Preliminary and Incomplete

Risk Sharing in Human Capital Models with Limited Enforcement

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Abstract

This paper develops a tractable model of economic growth with human capital risk and limited enforcement/commitment. The paper shows how recursive equilibria can be exactly computed by solving a convex, finite-dimensional fixed point problem, and proves the existence of recursive equilibria by proving the existence of a solution the finite-dimensional fixed point problem. The paper also shows that a calibrated version of the model is consistent with some important empirical facts about individual income and consumption. In particular, the model replicates the strong response of individual consumption to permanent (highly persistent) income shocks that has been documented by recent micro studies. The calibrated model implies that any government policy that improves the enforcement of risk sharing arrangements/contracts has substantial effects on economic growth and welfare.

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I. Introduction

There is by now a large body of empirical work documenting that individual households face a substantial amount of labor income risk, and that this risk has non-negligible effects on consumption and welfare[1]. In other words, the hypothesis of perfect risk sharing (complete markets) is strongly rejected by the data. A recent literature has suggested that this lack of risk sharing may be explained by enforcement problems (Alvarez and Jermann (2000), Kehoe and Levine (1993), Kocherlakota (1996), Ligon, Thomas, and Worrall (2002), and Thomas and Worrall (1988)). This literature has commonly assumed that the labor income (wage) process is exogenous, an assumption that considerably limits the range of issues that can be addressed within the framework. In this paper, we develop and analyze a model of economic growth in which labor income is endogenously determined through human capital choices and insurance against idiosyncratic labor income (human capital) shocks is limited by enforcement problems.

Our model is a production economy with an aggregate constant-returns-to-scale production function using physical and human capital as input factors. There are a large number of individual households who can invest in risk-free physical capital and risky human capital. Human capital investment is risky due to idiosyncratic shocks to the stock of human capital. Financial intermediaries provide debt and insurance contracts to individual households in competitive markets. Financial contracts (risk sharing agreements) have to be self-enforcing in the sense that at any point in time it must be in the best interest of households to honor their promises. Following the literature, we assume that a household who defaults will be excluded from financial markets (risk sharing) in the future.

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[1] For the estimation of income risk, see, for example, MaCurdy (1982), Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten, Telmer, and Yaron (2004). For the consumption response, see, for example, Cochrane (1990), Flavin (1981), Townsend (1994), and Blundell, Pistaferri, and Preston (2008).

We use our model to address three fundamental problems that have hindered progress in the literature. First, stochastic models with heterogeneous households are in general difficult to analyze since the endogenous wealth distribution becomes a relevant state variable. Thus, even if we confine attention to Markov shock processes with a finite support, recursive equilibria are in general the solution to a complicated infinite-dimensional fixed point problem. For the model developed in this paper, we show that this tractability problem can be avoided. More specifically, we show that individual decision rules are linear in total wealth (financial plus human), which allows us to prove the equivalence between the complicated infinite-dimensional fixed point problem and a much simpler finite-dimensional fixed-point problem. We then prove the existence of recursive equilibria by proving the existence of a solution to the finite-dimensional fixed point problem.

Second, in models with capital accumulation (physical or human), limited enforcement of contracts gives rise to a participation constraint that may introduce non-convexities into the choice sets of individual households. Thus, individual demand correspondences may not be convex-valued and first-order conditions may not be sufficient. In our model, this non-convexity problem can be avoided. More precisely, we show that after we have transformed the infinite-dimensional fixed point problem into an equivalent finite-dimensional problem, the resulting choice sets are convex.

Finally, we use our model to address an important issue of economic substance. Recent work in the literature has shown that realistically calibrated models with physical capital and production yield almost perfect risk sharing (Krueger and Perri, 2006). In other words, the theory fails to explain the strong response of individual consumption to (permanent) income shocks found by the empirical micro literature. We show that this result crucially depends on one assumption made in the previous literature, namely that households who renege/default are punished by exclusion from future risk sharing *and* confiscation of all their capital. Intuitively, in these models the average household holds a significant amount of capital and the threat of taking it all away is sufficient to sustain almost complete risk sharing. In contrast, when we allow reneging/defaulting households to keep around half of their physical capital (and all of their human capital), we find that our model implies only a limited amount of risk sharing. In particular, the model is consistent with the response of consumption to permanent income shocks recently estimated by Blundell, Pistaferri, and Preston (2008) using household-level data on consumption and income.

Even though our analysis implies that risk sharing is far from perfect, we still find that a significant amount of insurance can be sustained in equilibrium. Moreover, the welfare gains from this insurance are quite large. In our baseline model, these gains are around 7 percent of lifetime consumption, which is two orders of magnitude larger than the welfare cost of business cycles (aggregate risk) found by Lucas (2003). Put differently, a country with weak contract enforcement can reap large benefits from improving the enforcement of various risk sharing arrangements.

II. Model

In this section, we develop the model and define the relevant equilibrium and efficiency concepts. With its emphasis on physical and human capital investment within a convex framework, the model is similar to Krebs (2003), but Krebs (2003) assumes exogenous market incompleteness. Wright (2003) considers an "AK" model with limited enforcement that bears some resemblance to the current set-up.

a) Time and Uncertainty

Time is discrete and indexed by $t = 0, 1, \ldots$. Aggregate variables are denoted by upper-case letters and individual-specific variables by lower-case letters. There is no aggregate risk and we confine attention to stationary equilibria. Idiosyncratic risk is represented by an i.i.d. shock process with realizations, s_t, that take on a finite number of possible values. We denote by $s^t = (s_1, \ldots, s_t)$ the history of idiosyncratic shocks up to period t (date-event, node) and let $\pi(s^t) = \pi(s_1) \ldots \pi(s_t)$ stand for the probability that s^t occurs. At time $t = 0$, the type of an individual household is characterized by his initial state, $x_0 = (k_0, h_0, s_0)$, where s_0 denotes the initial shock, k_0 the initial stock of physical capital, and h_0 the initial stock of human capital (note that s_0 is not included in s^t). We take as given an initial measure, μ, over initial types.

b) Production

There is one all-purpose good that can be consumed, invested in physical capital, or invested in human capital. Production of this one good is undertaken by one firm (a large number of identical firms) that rents capital and labor in competitive markets and uses these input factors to produce output, Y_t, according to the aggregate production function $Y_t = F(K_t, H_t)$. Here K_t and H_t are the (aggregate) levels of physical and human capital employed by the firm. The production function, F, is a standard neoclassical function, that is, it has constant-returns-to-scale, satisfies a Inada condition, and is continuous, concave, and strictly increasing in each argument. Given these assumptions on F, the derived intensive-form production function, $f(\tilde{K}) = F(\tilde{K}, 1)$, is continuous, strictly increasing, strictly concave, and satisfies a corresponding Inada condition, where we introduced the "capital-to-labor ratio" $\tilde{K} = K/H$. Given the assumption of perfectly competitive labor and capital markets, profit maximization implies:

$$\begin{aligned} r_k &= f'(\tilde{K}) \qquad (1) \\ r_h &= f(\tilde{K}) + f'(\tilde{K})\tilde{K} \ , \end{aligned}$$

where r_k is the rental rate of physical capital and r_h is the rental rate of human capital. Note that r_h is simply the wage rate per unit of human capital and that we dropped the time index because of our stationarity assumption. Clearly, (1) defines rental rates as functions of the capital to labor ratio: $r_k = r_k(\tilde{K})$ and $r_h = r_h(\tilde{K})$.

c) Preferences

There are a large number of infinitely-lived, risk-averse households who have well-defined preferences over consumption allocations, $\{c_t\}$, where $\{c_t\}$ denotes a sequence of functions (random variables), c_t, mapping initial types, x_0, and histories, s^t, into consumption levels, $c_t(x_0, s^t)$. Similar notation will be used for investment plans (see below). Preferences are individualistic in the sense that a household of type x_0 only cares about his own consumption plan, $\{c_t(x_0)\}$. Moreover, preferences allow for a time-additive expected utility representation:

$$U(\{c_t\}) \doteq E\left[\sum_{t=0}^{\infty} \beta^t u(c_t) | x_0\right] , \tag{2}$$

where the expectations in (2) is taken over all histories, s^t, keeping the initial type, x_0, fixed. We assume that the one-period utility function exhibits constant relative risk aversion: $u(c) = \frac{c^{1-\gamma}}{1-\gamma}$ for $\gamma \neq 1$ and $u(c) = lnc$ otherwise. In other words, we assume that preferences are homothetic.

d) Budget Constraint

Each household can invest in physical capital, k, or human capital, h. In addition, he can buy and sell a complete set of financial contracts (assets) with state-contingent payoffs. More specifically, there is one contract (Arrow security) for each state, and we denote by $a_{t+1}(s_{t+1})$ the quantity bought in period t (sold if negative) of the contract that pays off one unit of the good in period $t+1$ if s_{t+1} occurs. Given his initial type, $x_0 = (s_0, k_0, h_0)$, a household chooses a plan, $\{c_t, k_t, h_t, \vec{a}_t\}$, where the notation $\vec{a}$ indicates that in each period the household chooses a vector of contract holdings. A budget-feasible plan has to satisfy the sequential budget constraint

$$
\begin{aligned}
c_t + k_{t+1} + h_{t+1} + \sum_{s_{t+1}} a_{t+1}(s_{t+1}) q(s_{t+1}) &= (1 + r_k - \delta_k) k_t + (1 + r_h - \delta_h(s_t)) h_t + a_t(s_t) \\
c_t \geq 0 \ , \ \ k_{t+1} \geq 0 \ , \ \ h_{t+1} \geq 0 \qquad & , \qquad (3)
\end{aligned}
$$

where $q(s_{t+1})$ is the price of a financial contract that pays off if s_{t+1} occurs and δ_k and $\delta_h(s_t)$ are the depreciation rates of physical and human capital, respectively. Note that (3) has to hold in realizations, that is, it has to hold for all histories, s^t. We focus on equilibria with risk neutral financial market prices:

$$
q(s_{t+1}) = \frac{\pi(s_{t+1})}{1 + r_f} \qquad (4)
$$

where r_f is the interest rate on financial transactions. The pricing condition (4) can also be interpreted as a zero-profit-condition for financial intermediaries that borrow at the risk-free rate r_f and use the proceed to sell insurance contracts to households at prices (4).

The budget constraint (3) assumes that physical capital can be accumulated by investing $x_{kt} = k_{t+1} - (1 - \delta_k))k_t$. Similarly, human capital can be accumulated by investing $x_{ht} = h_{t+1} - (1 - \delta_h(s_t))h_t$. The budget constraint (3) makes three implicit assumptions about the accumulation of human capital. First, it lumps together general human capital (education, health) and specific human capital (on-the-job training). Second, it neglects the decision of households to allocate a fixed amount of time across different activities. Third, (3) does not impose a non-negativity constraint on human capital investment ($x_{hit} \geq 0$).

The random variable δ_{ht} represents uninsurable idiosyncratic labor income risk. A negative human capital shock, $\delta_h(s_t) < \sum_s \delta_h(s)\pi(s)$, can occur when a worker loses firm- or sector-specific human capital subsequent to job termination (worker displacement). In order to preserve the tractability of the model, the budget constraint (3) rules out extended periods of unemployment because it assumes that the wage payment is received in each period. Thus, the emphasis is on earnings uncertainty, not employment uncertainty. A decline in health (disability) provides a second example for a negative human capital shock. In this case, both general and specific human capital might be lost. Internal promotions and upward movement in the labor market provide two examples of positive human capital shock.

It is convenient to introduce new variables that emphasize that individual households solve a standard inter-temporal portfolio choice problem with additional participation constraints. To this end, introduce the following variables:

$$\begin{aligned} w_t &= k_t + h_t + \sum_{s_t} q_{t-1}(s_t) a_t(s_t) \\ \theta_{kt} &= \frac{k_t}{w_t} \ , \ \theta_{ht} = \frac{h_t}{w_t} \ , \ \theta_{at}(s_t) = \frac{a_t(s_t)}{w_t} \\ 1 + r_t &= (1 + r_k - \delta_k)\theta_{kt} + (1 + r_h - \delta_h(s_t))\theta_{ht} + \theta_{at} \end{aligned} \tag{5}$$

In (5) the variable w_t stands for beginning-of-period wealth consisting of real wealth, $k_t + h_t$, and financial wealth, $\sum_{s_t} q_{t-1}(s_t) a_t(s_t)$. The variable $\theta_t = (\theta_{kt}, \theta_{ht}, \vec{\theta}_{at})$ denotes the vector of portfolio shares and $(1 + r)$ is the gross return to investment. Using the new notation, the budget constraint (5) reads

$$\begin{aligned} w_{t+1} &= [1 + r(\theta_t, s_t)]\, w_t - c_t \\ 1 &= \theta_{kt} + \theta_{ht} + \sum_{s_t} q_{t-1}(s_t) \theta_{at}(s_t) \\ c_t &\geq 0 \ , \ w_t \geq 0 \ , \ \theta_{kt} \geq 0 \ , \ \theta_{ht} \geq 0 \ . \end{aligned} \tag{6}$$

Clearly, (6) is the budget constraint corresponding to an intertemporal portfolio choice problem with linear investment opportunities and no exogenous source of income. Note that r not only depends on the individual choice of θ, but also on the aggregate variables $\tilde{K}$ and q, but we will suppress this dependence until we turn to the general equilibrium analysis.

So far, we have not imposed any restrictions on trading of financial assets. In this paper, we augment the sequential budget constraint by the following short-sale constraints:

$$\theta_{at}(s_t) \geq -\bar{\theta}_a(s_t) \ , \tag{7}$$

where $\bar{\theta}(s_t)$ is a number that will be chosen large enough so that it will not bind in equilibrium. In this case, (7) is equivalent to a no-Ponzi-scheme condition if $r_f > 0$. However, in contrast to the no-Ponzi-scheme condition, the short-sale constraint (7) has three advantages. First, it allows us to consider equilibria with $r_f < 0$. Second, it nicely fits into a recursive formulation of the problem. Finally, it will be useful for the proof of proposition 1.

e) Participation Constraint

In addition to the standard budget constraint, the household has to satisfy a sequential participation constraint, which ensures that at no point in time individual households have an incentive to default on their financial obligations. We assume that the penalty of defaulting is exclusion from risk sharing (financial market participation) in the future. Further, we assume that defaulting households keep all their human capital and a fraction $(1-\phi)$ of their physical capital. Finally, we make the assumption that after default households use their physical and human capital to produce at home using the same technology as before. Below we show that this leads to a value function for defaulting households that takes the form

$$V_d\left(w_t, (1-\phi)\theta_{kt}, \theta_{ht}, s_t\right) = \tilde{V}_d\left(1 + r((1-\phi)\theta_k, \theta_h, 0, s)\right)^{1-\gamma} w^{1-\gamma} , \tag{8}$$

where r is the return function defined above and $\tilde{V}_d$ is a real number. Using the autarky value function (8), the sequential participation constraint reads

$$E\left[\sum_{n=0}^{\infty} \beta^n u(c_{t+n}) | x_0, s^t\right] \geq \tilde{V}_d\left(1 + r((1-\phi)\theta_{kt}, \theta_{ht}, 0, s)\right)^{1-\gamma} w_t^{1-\gamma} . \tag{9}$$

We now specify how a household employs his physical and human capital after default, which leads to the particular autarky value function (8). We assume that the household can produce output using the production function $y = F(k, h)$, where F is the production function introduced before so that there are no technological differences. Similarly, the household still faces the same types of human capital shocks in autarky. Hence, the sequential budget constraint of a household who defaulted in period t becomes:[2]

$$\begin{aligned} c_{d,t+n} + k_{d,t+n+1} + h_{d,t+n+1} &= F(k_{d,t+n}, h_{d,t+n}) + (1-\delta_k)k_{d,t+n} + (1-\delta_h(s_{t+n}))h_{d,t+n} \\ c_{d,t+n} \geq 0 \quad , &\quad k_{d,t+n+1} \geq 0 \;\; , \;\; h_{d,t+n+1} \geq 0 \; . \end{aligned} \tag{10}$$

After default, a household chooses a continuation plan, $\{c_{d,t+n}, k_{d,t+n}, h_{d,t+n}\}$, that maximizes expected lifetime utility (2) subject to the sequential budget constraint (10). The resulting choice problem of households is a standard utility maximization problem with CRRA-preferences and linear investment opportunities, for which the solution is well-known. The solution is given by (8), where the number $\tilde{V}_d$ solves the intensive-form Bellman equation

$$\tilde{V}_d = \max_{\tilde{c}_d, \theta_{d,k}} \left\{ \frac{\tilde{c}_d^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c}_d)^{1-\gamma}\tilde{V}_d \sum_{s'} (1 + r_d(\theta_d, s'))^{1-\gamma}\pi(s') \right\} . \tag{11}$$

with $r_d(\theta_d) = F(\theta_d, 1-\theta_d) - \theta_d\delta_k - (1-\theta_d)\delta_h(s)$.

Using a standard contraction mapping argument, it is straightforward to show that (11) has a unique solution if the solution, θ_d, to

$$\sum_{s'} \frac{(F_2(\theta_d, 1-\theta_d) - \delta_h(s')) - (F_1(\theta_d, 1-\theta_d) - \delta_k)}{(1 + r_d(\theta_d, s'))^{\gamma}} \pi(s') = 0 \; , \tag{12}$$

satisfies the condition

$$\beta \sum_{s'} (1 + r_d(\theta_d, s')\pi(s'))^{1-\gamma} < 1 \; . \tag{13}$$

Equation (12) says that the marginal utility weighted expected return on both investment opportunities is equalized. Using the standard properties of neoclasscial production function, one can show that (12) always has a unique solution (Krebs 2006), though condition (13) is an extra condition that is not always satisfied.

g) Equilibrium

[2]For the period in which the household defaults, we assume that he still supplies his physical and human capital to the market, so that his return (income minus depreciation) in that period is $r((1-\phi)\theta_k, \theta_h, 0, s)$.

In equilibrium, each type of household, $x_0 = (w_0, \theta_0, s_0)$, chooses a plan, $\{c_t(x_0), w_t(x_0), \theta_t(x_0)\}$, where $\theta_t = (\theta_{kt}, \theta_{ht}, \vec{\theta}_{at})$. The collection of plans for all initial types defines a global plan or allocation $\{c_t, w_t, \theta_t\}$. In equilibrium, the level of physical and human capital demanded by the firm must be equal to the corresponding aggregate levels supplied by households. Because of the constant-returns-to-scale assumption, only the ratio of physical to human capital is pinned down by this market clearing condition. That is, in equilibrium we must have:

$$\tilde{K} = \frac{E\left[\theta_{kt} w_t\right]}{E\left[\theta_{ht} w_t\right]} . \tag{14}$$

Note that $\theta_{kt} w_t$ is simply the physical capital stock of an individual household and $\theta_{ht} w_t$ the corresponding human capital stock. Note further that in (14) the expectations is taken over initial types $x_0 = (w_0, \theta_0, s_0)$ and history of shocks, s^{t-1}.

The second market clearing condition requires that no resources are created or destroyed by trading in financial contracts:

$$E\left[\theta_{at}(s_t) w_t\right] \; = \; 0 \tag{15}$$

As before, the expectation in (15) is taken over initial types $x_0 = (w_0, \theta_0, s_0)$ and history of shocks, s^{t-1}. In addition, we take the expectation over types of assets, s_t.

Straightforward calculation shows that the two market clearing conditions in conjunction with the budget constraint (7) and the pricing condition (5) imply the standard aggregate resource constraint (goods market clearing):

$$E[w_{t+1}] = F(E[\theta_{kt} w_t], E[\theta_{ht} w_t]) + (1 - \delta_k) E[\theta_{kt} w_t] + (1 - \delta_h) E[\theta_{ht} w_t] - E[c_t] \tag{16}$$

where $\delta_h = \sum_s \delta_h(s)\pi(s)$ stands for the average depreciation rate of human capital.

Our definition of equilibrium with financial markets is as follows

Definition 1. A stationary equilibrium is an allocation, $\{c_t, w_t, \theta_t\}$ and $\tilde{K}$, together with rental rates, (r_k, r_h), and financial market prices, $q(.)$, so that

i) Utility maximization: for each household type, (w_0, θ_0, s_0), the corresponding plan, $\{c_t(x_0), w_t(x_0), \theta_t(x_0)\}$ maximizes expected lifetime utility (2) subject to the sequential budget constraint (6), the short-sale constraint (7), and the sequential participation constraint (9).

ii) Profit maximization: the aggregate capital-to-labor ratio and rental rates satisfy the first-order conditions (1).

iii) Market clearing: equations (14) and (15) hold.

III. Theoretical Results

In this section, we present the main theoretical results. We begin with the principle of optimality for the individual household problem (propositions 1) and the equivalence between intensive-form Bellman equation and extensive-form Bellman equation (proposition 2). We then show the equivalence between stationary recursive equilibria and intensive-form equilibria. Further, we show the equivalence between intensive-form equilibria and an intensive-form social planner problem. We conclude with the existence of a a maximal solution to the intensive-form social planner problem, which implies that a stationary recursive equilibrium exists.

a) Principle of Optimality

The budget constraint (6) and the participation constraint (9) suggest that the utility maximization problem of an individual household is recursive in the state variable (w, θ, s). More precisely, consider the Bellman equation

$$\begin{aligned} V(w,\theta,s) &= \max_{c,w',\theta'} \left\{ \frac{c^{1-\gamma}}{1-\gamma} + \beta \sum_{s'} V\left(w',\theta',s'\right)\pi(s') \right\} \qquad (17) \\ s.t.\ \ w' &= (1 + r(\theta',s'))w - c \\ 1 &= \theta'_k + \theta'_h + \sum_{s'} \frac{\pi(s')\theta'_a(s')}{1+r_f} \end{aligned}$$

$$
\begin{aligned}
V(w', \theta', s') &\geq \tilde{V}_d \left(1 + r((1-\phi)\theta'_k, \theta'_h, 0, s')\right)^{1-\gamma} (w')^{1-\gamma} \\
c &\geq 0 \ , \ w' \geq 0 \ , \ \theta'_k \geq 0 \ , \ \theta'_h \geq 0 \ , \ \theta'_a(s') \geq \bar{\theta}_a(s') \ .
\end{aligned}
$$

Then under the condition that for all θ we have

$$
\beta \sum_s \left(1 + r(\theta, s)\right)^{1-\gamma} < 1 \tag{18}
$$

the principle of optimality holds and we can confine attention to the Bellman equation (17) when discussing the utility maximization problem of individual households.

Proposition 1. Suppose that condition (18) is satisfied. Then the values function, V, solves the Bellman equation (17). Conversely, the value function is the maximal solution of the Bellman equation (17) and is obtained as

$$
\lim_{t \to \infty} T^n V_0 = V \ ,
$$

where T is the operator associated with the Bellman equation (17) and V_0 is the solution to the corresponding Bellman equation without participation constraint.

Proof (outline). This is theorem 3.6 in Rustichini (1998), so it suffices to show that the conditions A1-A4 in Rustichini (1998) are satisfied. Bi-convergence holds if lifetime utility is finite (Streufert, 1991), and condition (18) ensures exactly this finiteness property. It is also straightforward to show that the feasibility correspondence defining the state transition is continuous and compact-valued. This proves the proposition.

b) Intensive-form Bellman equation

The next proposition is a direct consequence of proposition 1 and is the essential step for our transformation of a rather complex problem into a very simple problem. It states that instead of solving the Bellman equation (17), we can confine attention to the much simpler intensive-form Bellman equation

$$
\tilde{V} = \max_{\tilde{c}, \theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta (1-\tilde{c})^{1-\gamma} \tilde{V} \sum_{s'} \left(1 + r(\theta', s')\right)^{1-\gamma} \pi(s') \right\} \tag{19}
$$

$$
\begin{aligned}
s.t.:\quad 1 &= \theta_k' + \theta_h' + \sum_{s'} \frac{\theta_a'(s')\pi(s')}{1+r_f} \\
1 + r(\theta_k', \theta_h', \vec{\theta}_a', s') &\geq \left(\frac{\tilde{V}_d}{\tilde{V}}\right)^{\frac{1}{1-\gamma}} (1 + r((1-\phi)\theta_k', \theta_h', 0, s')) \\
1 \geq \tilde{c} \geq 0 \quad &, \quad \theta_k' \geq 0 \ , \ \theta_h' \geq 0 \ , \ \theta_a'(s') \geq \bar{\theta}_a(s')
\end{aligned}
$$

where the intensive-form variables are defined as follows:

$$
\begin{aligned}
V(w, \theta, s) &= \tilde{V}\,(1 + r(\theta, s))^{1-\gamma}\, w^{1-\gamma} \\
c(w, \theta, s) &= \tilde{c}(1 + r(\theta, s))w \ .
\end{aligned}
$$

Proposition 2. Suppose that condition (18) is satisfied. Then the value function is $V(w, \theta, s) = \tilde{V}\,(1 + r(\theta, s))^{1-\gamma}\, w^{1-\gamma}$, where $\tilde{V}$ is the maximal solution to the intensive-form Bellman equation (19).

Proof (outline). It is well-known that the value function of the maximization problem without participation constraint, V_0, has the functional form (for example, Krebs 2006). A tedious but straightforward argument shows that if $V_n = T^n V_0$ has this property, so has $V_{n+1} = TV_n$. The proposition then follows from proposition 1.

Note that proposition 2 cannot simply be proved by using substitution since there are in general multiple solutions to the Bellman equation (17). In other words, the operator associated with the Bellman equation is monotone, but not a contraction. However, proposition 2 ensures that we have indeed found the value function associated with the original utility maximization problem. Note further that the constraint set in (19) is linear since the return functions are linear in θ. Thus, the constraint set is convex and we have transformed the original utility maximization problem into a convex problem. In other words, the non-convexity problem alluded to in the introduction has been resolved.

c) Intensive-form equilibrium

Clearly, the maximization problem (19) has a strictly concave objective function and a convex choice set. Thus, the optimal portfolio choice, θ, is unique. Moreover, the optimal portfolio choice is independent of s and w. In other words, regardless of their initial type and history of shocks, all households choose the same portfolio shares. This property in conjunction with the financial asset pricing equation (5) allows us to re-write the market clearing conditions (14) and (15) in intensive form:

$$\begin{aligned} \tilde{K} &= \frac{\theta_k}{\theta_h} \\ \sum_s \theta_a(s)\pi(s) &= 0 \end{aligned} \tag{20}$$

To sum up, in our search for a stationary recursive equilibrium, we can confine ourselves to equations (19) and (20):

Proposition 3. Suppose that $(\theta, \tilde{c}, \tilde{K}, r_f)$ is an intensive-form equilibrium, that is, $(\theta, \tilde{K})$ solves the intensive-form market clearing conditions (20) and, for given $(r_k(\tilde{K}), r_h(\tilde{K}), r_f)$, the consumption-portfolio choice $(\tilde{c}, \theta)$ together with the value $\tilde{V}$ are the maximal solution to the intensive-form Bellman equation (19) satisfying condition (18). Then the corresponding allocation $\{c_t, w_t, \theta_t\}$ together with prices $(r_k(\tilde{K}), r_h(\tilde{K}), r_f)$ are a stationary recursive equilibrium.

Proof. From proposition 1 and 2 we know that implied consumption-investment plan, $\{c_t(x_0), w_t(x_0), \theta_t(x_0)\}$, is a solution to the household maximization problem for any household x_0. Further, the fact that individual plans are linear in wealth implies the equivalence between the market clearing conditions (14,15) and (20). This proves the proposition.

Note that the maximal solution to the intensive-form Bellman equation (19) can be obtained as the limit $\lim_{n\to\infty} T^n\tilde{V}_0$ (proposition 1).

d) Existence and Characterization of Equilibrium

In an intensive-form equilibrium, the maximization problem of individual households (19) has a concave objective function and a convex constraint set. Thus, first-order conditions are not only necessary, but also sufficient. Furthermore, prices enter into the constraint set only in a linear fashion. The convexity of the constraint set together with the linearity in prices implies that a version of the second welfare theorem holds. More precisely, define the intensive-form social planner problem as

$$\tilde{V} \;=\; \max_{\tilde{c},\theta'} \left\{ \frac{\tilde{c}^{1-\gamma}}{1-\gamma} + \beta(1-\tilde{c})^{1-\gamma}\tilde{V} \sum_{s'} \left(1 + r_{eff}(\theta', s')\right)^{1-\gamma} \pi(s') \right\} \quad (21)$$

$$s.t. : \;\; \theta'_k + \theta'_h = 1 \quad ; \quad \sum_{s'} \theta'_a(s')\pi(s') = 0$$

$$1 + r_{eff}(\theta'_k, \theta'_h, \vec{\theta}'_a, s') \;\geq\; \left(\frac{\tilde{V}_d}{\tilde{V}}\right)^{\frac{1}{1-\gamma}} (1 + r_{eff}((1-\phi)\theta'_k, \theta'_h, 0, s'))$$

$$1 \geq \tilde{c} \geq 0 \quad , \quad \theta'_k \geq 0 \;\; , \;\; \theta'_h \geq 0 \;\; , \;\; \theta'_a(s') \geq \bar{\theta}_a(s') \; ,$$

where r_{eff} is the investment return of the social planner defined as

$$r_{eff}(\theta_k, \theta_h, \vec{\theta}_a, s) = F(\theta_k, \theta_h) - \delta_k\theta_k - \delta_h(s)\theta_h + \theta_a(s) \; .$$

The following proposition says that an intensive-form equilibrium can be found by solving the intensive-form social planner problem (21). The proof is based on a comparison of first-order conditions. These first-order condition also imply some familiar and not so familiar properties of equilibrium, which are also stated in the proposition:

Proposition 4. Suppose $(\theta, \tilde{c}, \tilde{K})$ is a maximal solution to the intensive-form social planner problem (21). Then $(\theta, \tilde{c}, \tilde{K}, r_f)$ in an intensive-form equilibrium, where the interest rate on financial transactions is given by

$$r_f = \frac{1}{q_f} - 1 = \frac{1}{\beta(1 + r(\theta, \bar{s}))} - 1 \; ,$$

with $\bar{s}$ being any state for which the participation constraint is not binding. Further, the intensive-form equilibrium has the following properties:

i) Investment returns are equalized for all states, s, s', with non-binding participation constraint: $r(\theta, s) = r(\theta, s')$

ii) The interest rate on financial transactions does not exceed the return to physical capital, $r_f \leq r_k - \delta_k$, where the inequality is strict when there is less than perfect risk sharing in equilibrium.

Proof (outline) As mentioned before, first-order conditions are sufficient in both the social planner problem and the individual household problem. For the problem of an individual household, we find the following Kuhn-Tucker conditions:

$$\begin{aligned}
\tilde{c}^{-\gamma} &= \beta\tilde{V}(1-\tilde{c})^{-\gamma}\sum_{s'}(1+r(\theta',s'))^{1-\gamma}\pi(s') \qquad (22)\\
\mu &= (1+r_k-\delta_k)\sum_{s'}(1+r(\theta',s'))^{-\gamma}\pi(s') + (1+r_k-\delta_k)\left(\left(\frac{V}{V_{aut}}\right)^{\frac{1}{1-\gamma}} - (1-\phi)\right)\sum_{s'}\lambda(s')\\
\mu &= \sum_{s'}(1+r_h-\delta_h(s'))(1+r(\theta',s'))^{-\gamma}\pi(s') + \left(\left(\frac{V}{V_{aut}}\right)^{\frac{1}{1-\gamma}} - 1\right)\sum_{s'}(1+r_h-\delta_h(s')\lambda(s')
\end{aligned}$$

$$\begin{aligned}
\mu\frac{\pi(s')}{1+r_f} &= (1+r(\theta',s'))^{-\gamma} + \left(\frac{V}{V_d}\right)^{\frac{1}{1-\gamma}}\lambda(s')\\
1 &= \theta_k + \theta_h + \sum_{s'}\theta'(s')\frac{\pi(s')}{1+r_f}\\
1+r(\theta'_k,\theta'_h,\vec{\theta}'_a,s') &\geq \left(\frac{\tilde{V}_d}{\tilde{V}}\right)^{\frac{1}{1-\gamma}}(1+r((1-\phi)\theta'_k,\theta'_h,0,s'))\\
0 &= \sum_{s'}\lambda(s')\left(1+r(\theta'_k,\theta'_h,\vec{\theta}'_a,s') - \left(\frac{\tilde{V}_d}{\tilde{V}}\right)^{\frac{1}{1-\gamma}}(1+r((1-\phi)\theta'_k,\theta'_h,0,s'))\right)\\
\lambda(s') &\geq 0\ ,
\end{aligned}$$

where the Lagrange multipliers λ and μ have already been re-scaled by the factor $\beta(1-\tilde{c})^{1-\gamma}\tilde{V}$ and we have assumed that the constraints $0 \leq \tilde{c} \leq 1$ and $\theta_k \geq 0, \theta_h \geq 0$ are non-binding in equilibrium. The first equality in (22) expresses the optimality of the consumption-saving choice, the next two equalities the optimality of the real investment decisions θ_k and θ_h, and the fourth equality ensures that financial decisions are made optimally. Similar Kuhn-Tucker conditions can be derived for the social planner problem, where the rental rates r_h and r_k are replaced by their respective marginal products and the total investment return r by r_{eff}. Further, simple algebraic manipulation show that the two sets of Kuhn-Tucker conditions are equivalent if r_f is chosen as specified. Finally, the properties i) and ii) follow directly from the first-order conditions (22).

Propositions 3 and 4 imply that any solution to the intensive-from social planner problem (21) that satisfies (18) defines a stationary recursive equilibrium:

Corollary Suppose $(\theta, \tilde{c}, \tilde{K})$ is a maximal solution to the intensive-form social planner problem (21) satisfying (18). Define r_f as in proposition 4. Then $(\theta, \tilde{c}, \tilde{K}, r_f)$ generates a stationary recursive equilibrium with allocation, $\{c_t, w_t, \theta_t\}$, where the consumption-wealth plan, $\{c_t, w_t\}$, is defined as in (19).

Finally, we show that under a rather weak condition, a solution to the social planner problem satisfying (18) exists, which implies that a stationary recursive equilibrium exists. The condition we need to prove existence is that a solution to the maximization problem with no insurance and full insurance (perfect risk sharing) exists. The existence condition for the no-insurance case is (13) and the corresponding condition for the full insurance case is

$$\beta \left(1 + r_{full}(\theta_{full})\right)^{1-\gamma} < 1 \ , \tag{23}$$

where $r_{full} = F(\theta_{full}, 1-\theta_{full}) - \delta_k \theta_{full} - \sum_s \pi(s)\delta_h(s)(1-\theta_{full})$ and θ_{full} is the unique solution to

$$F_2(\theta_{full}, 1-\theta_{full}) - \sum_s \pi(s)\delta_h(s) = F_1(\theta_{full}, 1-\theta_{full}) - \delta_k \ . \tag{24}$$

We have the following result:

Proposition 5. Suppose that condition (13) and (23) holds. Then there exists a maximal solution to the intensive-form social planner problem (21) satisfying (18). This solution is obtained as the limit

$$\lim_{t\to\infty} \tilde{T}^n \tilde{V}_0 = \tilde{V} \ ,$$

where $\tilde{T}$ is the operator associated with the intensive-form social planner problem (21) and $\tilde{V}_0$ is the solution to the corresponding intensive-form social planner problem (21) without participation constraint.

Proof (outline) Condition (13) and (23) imply that for all θ that lie in the social planner constraint set, we have

$$\beta \sum_s \pi(s)\,(1 + r(\theta, s))^{1-\gamma} < 1 \ . \tag{25}$$

As in the proof of proposition 1, we can then apply theorem 3.6 in Rustichini (1998).

IV. Quantitative Results

We now discuss the quantitative implications of the model. First, we specify functional forms for utility and production functions and then assigns values to all relevant parameters of the model (calibration). We then report how much insurance is provided in equilibrium, how consumption responds to uninsured income shocks, and how much would be lost in an economy with no insurance.

a) Calibration

The quantitative analysis is based on an economy where workers have logarithmic utility functions: $u(c) = logc$. Further, we assume that human capital shocks are normally distributed, $\eta \sim N(0, \sigma_\eta^2)$, and that the production function is Cobb-Douglas: $f(\tilde{k}) = A\tilde{k}^\alpha$. We use $\alpha = .36$ to match capital's share in income and $\delta = .06$ (annually) as a compromise between the higher depreciation rate of physical capital used in the literature (but see also Cooley and Prescott (1995) for an argument that $\delta_k = .05$) and the probably lower depreciation rate of human capital. The values of the fundamental parameters A, σ_η^2, and β are chosen so that the model is roughly consistent with the US evidence along three dimensions: saving, growth, and labor income risk. More specifically, we require that per capita consumption growth satisfies $\mu_g = E[c_{i,t+1}/c_{it}] - 1 = .02$ and that the implied saving rate is $s_k = x_{kt}/y_t = .20$. For the annual US data on saving and growth, see Summers and Heston (1991). Finally, we match observed labor income risk (before transfer payments) by requiring $\sigma_y = \sigma_\eta/(1+\tilde{k}) = .15$. To simplify the analysis, we initially calibrate an economy without insurance. This approach yields $A = .267$, $\beta = .935$, and $\sigma_\eta = .252$.

The choice of $\sigma_y = \sigma_\eta/(1+\tilde{k}) = .15$ is made to ensure consistency with the empirical results of a number of micro studies on labor income risk. More specifically, in the model economy log-labor income of household i, y_{hit}, is given by $y_{hit} = (r_h + \delta)\, h_{it}$. Using the equilibrium condition $h_{i,t+1} = \beta\,[1 + \theta r_k + (1-\theta)(r_h + \eta_{it}]\, h_{it}$, we find

$$\begin{aligned} log\, y_{hi,t+1} - log\, y_{hit} &= log h_{i,t+1} - log h_{it} \\ &= log\beta + log\,(1 + \theta r_k + (1-\theta)(r_h + \eta_{it})) \\ &\approx d + \tilde{\eta}_{it}\ , \end{aligned} \tag{26}$$

where $d = log\beta + \theta r_k + (1-\theta) r_h$ and $\{\tilde{\eta}_{it}\}$ is a sequence of i.i.d. random variables with $\tilde{\eta}_{it} = (1-\theta)\eta_{it}$. Hence, the logarithm of labor income follows (approximately) a random walk with drift d and error term $\tilde{\eta}_{it} \sim N(0, \sigma_y^2), \sigma_y = (1-\theta)\sigma_\eta$.[3] The random walk specification is often used by the empirical literature to model the permanent component of labor income risk (Carroll and Samwick (1997), Meghir and Pistaferri (2004), and Storesletten et al. (2004)). Thus, their estimate of the standard deviation of the error term for the random walk component of annual labor income corresponds to the value of $\sigma_y = (1-\theta)\sigma_\eta$. In our

[3]We have $\tilde{\eta}_{it}$ instead of $\tilde{\eta}_{i,t+1}$ in equation (24), and the latter is the common specification for a random walk. However, this is not a problem if the econometrician observes the idiosyncratic depreciation shocks with a one-period lag. In this case, (24) is the correct equation from the household's point of view, but a modified version of (22) with $\tilde{\eta}_{i,t+1}$ replacing $\tilde{\eta}_{it}$ is the specification estimated by the econometrician.

baseline model we use $\sigma_y = .15$, which lies on the lower end of the spectrum of estimates found by the empirical literature. For example, Carroll and Samwick (1997) find .15, Meghir and Pistaferri (2004) estimate .19, and Storesletten et al. (2004) have .25 (averaged over age-groups and, if applicable, over business cycle conditions). All these studies use labor income before transfer payments, which is the relevant variable from our point of view.

There are at least two reasons why the above approach might underestimate human capital risk. First, a constant $\sigma_y = .15$ represents less uncertainty than a σ_y that fluctuates with business cycle conditions and has a mean of .15. Second, the assumption of normally distributed innovations understates the amount of idiosyncratic risk households face if the actual distribution has a fat lower tail. For strong evidence for such a deviation from the normal-distribution framework, see Geweke and Keane (2000). Further, the literature on the long-term consequences of job displacement (Jacobson, LaLonde, and D. Sullivan (1993)) has found wage losses of displaced workers that are somewhat larger than suggested by our mean-variance framework.

There are, however, also arguments that the current approach might overestimate human capital risk. First, we assumes that all of labor income is return to human capital investment. However, if some component of labor income is independent of human capital investment and if this component is random (random endowment of genetic skills), then some part of the variance of labor income is not human capital risk. Second, by ignoring job mobility the empirical literature cited above attributes wage hikes due to improved firm-worker matches to income risk, a point that has been emphasized by Low, Meghir, and Pistaferri (2008).

The above calibration procedure ensures that the model economy matches as many features of the US economy as there are free parameters. It is also interesting to investigate how the calibrated model performs in matching additional features of the U.S. economy. For example, the implied values for the average return on physical and human capital are

$r_k = 5.52\%$ and $r_h = 9.47\%$, respectively. The return $r_k = 5.52\%$ is higher than the observed real interest rate on short-term U.S. government bonds (1%), but lower than the observed real return on US equity (8%). Given that there is no aggregate risk, and therefore no equity premium, in the model, it is not clear which one of the many financial return variables should be used as a basis for calibration, and we therefore conclude that the implied value is within the range of reasonable values.[4] The implied average return on investment in human capital, $r_h = 9.47\%$, is in line with the estimates of rate of returns to schooling.[5] Notice that the implied excess return on human capital investment is $r_h - r_k = 3.95\%$. Thus, the model generates a substantial "human capital premium".

Finally, we introduce a stochastic probability, q, that a household who lives in autarky can return to the formal sector and participate in risk sharing. This modification of the model changes the participation constraint so that the discount factor used when calculating the continuation utility is $\tilde{\beta} = (1-q)\beta$. We choose q so that on average a defaulting household would spend seven years in autarky.

b) Quantitative Results

Table 1 shows the results for different values of ϕ, which is the fraction of physical capital seized upon default and measures the extent to which contracts are enforced. In the case that no physical capital is seized, $\phi = 0$, we have no risk sharing at all, that is, before and after transfer payment income volatility is the same.[6] For $\phi = 0.2$, the income volatility is reduced from $\sigma_y = .15$ to $\sigma_y = .14$. When we increase the enforcement parameter to $\phi = .4$,

[4]The RBC literature usually strikes a compromise and chooses the parameter values so that the implied return on capital is 4%, which is somewhat lower than the value used here.

[5]The estimates vary considerably across households and studies, with an average of about 10% (Krueger and Lindhal 2001).

[6]There is some small amount of risk sharing since very large negative income shocks are insured, but the probability of these shocks is so small that they have a negligible effect on the variance.

we find $\sigma_y = .10$, that is, two-third of income risk is insured. The value of $\sigma_y = .10$ is in line with the estimate of Blundell, Pistaferri, and Preston (2008) of permanent labor income risk after taxes and transfers are taken into account. Finally, if we have $\phi = 1$ and the entire physical capital is seized, then there is complete insurance against labor income risk. The case $\phi = 1$ is the one considered in Krueger and Perri (2006), and our perfect risk sharing result is, mutatis mutandis, the result Krueger and Perri (2006) found for the majority of households.

The second column in table 1 shows to what extent the uninsured part of labor income shocks translate into consumption changes. Independently of the value of ϕ, the model predicts that roughly two-thirds of any uninsured labor income shock translate into consumption changes. Intuitively, labor income is around two-thirds of total income, so that this is exactly the fraction of the labor income shocks that translates into consumption changes. Further, this level of consumption response to permanent income shocks is very much in line with the empirical estimates reported in Blundell, Pistaferri, and Preston (2008).

Finally, the third column shows the welfare consequences of incomplete insurance. More precisely, we report the welfare gains from having enforcement ϕ expressed as a percentage of lifetime consumption. The difference between different rows measures the welfare gains from improving the enforcement of risk sharing. For example, if we move from no enforcement ($\phi = 0$, the developing country) to a medium level of enforcement ($\phi = 0.4$, our prefered value for the US economy), we have a welfare gain of 5 percent of lifetime consumption. In other words, any government policy that improves the enforcement of risk sharing arrangements has a very strong effect on welfare.

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Table I. Insurance, consumption, and welfare for different values of the enforcement parameter[7]

	$\tilde{\sigma}_y$	σ_c	ΔU
$\phi = 0$	.15	.10	0 %
$\phi = .2$	.12	.08	3%
$\phi = .4$	.10	.07	5 %
$\phi = 1$	0	0	10 %

[7] ϕ is the fraction of capital seized. $\tilde{\sigma}_y$ is the standard deviation of uninsured income shocks. σ_c is the standard deviation of individual consumption growth. ΔU is the welfare gain from having contract enforcement ϕ, expressed as percent of lifetime consumption.