* $(\sigma, \phi) \in \mathcal{D}$ *and that* $K(0) > \underline{K}$. *Then there are three stationary equilibria:* (C^*_M, K^*_M) *of the saddle-point type with monopolistic competition,* (C^*_H, K^*_H) *a unstable focus and* (C^*_L, K^*_L) *a saddle-point, both with CMC. If* $K(0) > K^*_L$ *then the equilibrium path* $\{\Phi(t, K(0)\ t \geq 0\}$, *will converge to the low-markup equilibrium* (C^*_L, K^*_L). *If* $K(0) < K^*_M$ *the equilibrium path will converge to the monopolistic-competition equilibrium* (C^*_M, K^*_M). *If* $K^*_H < K(0) < K^*_L$ *then the equilibrium path convergence to* (C^*_L, K^*_L) *along the smooth heteroclinic trajectory* Γ_{HL}. *If* $K^*_M < K(0) < K^*_H$ *then the economy will converge to* (C^*_M, K^*_M) *along the piecewise-smooth heteroclinic trajectory*Γ_{HM}.

Proof See the appendix.

`Figure 7 around here`

Figure 7 illustrates proposition 9. Depending on the initial level of the stock of capital, the economy "chooses" a MC equilibrium (C^*_M, K^*_M) or a CMC equilibrium with a low markup (C^*_L, K^*_L). The dividing barrier is given by the CMC equilibrium with a high markup (C^*_H, K^*_H). Off course, the economy may be trapped in the high markup CMC equilibria, which is totally unstable. A small shitf in any parameter will make the economy converge to one of the two saddle point stable equilibria. In some sense there will be a kind of juxtaposition of the two previous cases with unique stationary equilibria if the economy is located in one of the sides of the barrier. There is though a difference: there are restrictions on the possibility of regime shifts along transition paths. If the economy starts with a MC equilibrium it will never converge to a CMC stationary equilibrium, as in the case of proposition 8. However, the converse is not true: if the initial stock of capital is associated with a CMC dynamics such that $\tilde{K} < K(0) < K_H^*$ then the economy will converge to a MC equilibrium. This asymmetry is related to the fact that in this case there is a small elasticity of substitution in the demand for intermediary goods, which can be interpreted as a case in which there is an overall low measure of flexibility in the economy.

The necessity of the existence of heteroclinic orbits, which have a global nature, is obvious: the basins of atraction of the two saddle point stable equilibria should be bounded.

Proposition 10 *: Assume that $(\sigma, \phi) \in \mathcal{C}$and that $K(0) > K_H^*$. Then there are two stationary equilibria: (C_H^*, K_H^*) a unstable focus and (C_L^*, K_L^*) a saddle-point, both with CMC. If $K(0) > K_H^*$ then the equilibrium path $\{\Phi(t, K(0))\ t \geq 0\}$ will converge to the low-markup equilibrium through a smooth heteroclinic trajectory, Γ_{HL}.*

Proof See the appendix.

`Figure 8 around here`

7 Relation to existing literature

Endogenous mark-ups have been a matter of interest in macroeconomics, especially from the middle 1990's onwards[8]. There a variety of mechanisms that have been explored. In monetary models, nominal rigidities have been seen as a source of endogenous markups in the New Neoclassical/Keynesian Synthesis approach (Clarida et al. (1999) and Goodfriend and King (1997))[9]. Second there are models in which the elasticity of demand varies over time. This can be due to the composition of demand that

[8]Despite the fact we can find older references to endogenous markups in macroeconomics, especially in Dunlop (1938) critique to Keynes' counter-cyclical real wage due to demand shocks, the generalised interest was established with the seminal works of Rotemberg and Woodford (1991) and Rotemberg and Woodford (1995).

[9]Sticky prices are not the only source of endogenous markups, and they may not be the most important one. Furthermore, the interaction between sources of markup variation and with other real rigidities may play an important role in explaining the business-cycle phenomena. For a survey of the literature refer to Rotemberg and Woodford (1999)

varies[10], or entry altering the variety of intermediate inputs (Galí (1995) assumes entry increases the elasticity of substitution, hence reducing the markup[11]). Ravn et al. (2006) use the "deep habit" model with lower elasticities of demand when output expands, i.e. habitual consumption has a lower elasticity than additional consumption: Ravn et al. (2008) use variety-specific subsistence levels to generate procyclical elasticities for each variety[12]. Variations in the degree of collusion as the interest rate (discount rate) varies have also been put forward (Rotemberg and Woodford (1991) and Rotemberg and Woodford (1992)). There, it is assumed that Bertrand competitors in each industry have an implicit cartel contract with punishments for those who deviate from it. Since firms have a high incentive to deviate during booms, then the cartel decreases its markup in order to eliminate this incentive.

Related to the Cournotian approach of this paper is Linneman (2001) model of entry in monopolistic competition as used in Jaimovich (2007) and Jaimovich and Floetotto (2008), Bilbiie et al. (2007). Entry reduces the market share of firms, and hence reduces the "own price effect" of the monopolist on the aggregate price index, which increases the elasticity of demand (see Yang and Heidra (1993)). Other papers that consider a variety of aggregate feedback mechanisms are or D'Aspremont et al. (1989), Wu and Zhang (2000) and Costa (2001).

Another interesting feature of endogenous-markups models, also shared by models with production externalities, is the possibility of generating local indeterminacy and consequently allowing for shocks in expectations to drive fluctuations in sunspot equilibria. dos Santos Ferreira and Dufour (2006) and Jaimovich (2007) are only two recent examples of this strand of literature, and Benhabib and Farmer (1999) and Benhabib and Galí (1995) supply useful surveys for this topic.

8 Conclusion

In this paper we develop a simple dynamic general-equilibrium model with Cournotian Monopolistic Competition and instantaneous free entry. The model endogenously generates two regimes with different economic and dynamic features: i) a stable monopoly regime associated with very high markup levels and low welfare and ii) a oligopoly regime that may produce one or two equilibria, a stable low-markup and an unstable high-markup one, where the latter works as a threshold between regimes.

[10] Galí (1994b) and Galí (1994a) assume the elasticities of demand for investment and consumption differ, so the aggregate elasticity varies along the business cycle due to changes in the composition of aggregate demand.

[11] see also Bilbiie et al. (2007).

[12] This approach provides a microeconomic foundation for a similar "ad hoc" effect in Chevalier and Scharfstein (1996) and Rotemberg and Woodford (1991).

We provide a rigorous study of non-smooth dynamics and we also analyse the global dynamics of the model, demonstrating the model exhibits robust heteroclinic orbits, either of the smooth or the non-smooth type.

We show that two economies that exhibit the same fundamental parameters may behave very differently, as they may be in two different regimes. Additionally, it is possible for one economy to change regime along its convergence to a stable long-run equilibrium. Fixed costs and elasticities of demand, play a crucial role in this model and changing their values may alter the dynamics in a radical way, either by inducing a discontinuous transition or a discontinuous hysteresis.

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A Technical appendix

A.1 An alternative formulation

Consider the CES production function of final good is given by the following expression instead of the one in equation (5):

$$Y(t) = \left[\int_0^{z(t)} y(v,t)^{\frac{\sigma-1}{\sigma}} dv \right]^{\frac{\sigma}{\sigma-1}},$$

where $0 < z(t) \leq 1$ is the mass of the continuum of industries at moment t. Here, we assume it is bounded above by a technological frontier[13]. Is, as in Costa and Dixon (2007), we assume new firms prefer to be monopolists to share existing industries, free entry will produce a monopolistic-competition regime for $0 < K^* < \tilde{K}$ and the lower bound $\underline{K}$ is simply compressed to zero. In this case, there is no need for an extra coordination mechanism in order to sustain a monopolistic-competition equilibrium.

For S_1, that is for $0 < K < \tilde{K}$, the GE model is given by

$$\begin{aligned} \dot{C} &= \Big((1-1/\sigma)F'(K) - (\rho+\delta)\Big)C, \\ \dot{K} &= (1-1/\sigma)F(K) - C - \delta K. \end{aligned}$$

Notice all the qualitative results remain the same, as the model is still piecewise-smooth and it still behaves like a constrained Ramsey model in S_1.

A.2 Proofs

Proof of Lemma 1. When we compare equations (25) and (28), we see that

$$\overline{K} = \left(\frac{\tilde{\sigma}}{\sigma}\right)^{2/\alpha} \tilde{K}.$$

■

Proof of Proposition 1. Consider function $R(K)$ as in equation (26). First, it is continuous in all its domain because $R_1(K)$ is continuous if $K \in [\underline{K}, \tilde{K})$ and $R_2(K)$ is continuous if $K \in [\tilde{K}, \infty)$, and it is also continuous if $K = \tilde{K}$, as $R_1(\tilde{K}) = R_2(\tilde{K})$, because $m(\tilde{K}) = m(K)|_{K=\tilde{K}} = 1/\sigma$. Also, we readily see that $R_1(K) > 0$ and that $R_2(K) \geq 0$ if $m(K) \leq 1$. But as this condition holds only if $K \geq K^{(m)} = (\phi/(\sigma A))^{1/\alpha}$

[13] We ignore the product-innovation process here in order to simplify the model. However, exploring a Schumpeterian version of this model is a matter for further research.

and $\underline{K} > K^{(m)} \equiv \{K : m(K) = 1\}$ then $R_2(K) > 0$ if $K > \underline{K}$. Then $R_2(K) > 0$ if $K \in S_2$.

To study its slope, consider first the two branches in isolation. The (classical) derivatives for the two branches are

$$R_1^{'}(K) = -(1-\alpha)(1-1/\sigma)F^{'}(K)/K,$$

then $R_1^{'}(K) < 0$ and has the Inada properties, for all K, and

$$R_2^{'}(K) = -(1-\alpha)\left(1-\overline{\sigma}m(K)\right)F^{'}(K)/K.$$

We see that $R_2^{'}(K) > 0$ if $1/\overline{\sigma} < m(K)$, $R_2^{'}(K) = 0$ if $m(K) = 1/\overline{\sigma}$ and $R_2^{'}(K) < 0$ if $m(K) < 1/\overline{\sigma}$. Equivalently, we can translate this condition over the domain K: if $\underline{K} < K < \overline{K}$ then $R_2^{'}(K) > 0$, if $K = \overline{K}$ then $R_2^{'}(K) = 0$, and if $K > \overline{K}$ then $R_2^{'}(K) < 0$, where $\overline{K}$ is given in equation (28).

Now, consider function $R(K)$, as defined in equation (26), in all its domain. We have, $R^{'}(K) = R_1^{'}(K)$ if $K \in [\underline{K}, \tilde{K})$ and $R^{'}(K) = R_2^{'}(K)$ if $K \in [\tilde{K}, \infty)$, and if $K = \tilde{K}$:

$$\begin{aligned} R_1^{'}(\tilde{K}) &= R_1^{'}(K)|_{K=\tilde{K}} = -(1-\alpha)(1-1/\sigma)F^{'}(\tilde{K})/\tilde{K} \\ R_2^{'}(\tilde{K}) &= R_2^{'}(K)|_{K=\tilde{K}} = -(1-\alpha)\left(1-\overline{\sigma}/\sigma\right)F^{'}(\tilde{K})/\tilde{K} \end{aligned}$$

then $R_1^{'}(\tilde{K}) \neq R_2^{'}(\tilde{K})$ as

$$R_2^{'}(\tilde{K}) = R_1^{'}(\tilde{K}) + \frac{\alpha}{2\sigma}\frac{F^{'}(\tilde{K})}{\tilde{K}}.$$

Then: (a) for $\tilde{K} < \overline{K}$ then $R_1^{'}(\tilde{K}) < 0$ and $R_2^{'}(\tilde{K}) > 0$; (b) for $\tilde{K} = \overline{K}$ then $R_1^{'}(\tilde{K}) < 0$ and $R_2^{'}(\tilde{K}) = 0$; and (c) for $\tilde{K} > \overline{K}$ then $R_1^{'}(\tilde{K}) < R_2^{'}(\tilde{K}) < 0$.

From the properties of $R_2(K)$ and from lemma **??**, it is easy to see that: if $\sigma > \overline{\sigma}$ then $\tilde{K} > \overline{K}$ and we have case $R^b(K)$ where $R^{b'}(K) < 0$ for $K \in [\underline{K}, \infty)$ $\{\tilde{K}\}$ and $R_1^{'}(\tilde{K}) < R_2^{'}(\tilde{K}) < 0$; if $\sigma < \overline{\sigma}$ then we have case $R^a(K)$ where $R^{a'}(K) = R_1^{'}(K) < 0$ for $K \in [\underline{K}, \tilde{K})$ and $R^{a'}(K) = R_2^{'}(K)$ for $K \in [\tilde{K}, \infty)$, where $R_2^{'}(K) > 0$ for $K \in (\tilde{K}, \overline{K})$, $R_2^{'}(K) = 0$ for $K = \overline{K}$ and $R_2^{'}(K) < 0$ for $K > \overline{K}$. ■

Proof of Proposition 2. Continuity: we readily see that $Y_1(\tilde{K}) - Y_2(\tilde{K}) = F(\tilde{K})/\sigma - \phi = 0$. For the derivatives in the two branches, we get

$$\begin{aligned} Y_1^{'}(K) &= F^{'}(K) \\ Y_2^{'}(K) &= \left(1 - \frac{m(K)}{2}\right)F^{'}(K). \end{aligned}$$

which are both positive for any $K > \underline{K}$, then the two branches are monotonously increasing. In the switching point $K = \tilde{K}$ we have $Y_1^{'}(\tilde{K}) - Y_2^{'}(\tilde{K}) = F^{'}(\tilde{K})/(2\sigma) > 0$. ■

Proof of Proposition 3. We start with the equilibrium condition $R(K^*) = \rho + \delta$. Again, there are two cases related to the difference between σ and $\tilde{\sigma}$:

1. Let $\sigma \geq \bar{\sigma}$. In this case $R(K) = R^b(K)$, and as the return function is monotonically decreasing we have three possible cases:

   (a) if $\rho + \delta \geq R_1(\underline{K})$ then there is no feasible equilibrium;

   (b) if $R_1(\underline{K}) < \rho + \delta < R_1(\tilde{K}) = R_2(\tilde{K})$ then there is a unique equilibrium $K^* \in S_1$ and the equilibrium condition is $R_1(K) = \rho + \delta$;

   (c) if $\rho + \delta > R_1(\tilde{K}) = R_2(\tilde{K})$ then there is a unique equilibrium $K^* \in S_2$ and the equilibrium condition is $R_2(K) = \rho + \delta$. There is an unique boundary equilibrium if $R_1(\tilde{K}) = R_2(\tilde{K}) = \rho + \delta$.

   It is easy to see that case (a) holds if and only if $\phi < \underline{\phi}$, condition (b) holds if and only if $\underline{\phi} > \phi > \tilde{\phi}$, and condition (c) holds if and only if $\phi < \tilde{\phi}$. There is a boundary equilibrium if and only if $\phi = \tilde{\phi}$.

2. Let $1 < \sigma < \bar{\sigma}$. In this case $R(K) = R^a(K)$ and there may be multiplicity, because the return function is non-monotonous, as it decreases if $K < \tilde{K}$, has a jump in the slope at $K = \tilde{K}$, increases if $\tilde{K} < K < \overline{K}$, reaches a local maximum at $K = \overline{K}$ and is decreasing again if $K > \overline{K}$. Then five generic cases are possible:

   (a) if $R_1(\underline{K}) < R_2(\overline{K}) < \rho + \delta$, and $R_2(\overline{K}) < R_1(\underline{K}) < \rho + \delta$ then there is no equilibria such that $K^* > \underline{K}$;

   (b) if $R_1(\underline{K}) < \rho + \delta \leq R_2(\overline{K})$ then there is one or two equilibria belonging to S_2 if the second relation holds as an equality or as an inequality, respectively;

   (c) if $R_1(\underline{K}) > \rho + \delta > R_2(\overline{K})$ then there is a single equilibrium belonging to S_1;

   (d) if $R_1(\underline{K}) > R_2(\overline{K}) > \rho + \delta \geq R_1(\tilde{K}) = R_2(\tilde{K})$ then there are three equilibria if the inequality holds, one in S_1 and two in S_2, or two equilibria, one belonging to Σ and one to S_2;

   (e) if $R_1(\tilde{K}) = R_2(\tilde{K}) > \rho + \delta > 0$ then there is again an unique equilibrium belonging to S_2.

   Case (a) holds if and only if $\phi > \max\{\underline{\phi}, \overline{\phi}\}$, case (b) holds if and only if $1 < \sigma < \sigma_a$ and $\overline{\phi} \geq \phi > \underline{\phi}$, case (c) holds if and only if $\sigma_a < \sigma < \bar{\sigma}$ and $\overline{\phi} < \phi < \underline{\phi}$, case (d) holds if and only if $\min\{\underline{\phi}, \overline{\phi}\} < \phi \leq \tilde{\phi}$, and case (e) holds if and only if $\tilde{\phi} > \phi > 0$.

Now, we have to consider condition $C(K^*) = Y(K^*) - \delta K^* \geq 0$. For the equilibria belonging to branch S_2, we obtain $C_2(K_2^*) = (1 - m(K_2^*))F(K_2^*) - \delta K_2^* = K_2^*(\rho + (1-\alpha)\delta)/\alpha > 0$, where $K_2^* = \{K : R_2(K) = \rho + \delta\}$, for any admissible values of the parameters. Therefore, there is no new restriction imposed here. For the equilibrium belonging to branch S_1, we determine $C_1(K_1^*) = Y_1(K_1^*) - \delta K_1^* = (\sigma(\rho+\delta)/(\alpha(\sigma-1)) - \delta)\, K_1^* - \phi = \phi_c - \phi$. Then $C_1(K_1^*) \geq 0$ if $\phi \leq \underline{\phi}_c$. As $\underline{\phi}_c < \underline{\phi}$, this imposes a more stringent condition for the equilibria belonging to branch S_1. Therefore, in the conditions relating to the function $R(K)$, in the branch S_1, we have to substitute $\underline{\phi}$ by $\underline{\phi}_c$. ■

Proof of Lemma 5. If the equilibrium point belongs to branch S_1, we find that

$$\mathrm{tr}(J_1(K_1^*)) = C_1^{'}(K_1^*) = F^{'}(K_1^*) - \delta = \frac{\sigma(\rho+\delta)}{\sigma - 1} - \delta = \frac{\sigma\rho + \delta}{\sigma - 1} > 0$$

and

$$\det(J_1(K_1^*)) = C_1(K_1^*)R_1^{'}(K_1^*) = -(1-\alpha)(\rho+\delta)\left(\underline{\phi}_c - \phi\right)/K_1^* < 0.$$

Then $\lambda_1^- < 0 < \lambda_1^+$. ■

Proof of Lemma 6. If the equilibrium point belongs to branch S_2 then

$$\begin{aligned} C_2(K_2^*) &= Y_2(K_2^*) - \delta K_2^* = (1 - m(K_2^*))F(K_2^*) - \delta K_2^* = \\ &= \left((1 - m(K_2^*))\frac{F^{'}(K_2^*)}{\alpha} - \delta\right) K_2^* = \beta K_2^*, \end{aligned}$$

where

$$\beta \equiv \frac{\rho + \delta(1-\alpha)}{\alpha} > 0,$$

and

$$\begin{aligned} R_2^{'}(K_2^*)K_2^* &= -\left[\, 1 - \alpha - (2-\alpha)\frac{m(K_2^*)}{2}\,\right] F^{'}(K_2^*) = \\ &= -(1 - m(K_2^*))F^{'}(K_2^*) + \alpha\left(1 - \frac{m(K_2^*)}{2}\right) F^{'}(K_2^*) = \\ &= -R_2(K_2^*) + \alpha\delta + \alpha C_2^{'}(K_2^*) \\ &= \alpha\left(C_2^{'}(K_2^*) - \frac{C_2(K_2^*)}{K_2^*}\right) = \\ &= \alpha\left(C_2^{'}(K_2^*) - \beta\right). \end{aligned}$$

Then

$$\mathrm{tr}(J_2(K_2^*)) = C_2^{'}(K_2^*) = \beta + R_2^{'}(K_2^*)K_2^*/\alpha,$$

and

$$\det(J_2(K_2^*)) = C_2(K_2^*)R_2^{'}(K_2^*) = \beta R_2^{'}(K_2^*)K_2^*.$$

If $R_2^{'}(K_2^*) < 0$ then $\det(J_2(K_2^*)) < 0$ and the equilibrium point is saddle point stable, $\lambda_2^- < 0 < \lambda_2^+$. If $R_2^{'}(K_2^*) = 0$ then $\det(J_2(K_2^*)) = 0$ and $\lambda_2^- = 0 < \lambda_2^+ = \beta$. If $R_2^{'}(K_2^*) > 0$ then $\mathrm{tr}(J_2(K_2^*)) > \beta > 0$ and $\det(J_2(K_2^*)) > 0$, and, as

$$0 < \left(\frac{\beta - R_2^{'}(K_2^*)K_2^*/\alpha}{2}\right)^2 < \Delta(J_2) < \left(\frac{\beta + R_2^{'}(K_2^*)K_2^*/\alpha}{2}\right)^2,$$

then the eigenvalues are real and verify $\lambda_2^+ > \lambda_2^- > 0$. As a necessary condition for $R_2^{'}(K_2^*) = 0$ is that $1 < \sigma < \overline{\sigma}$ and is verified if ϕ crosses continuously the value of $\overline{\phi}$ then the point K_2^* is a continuous bifurcation point (see (Leine, 2006, p. 126)). ■

Proof of Lemma 7. As $R_1(\tilde{K}) = (1 - 1/\sigma)F^{'}(\tilde{K}) = \rho + \delta$, if $K^* = \tilde{K}$, then $F^{'}(\tilde{K}) = (\rho + \delta)\sigma/(\sigma - 1)$, then

$$\mathrm{tr}(J_1(\tilde{K})) = C_1^{'}(\tilde{K}) = F^{'}(\tilde{K}) - \delta = \frac{\sigma\rho + \delta}{\sigma - 1}.$$

As $C_1(\tilde{K}) = F(\tilde{K}) - \phi - \delta\tilde{K} = \left((1 - 1/\sigma)F(\tilde{K})/\tilde{K} - \delta)\right)\tilde{K} = \beta\tilde{K}$, because $F(\tilde{K}) = \sigma\phi$, and $R_1^{'}(\tilde{K}) = -(1 - \alpha)R_1(\tilde{K})/\tilde{K}$, then

$$\det(J_1(\tilde{K})) = C_1(\tilde{K})R_1^{'}(\tilde{K}) = -\beta(1 - \alpha)(\rho + \delta) < 0.$$

For the boundary regarding branch S_2, we have $m(\tilde{K}) = 1/\sigma$ then $R_2(\tilde{K}) = R_1(\tilde{K})$, $C_1(\tilde{K}) = C_2(\tilde{K})$, and also $C_2^{'}(\tilde{K}) = \left(1 - \frac{m(\tilde{K})}{2}\right)F^{'}(\tilde{K}) - \delta = F^{'}(\tilde{K}) - \delta - \frac{m(\tilde{K})}{2}F^{'}(\tilde{K}) = C_1^{'}(\tilde{K}) - F^{'}(\tilde{K})/2\sigma$. Then, as $\mathrm{tr}(J_2(\tilde{K})) = C_2^{'}(\tilde{K})$, we have

$$\mathrm{tr}(J_2(\tilde{K})) = \mathrm{tr}(J_1(\tilde{K})) - \frac{\rho + \delta}{2(\sigma - 1)} = \frac{\rho(2\sigma - 1) + \delta}{2(\sigma - 1)} > 0.$$

Considering that

$$\begin{aligned} R_2^{'}(\tilde{K}) &= -(1 - \alpha)\left[1 - \bar{\sigma}m(\tilde{K})\right]\frac{F^{'}(\tilde{K})}{\tilde{K}} = \\ &= -\left(\frac{2\sigma(1 - \alpha) + \alpha - 2}{2\sigma}\right)\frac{\sigma(\rho + \delta)}{\sigma - 1}\frac{1}{\tilde{K}} = \\ &= -(1 - \alpha)(\rho + \delta)\frac{(\sigma - \bar{\sigma})}{\sigma - 1}\frac{1}{\tilde{K}}, \end{aligned}$$

then

$$\det(J_2(\tilde{K})) = C_2(\tilde{K})R_2^{'}(\tilde{K}) = -\beta(1 - \alpha)(\rho + \delta)(\sigma - \bar{\sigma})/(\sigma - 1).$$

■

Proof of Lemma 8. The trace and the determinant of Jacobian J_q are given by

$$\begin{aligned} \mathrm{tr}(J_q(\tilde{K})) &= (1-q)\mathrm{tr}(J_1(\tilde{K})) + q\mathrm{tr}(J_2(\tilde{K})), \\ \det J_q(\tilde{K}) &= (1-q)\det(J_1(\tilde{K})) + q\det(J_2(\tilde{K})). \end{aligned}$$

Using the previous results, we find

$$\mathrm{tr}(J_q(\tilde{K})) = \frac{2(\sigma\rho+\delta) - q(\rho+\delta)}{2(\sigma-1)} > 0,$$

for any value of $q \in [0,1]$, and

$$\det(J_q(\tilde{K})) = -\beta(1-\alpha)(\rho+\delta)\left(1 - q\left(\frac{\bar{\sigma}-1}{\sigma-1}\right)\right),$$

which can be zero if and only if $q = \frac{\sigma-1}{\bar{\sigma}-1}$ which can only belong to the interval $[0,1]$ if $\sigma \leq \bar{\sigma}$. ■

Proof of Proposition 4. As

$$\mathrm{tr}(J_1(\tilde{K}))\mathrm{tr}(J_2(\tilde{K})) > 0,$$

a necessary condition for the existence of limit cycles does not hold, see Leine (2006) and di Bernardo et al. (2008). If $\sigma > \bar{\sigma}$ then

$$\det(J_1(\tilde{K}))\det(J_2(\tilde{K})) > 0,$$

and there is persistence. If $1 < \sigma < \bar{\sigma}$ then

$$\det(J_1(\tilde{K}))\det(J_2(\tilde{K})) < 0,$$

and the system undergoes a discontinuous bifurcation, where two admissible equilibria (for $\phi > \tilde{\phi}$) become virtual (for $\phi < \tilde{\phi}$) when ϕ crosses $\tilde{\phi}$. ■

Proof of Proposition 5. This proposition is a consequence of Proposition 3 and Lemmas 5 and 6. In the first subset in the parameter space, if $(\phi,\sigma) \in \mathcal{B} \cup \mathcal{E}$, the steady state is unique by Proposition 3. If $K^* \in S_1$ then it is saddle point stable by Lemma 5. If n addition, if $K^* \in S_2$ the steady state can only be unique when the $R(K)$ function is locally decreasing, both in cases in which $R(K) = R^a(K)$ or $R(K) = R^b(K)$, and therefore is saddle point stable by Lemma 6. In the second case, $(\phi,\sigma) \in \mathcal{D}$, there are three stationary equilibria, one of the equilibrium lies in the MC area and the other two in the CMC area. In this case we have necessarily $R(K) = R^a(K)$. Therefore, the first is a saddle point, by Lemma 5. The other two should necessary be a saddle, for high levels of K and a unstable focus, for low levels of K, by Lemma 6. In the last case, if $(\phi, \sigma) \in \mathcal{C}$, there will be no admissible MC equilibria, because if it exists it would be associated to a negative consumption. As, again, $R(K) = R^a(K)$ the number and stability properties are similar to the CMC equilibria in the last case. ■

Proof of Proposition 6. 1. Is a consequence of Proposition 3, of the definition of $\overline{\phi}$ and of Lemmas 5 and 6. As $\mathcal{D}^s$ is in the boundary between regions $\mathcal{B}$ and $\mathcal{E}$ the number of equilibria passes from one to three if the when the parameter ϕ passes through the value $\overline{\phi}$ from superior values. If $\phi = \overline{\phi}$ a new CMC equilibrium arises such that $K_2^* = \overline{K}$ and $\det(\overline{K}) = 0$, which is a bifurcation point. As, for those values of the parameters, $\overline{K} \neq \tilde{K}$, this equilibrium is not a boundary equilibrium, then we have a classical or continuous subcritical pitchfork bifurcation, as in smooth ordinary differential equations bifurcation theory.

2. Is also a consequence of Proposition 3, of the definition of $\tilde{\phi}$, of Lemmas 7 and 8, and of Proposition 4. Similarly to the previous case, when ϕ increases from smaller values in the transition for $\mathcal{D}$ to $\mathcal{B}$ the number of equilibria passes from one to three. However, differently from the previous case when $\phi = \tilde{\phi}$ the system undergoes a non-smooth bifurcation. In this point there is a generalized eigenvalue such that it contains a zero eigenvalue. This is a discontinuity induced bifurcation of the pitchfork type because in this point the number of equilibria passes from one to three in a discontinuous way.

3. This case occurs for a single point in the parameter space and corresponds to a co-dimension two discontinuous bifurcation, and is a limit case of the two previous bifurcations. ■

Proof of Proposition 7. The fact that the stationary equilibrium is unique, belongs to set S_1 (i.e., it is MC), and is saddle point stable should be obvious from Propositions 3 and 5. The local dynamics for initial points belonging to set S_1 is standard. The transversality condition (32) is met if the GE trajectories $\Phi(t)$ are tangent to the stable manifold passing through (C_M^*, K_M^*), $W_M^s = W^s(C_M^*, K_M^*)$. Locally, W_M^s is tangent to the linear subspace E_M^s whose slope is steeper than that of the isocline $\dot{K} = 0$,

$$0 < \left.\frac{dC}{dK}\right|_{\dot{K}=0} = C_1'(K_M^*) = \mathrm{tr}(J_1(K_M^*)) < \left.\frac{dC}{dK}\right|_{E_M^s} = \lambda_M^+.$$

As the isocline $\dot{C} = 0$ has slope $\frac{dC}{dK}|_{\dot{C}=0} = \infty$, this means that the local dynamics is similar to the Ramsey model. If the initial point belongs to set S_2 the proof involves non-smooth dynamic analysis. There are several possible different ways in which a out-of-equilibrium trajectory behaves in the neighborhood of a switching boundary. In our model, the local dynamics in of the switching boundary is such that a trajectory for $(C(t), K(t))$ starting with a level of $K(0)$ belonging to S_2 has an associated level of consumption such that $\dot{K} < 0$, approaches the switching boundary Σ from above the isocline $\dot{K} = 0$, and passes through Σ (and does not slide or ricochet back to S_2, see di Bernardo et al. (2008)), changes its rate of growth, and continues to decrease. The initial level of consumption, in the branch S_2 should be chosen such that the transversality condition (32) , determined in branch S_1, holds. This implies that there is only one level of consumption $\Phi_C(\tilde{K})$ in the switching boundary such that the transversality condition holds and it belongs to the stable manifold associated with equilibrium point (C^*_M, K^*_M), $W^s_M = W^s(C^*_M, K^*_M)$. Therefore the initial level of consumption $C(0)$ is such that the point $(\Phi_C(\tilde{K}), \tilde{K})$ is reached at time $t = \tau > 0$ when the initial capital stock is given by $K(0)$ at time $t = 0$. We observe that the stable manifold $W^s_M = W^s(C^*_M, K^*_M)$ is extended to branch S_2 and is therefore non-smooth. ■

Proof of Proposition 8. The proof is similar to the proof of Proposition 7, *mutatis mutandis*. In this case, observe that the equilibrium point is (C^*_L, K^*_L) and transversality condition is met if all the trajectories, for every admissible initial capital stock $K(0)$ belong to the the stable manifold $W^s_L = W^s(C^*_L, K^*_L)$, which is again non-smooth and extends to branch S_1. Asymptotically the GE trajectories converge to the subspace tangent to the stable manifold in the neighborhood of (C^*_L, K^*_L), E^s_L, which has a slope which is again steeper than the isocline $\dot{K} = 0$ as

$$0 < \left.\frac{dC}{dK}\right|_{\dot{K}=0} = C_2'(K^*_L) = \mathrm{tr}(J_2(K^*_L)) < \left.\frac{dC}{dK}\right|_{E^s_L} = \lambda^+_L.$$

■

Proof of Lemma 9. Assume that $1 < \sigma < \tilde{\sigma}$ and $\overline{\phi} < \phi < \tilde{\phi}$. In this case we have $R(K) = R^a(K)$ and we consider the dynamic system in a sub-region of region S_2 such that $K^*_H \leq K \leq K^*_L$. The dynamics is then given by equations $\dot{C} = C(R_2(K) - (\rho + \delta))$ and $\dot{K} = Y_2(K) - C - \delta K$. We study the local dynamics in that subregion and next we study global dynamics.

Let us consider the two loci:

$$\begin{aligned} L_1 &= \{(C, K) : \ C = Y_2(K) - \delta K, \ K^*_H < K < K^*_L\} \\ L_2 &= \{(C, K) : \ C = \beta K, \ K^*_H < K < K^*_L\}, \end{aligned}$$

the first locus corresponds to isocline $\dot{K} = 0$ and the second to a line joining the two equilibria (see figure 9).

We already know that the two equilibria, (C^*_H, K^*_H) and (C^*_L, K^*_L), are locally an unstable source and a saddle point, respectively. The Jacobian J_2 associated to the (C^*_L, K^*_L) has eigenvalues

$$\lambda^-_L = \frac{\mathrm{tr}(J_2(K^*_L))}{2} - \Delta(J_2(K^*_L))^{1/2} < 0 < \lambda^+_L = \frac{\mathrm{tr}(J_2(K^*_L))}{2} + \Delta(J_2(K^*_L))^{1/2}.$$

In addition, $E_L^s = (\lambda_L^+, 1)$ is the eigenvector associated to λ_L^- and $E_L^u = (\lambda_L^-, 1)$ is the eigenvector associated to λ_L^+. The stable subspace E_L^s is tangent to the stable manifold $W_L^s = W^s(K_L^*, C_L^*)$.

As

$$\left.\frac{dC}{dK}\right|_{L_1,(C,L)=(C_L^*,K_L^*)} = C_2'(K_L^*) = \text{tr}(J_2(K_L^*))$$

then the stable manifold in the neighborhood of (C_L^*, K_L^*) will be delimited by lines L_1 and L_2,

$$\left.\frac{dC}{dK}\right|_{L_2} = \beta > \left.\frac{dC}{dK}\right|_{E_L^s} = \lambda_L^+ > \left.\frac{dC}{dK}\right|_{L_2} = 0 > \left.\frac{dC}{dK}\right|_{E_L^u}$$

because, from Lemma 6, $0 < \text{tr}(J_2(K_L^*)) < \lambda_L^+ < \beta$.

The eigenvalues and the eigenvectors of the jacobian J_2 associated to (C_H^*, K_H^*) are

$$0 < \lambda_H^- = \frac{\text{tr}(J_2(K_H^*))}{2} - \Delta(J_2(K_H^*))^{1/2} < \lambda_H^+ = \frac{\text{tr}(J_2(K_H^*))}{2} + \Delta(J_2(K_H^*))^{1/2},$$

and $E_H^{u,+} = (\lambda_H^-, 1)$, is the eigenvector associated to λ_H^+, and $E_H^{u,-} = (\lambda_H^+, 1)$, is the eigenvector associated to λ_H^-. They span the subsace tangent to the unstable manifold $W_H^u = W^u(K_H^*, C_H^*)$.

As

$$\left.\frac{dC}{dK}\right|_{L_1,(C,L)=(C_H^*,K_H^*)} = C_2'(K_H^*) = \text{tr}(J_2(K_H^*))$$

then, in the neighborhood of (C_H^*, K_H^*)

$$\left.\frac{dC}{dK}\right|_{E_H^{u,-}} = \lambda_H^+ > \left.\frac{dC}{dK}\right|_{L_1} = \beta > \left.\frac{dC}{dK}\right|_{E_H^{u,+}} = \lambda_H^-,$$

because, from Lemma 6, $0 < \lambda_H^- < \beta < \lambda_L^+ << \text{tr}(J_2(K_H^*))$. Then the tangent to the eigenspace associated to the eigenvalue which is smaller in absolute value will also be delimited by lines L_1 and L_2.

In order to study global dynamics, we consider the dynamics inside the trapping area delimited by lines L_1 and L_2. This approach is common in global dynamics analysis. The behaviour of the flows which solve the differential equation (31) -(31) in the boundary of the trapping area are as follows. The flows at side L_1 verify

$$\left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_1} = (Y_2(K) - \delta K)(R_2(K_H^*) - (\rho + \delta)) > 0,$$

$$\left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_1} = 0,$$

for all $K_H^* < K < K_L^*$, and therefore the slope of the flows generated by the differential equation (30) - (31) have slope

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{L_1} = \infty$$

and therefore point outward at side L_1. The flows at side L_2 verify

$$\left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_2} = \beta K(R_2(K) - (\rho+\delta)) > 0$$

and

$$\left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_2} = Y_2(K) - (\beta+\delta)K > 0$$

for all $K_H^* < K < K_L^*$. This implies that the slope of the flows generated by the differential equation (30) - (31) is

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{L_2} = \frac{\beta K(R_2(K) - (\rho+\delta))}{(R_2(K) - \alpha(\beta+\delta))K/\alpha} = \alpha\beta < \beta = \left.\frac{dC}{dK}\right|_{L_2},$$

that is the slope of the flows passing through L_2 is smaller than the slope segment of L_2 in space (K, C), which means that the flows which solve differential equation (30) -(31), also point outward at side L_2. As all flows point outward in the boundaries of the trapping area which do not coincide with stationary equilibria, this means that the stable manifold W_L^s should be located inside the trapping area and that there is a heteroclinic trajectory, Γ_{HL}, joining K_H^* and K_L^*. The heteroclinic trajectory is tangent to $E_H^{u,-}$ in the neighborhood of (K_H^*, C_H^*) and to E_L^s in the neighborhood of (K_L^*, C_L^*), which, as we saw, lies inside the trapping area. Geometrically, it separates the flows that leave side L_1 from those leaving from side L_2.

Then the general equilibrium path $(\Phi_C(t), \Phi_K(t))$, for any initial K such $K_H^* < K(0) < K_L^*$, follows the heteroclinic trajectory Γ_{HL} because it is the only flow starting inside the trapping area which verifies the transversality condition. ■

`Figure 9 around here`

Proof of Lemma 10. We apply the same method as in the previous lemma, but take into account the non-smoothness properties of the dynamic system, as the equilibrium path crosses two branches, S_1 and S_2. The main consequence is that the trapping region should be divided into two contiguous subsets belonging to each branch. We prove that there is an unique piecewise-smooth equilibrium trajectory inside that trapping region, connecting the stationary equilibria (C_H^*, K_H^*) and (C_M^*, K_M^*).

We have again $R(K) = R^a(K)$ and consider $K_M^* \leq K \leq K_H^*$. If $K_M^* \leq K \leq \tilde{K}$, then the dynamics is given by equations $\dot{C} = C(R_1(K) - (\rho + \delta))$ and $\dot{K} = Y_1(K) - C - \delta K$, and, if $\tilde{K} \leq K \leq K_M^*$, then $\dot{C} = C(R_2(K) - (\rho + \delta))$ and $\dot{K} = Y_2(K) - C - \delta K$. Consider the trapping area defined by lines (see figure 10)

$$
\begin{aligned}
L_1 &= \{(C,K):\ C = \beta K,\ K_M^* < K < K_H^*\}, \\
L_2 &= \{(C,K):\ \dot{K} = 0,\ K_M^* < K < K_H^*\} \\
L_3 &= \{(C,K):\ K = K_M^*,\ C_M^* < C \leq \beta K_M^*\}
\end{aligned}
$$

the first locus corresponds to the prejection of the $C = \beta K$ line, which links the two equilibria in Lemma 10, and the second line is the isocline $\dot{K} = 0$. We can partition the two segments L_1 and L_2 into the subsets lying on S_1 and S_2, as

$$
\begin{aligned}
L_{11} &= \{(C,K):\ C = \beta K,\ K_M^* < K \leq \tilde{K}\}, \\
L_{12} &= \{(C,K):\ C = \beta K,\ \tilde{K} \leq K < K_H^*\}, \\
L_{21} &= \{(C,K):\ C = Y_1(K) - \delta K,\ K_M^* < K < K_H^*\} \\
L_{22} &= \{(C,K):\ C = Y_2(K) - \delta K,\ \tilde{K} \leq K < K_H^*\},
\end{aligned}
$$

From Lemma 6 we know that the equilibrium (K_M^*, C_M^*) is a saddle point and from lemma 7 and 10 that equilibrium (K_H^*, C_H^*) is an unstable source. Again the stable manifold associated to (K_M^*, C_M^*), W_M^s, has a tangent, E_M^s, that lies inside the trapping area because

$$
0 < \left.\frac{dC}{dK}\right|_{L_{21}} = \operatorname{tr}(J_1(K_M^*)) < \left.\frac{dC}{dK}\right|_{E_M^s} = \lambda_M^+ < \left.\frac{dC}{dK}\right|_{L_3} = \infty
$$

and the eigenspace associated to the eigenvalue which is smaller in absolute value in a neighborhood of (K_H^*, C_H^*) also lies inside the trapping area because

$$
\left.\frac{dC}{dK}\right|_{L_{22}} = \operatorname{tr}(J_2(K_H^*)) > \left.\frac{dC}{dK}\right|_{E_H^{u,-}} = \lambda_H^- > \left.\frac{dC}{dK}\right|_{L_{12}} = \beta > 0.
$$

Next, we prove that the flows generated by differential equations (30) -(31), $\{(\Phi_C(t, K(0)), \Phi_K(t, K(0)):\ t \geq 0\}$, evaluated at the boundary of the trapping area, all point outwards:

1. in the side L_1 the flow points outwards because, for $K_M^* < K \leq \tilde{K}$

$$
\beta \tilde{K}(R_1(\tilde{K}) - (\rho + \delta)) < \left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_{11}} = \beta K(R_1(K) - (\rho + \delta)) < 0
$$

$$\left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_{11}} = Y_1(K) - (\delta+\beta)K < 0$$

and, then,

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_C(t,K)}\right|_{L_{11}} = \frac{\beta K(\rho+\delta) - R_1(K))}{(\delta+\beta)K - Y_1(K)} < \frac{\beta K(\rho+\delta) - R_1(K))}{(\delta+\beta)K - R_1(K)K/\alpha} < \alpha\beta < \left.\frac{dC}{dK}\right|_{L_{11}} = \beta$$

because $Y_1(K) = F(K) - \phi < (1 - 1/\sigma)F(K)$. In the second segment of L_1, for for $\tilde{K} \leq K < K_H^*$, we have

$$\beta\tilde{K}(R_2(\tilde{K}) - (\rho+\delta)) < \left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_{12}} = \beta K(R_2(K) - (\rho+\delta)) < 0$$

$$Y_2(\tilde{K}) - (\delta+\beta)\tilde{K} < \left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_{12}} = Y_2(K) - (\delta+\beta)K < 0$$

and, then,

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{L_{12}} = \frac{\beta K(\rho+\delta) - R_2(K))}{(\delta+\beta)K - Y_2(K)} = \alpha\beta < \left.\frac{dC}{dK}\right|_{L_{12}} = \beta.$$

therefore the flow $(\Phi_C(t), \Phi_K(t))$ crosses out both L_{11} and L_{12} with positive but smaller slopes than that of L_1 (i.e, β);

2. in the side L_2 the flow points outwards because,

$$\begin{aligned} C_1(K_M^*)(R_1(K_M^*) - (\rho+\delta)) = 0 \;&>\; \left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_{21}} = C_1(K)(R_1(K) - (\rho+\delta)) > \\ &>\; C_1(\tilde{K})(R_1(\tilde{K}) - (\rho+\delta)) \end{aligned}$$

$$\begin{aligned} C_2(\tilde{K})(R_2(\tilde{K}) - (\rho+\delta)) \;&<\; \left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_{22}} = C_2(K)(R_2(K) - (\rho+\delta)) < \\ &<\; C_2(K_H^*)(R_2(K_H^*) - (\rho+\delta)) = 0 \end{aligned}$$

$$\left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_{12}} = \left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_{22}} = 0$$

then the flow $(\Phi_C(t), \Phi_K(t))$ crosses out both L_{21} and L_{22} with slope

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{L_2} = \infty$$

3. in the side L_3 the flow points outwards because,

$$\left.\frac{d\Phi_C(t,K)}{dt}\right|_{L_3} = C(R_1(K^*_M) - (\rho+\delta)) = 0,$$
$$\left.\frac{d\Phi_K(t,K)}{dt}\right|_{L_3} = Y_1(K^*_M) - \delta K^*_M - C < 0,$$

because $C > C^*_H$, and crosses transversally L_3 as

$$\left.\frac{d\Phi_C(t,K)}{d\Phi_K(t,K)}\right|_{L_3} = 0, \; \left.\frac{dC}{dK}\right|_{L_3} = \infty.$$

All the flows that solve equations (30) -(31) , $\{(\Phi_C(t), \Phi_K(t)),\ t \geq 0\}$, have the same properties regarding continuity and smoothness as the functions in those equations. Therefore, they are continuous and piecewise-smooth, with a change in the time derivatives when they cross $\tilde{K}$.

Therefore, as in the case of Lemma 10, if $K^*_M < K(0) < K^*_H$ the GE equilibrium path, $\{(\Phi_C(t), \Phi_K(t)),\ t \geq 0\}$, which is the unique flow that verifies the transversality conditions, cannot cross the boundary of the trapping area. Then it is also unique and is a piecewise-smooth heteroclinic orbit joining (K^*_H, C^*_H) and (K^*_M, C^*_M), Γ_{HM}. ■

`Figure 10 around here`

Proof of Proposition 9. For the case $K^*_M < K(0) < K^*_L$ this is a consequence of lemmas 9 and 10. For the other initial points the proof is obvious. ■

Proof of Proposition 10. For the case $K^*_H < K(0) < K^*_L$ this is a consequence of lemma 9. For the other initial points the proof is obvious. ■

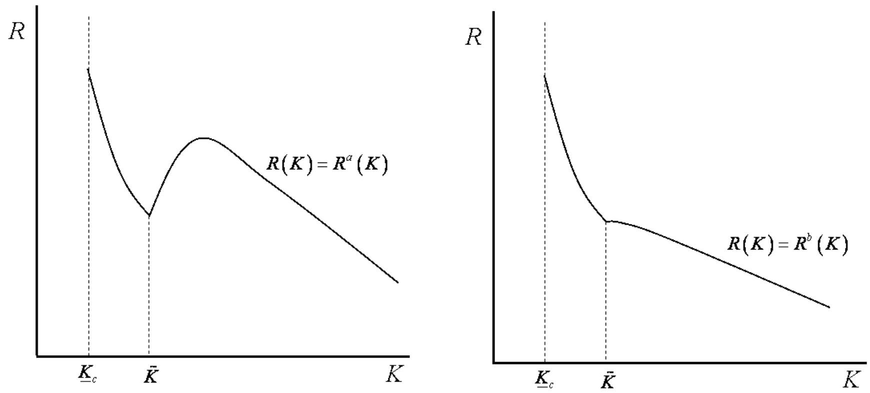

Figure 1: Alternative $R(K)$ functions. The left subfigure corresponds to the case $1 < \sigma < \overline{\sigma}$ and the right subfigure to the case $\sigma \geq \overline{\sigma}$.

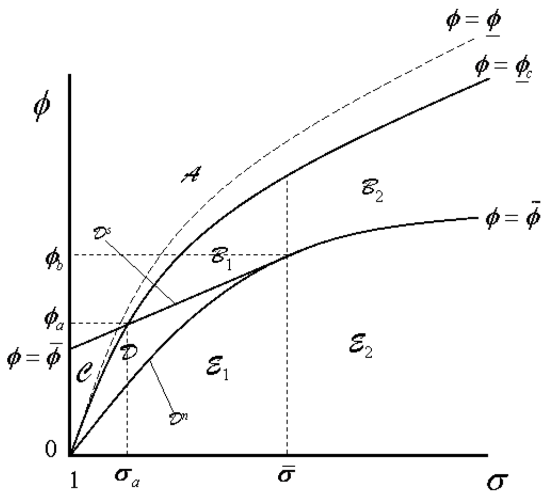

Figure 2: Stationary equilibria in the space (σ, ϕ): $\mathcal{A}$ no equilibria, $\mathcal{B}$ one MC equilibrium, $\mathcal{C}$ two CMC equilibria, $\mathcal{D}$ three equilibria, one MC and two CMC, $\mathcal{E}$ one CMC equilibrium. $\mathcal{D}^s$ local continuous bifurcations of CMC equilibria and $\mathcal{D}^n$ local discontinuity induced bifurcations of MC and CMC equilibria.

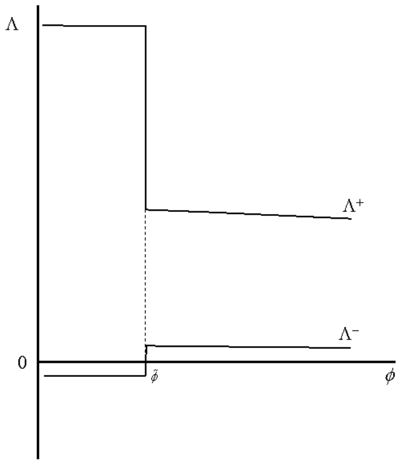

Figure 3: Generalized eigenvalues in a neighborhood of the discontinuous bifurcation when ϕ crosses the bifurcation value $\tilde{\phi}$, for $1 < \sigma < \bar{\sigma}$. If $\sigma = \bar{\sigma}$ the right part of the branch Λ^- will coincide with the abscissa.

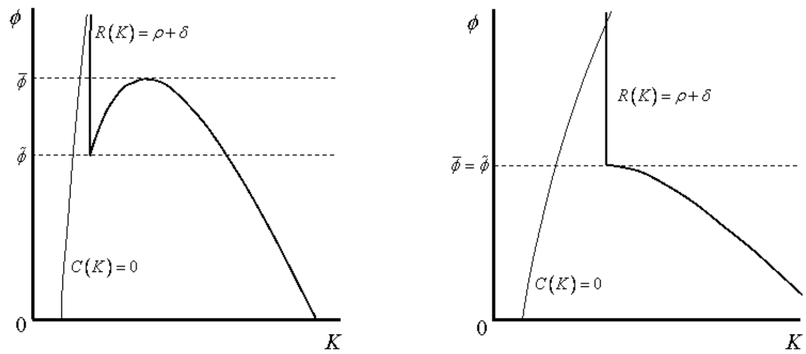

Figure 4: Bifurcation diagram (K, ϕ). The left subfigure is for case $1 < \sigma < \bar{\sigma}$ and displays a non-smooth hysteresis. Case $\mathcal{B}_1$ corresponds to the parameter values above line $\phi = \overline{\phi}$, $\mathcal{D}^s$ corresponds to line $\phi = \overline{\phi}$ and to a continous pitchfork bifurcation, between lines $\phi = \overline{\phi}$ and $\phi = \tilde{\phi}$ we have case $\mathcal{D}$, $\mathcal{D}^n$ corresponds to line $\phi = \tilde{\phi}$ and to a discontinous pitchfork bifurcation and the lower area corresponds to case $\mathcal{E}_1$. The right subfigure is for $\sigma \geq \bar{\sigma}$ and displays a non-smooth transition. The upper area corresponds to $\mathcal{B}_2$ and the lower area to $\mathcal{E}_2$.

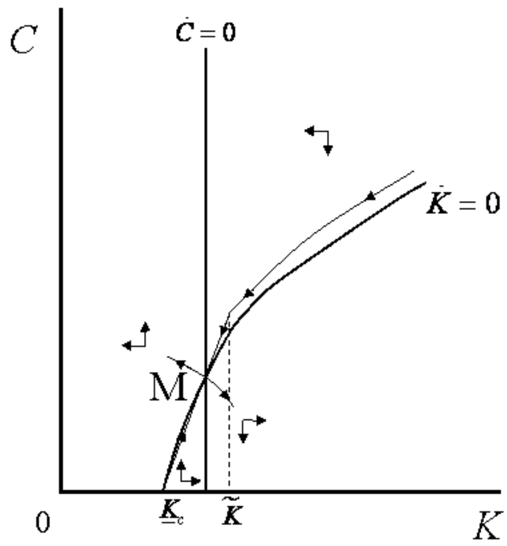

Figure 5: Phase diagram 1: $(\sigma, \phi) \in \mathcal{B}$ unique MC stationary equilibrium. The figure displays trajectories when the economy is initially in a CMC regime, or in a MC regime.

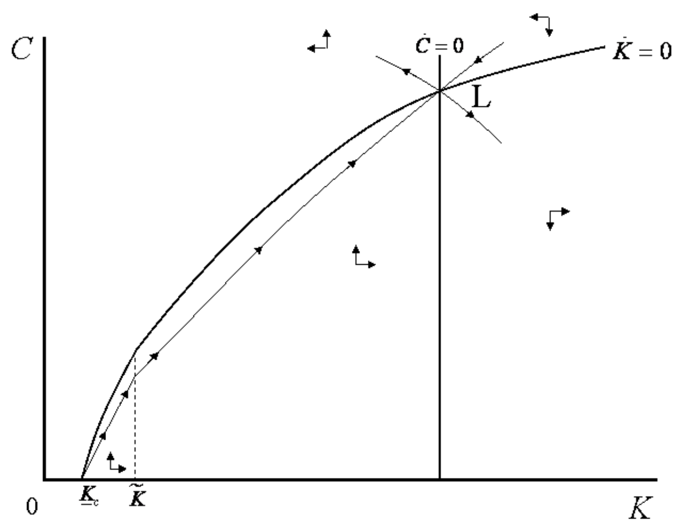

Figure 6: Phase diagram 2: $(\sigma, \phi) \in \mathcal{E}$ unique CMC stationary equilibrium. The figure displays trajectories when the economy is initially in a MC regime, or in a CMC regime.

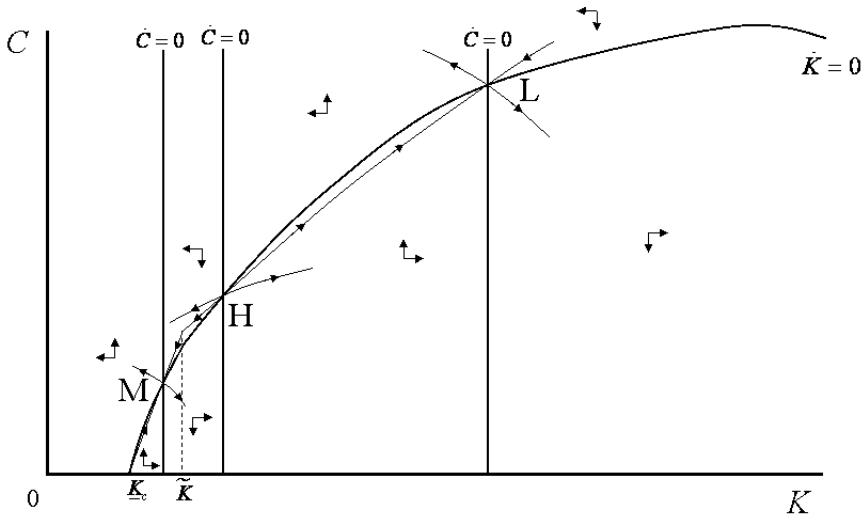

Figure 7: Phase diagram 3: $(\sigma, \phi) \in \mathcal{D}$ multiplicity of stationary equilibrium. The figure displays the smooth and non-smooth heteroclinic trajectories which correspond to the equilibrium adjustments for initial conditions smaller or larger that K_H^*.

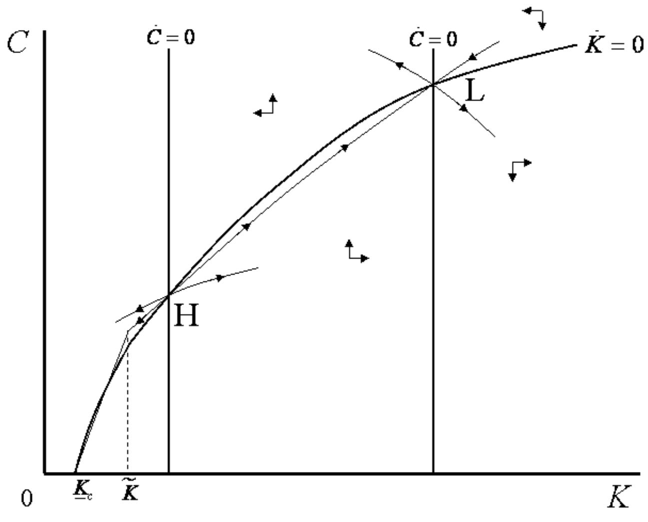

Figure 8: Phase diagram 4: $(\sigma, \phi) \in \mathcal{C}$ multiplicity of stationary equilibrium. The figure displays the smooth heteroclinic trajectory which correspond to the equilibrium adjustments for $K_H^* < K(0) < K_L^*$.

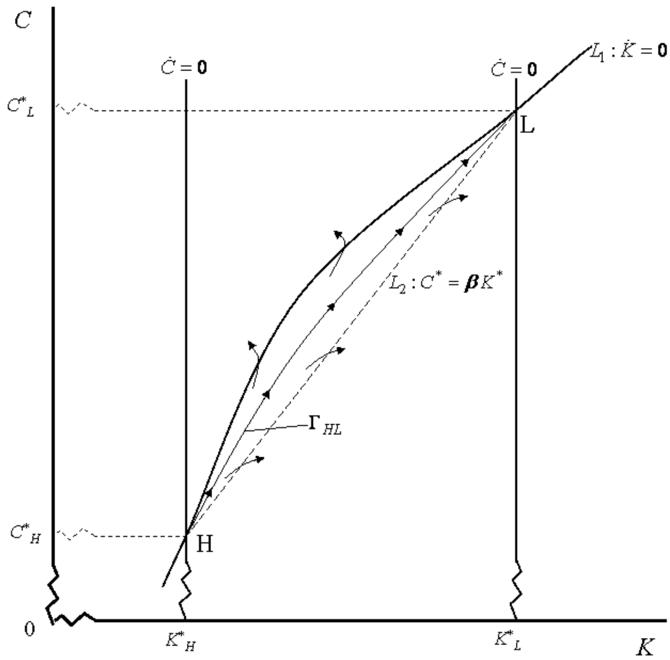

Figure 9: Graphical illustration of the proof of lemma 9

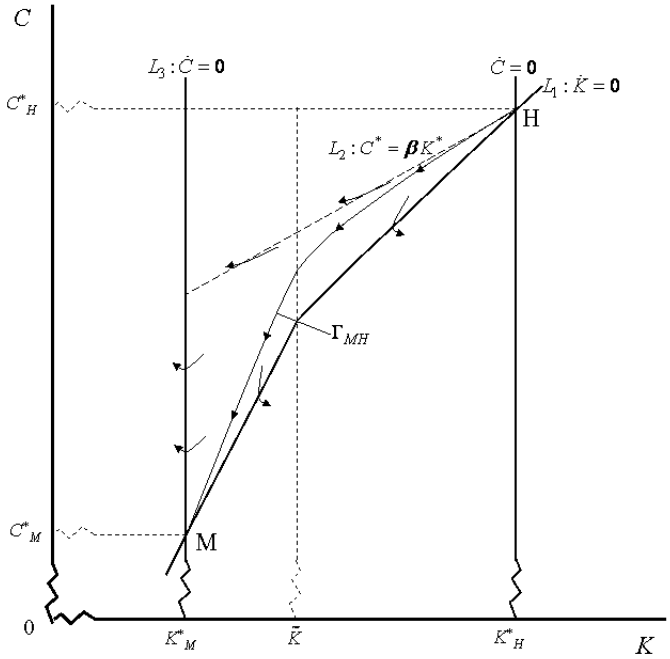

Figure 10: Graphical illustration of the proof of lemma 10