

On the Golden Rule of Capital Accumulation under Endogenous Longevity*

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Abstract

This note derives the Golden Rule of capital accumulation in a Chakraborty-type economy, i.e. a two-period OLG economy where longevity is endogenous. It is shown that the capital per worker maximizing steady-state consumption per head is inferior to the Golden Rule capital level prevailing under exogenous longevity as soon as health spending increase with capital per worker. We characterize also the Lifetime Golden Rule, that is, the capital per worker maximizing steady-state expected lifetime consumption per head, and show that this tends to exceed the Golden Rule capital level.

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1 Introduction

Introduced by Phelps (1961), the Golden Rule of capital accumulation states the condition under which the stock of capital per worker maximizes steady-state consumption per head. In a simple model with no technological progress, the Golden Rule states that steady-state consumption per head is maximized when the marginal productivity of capital equals the sum of the population growth rate and the rate of depreciation of capital. It does not depend on preferences.

Whereas the Golden Rule has given rise to various studies in growth theory (see Diamond, 1965; Phelps, 1965), no attention has been paid so far to the Golden Rule of capital accumulation in an economy where some resources are spent on health and agents's lifetime is endogenous, as in Chakraborty's (2004) OLG model. Under such an endogenous (finite) time horizon, does the Golden Rule capital level remain the same, or, on the contrary, does the endogeneity of lifetime modify the Golden Rule?

Despite the increasing body of recent literature on the relation between economic growth and survival conditions (Cervellati and Sunde (2005) and Chakraborty and Das (2006) are two highly cited examples. A survey is in Boucekine (2008)), that question has not been discussed so far, as the literature focused mainly on purely *descriptive* issues (e.g. multiplicity of equilibria), and did not consider the normative question of the optimal capital accumulation in that context.

Undoubtedly, studying the optimal capital accumulation in an economy where lifetime is endogenous consists of a most ambitious task, as the treatment of death, in welfare terms, remains problematic. Actually, the major difficulty concerns the definition of the utility level that should be assigned to death. However, given the growing body of recent literature with endogenous longevity, it is important, despite that difficulty, to study also the normative side of the growth/longevity relationship. This note, which aims at determining the Golden Rule capital level in the context of an economy with endogenous longevity, constitutes a first step in that direction. By focusing on consumption rather than utility, the present note will allow us to start the normative study of capital accumulation under endogenous longevity *without* having to make fragile postulates on the utility assigned to the death state.

Although one may be tempted to regard the study of the Golden Rule capital as irrelevant once longevity becomes a variable, it should be stressed, however, that consumption remains, even in the presence of longevity, the *unique* determinant of *temporal* welfare, that is, of the welfare associated to a particular period of life. This property comes from the singular nature of longevity as a dimension of welfare: its influence can be acknowledged only if a lifetime perspective is adopted (i.e. the complete life view). However, from an *instantaneous* welfarist point of view, the only piece of information that matters is the consumption per period, as studied by the Golden Rule. Naturally, one is not forced to adopt such an instantaneous point of view, but it is far from obvious that one can reject *a priori* a study of the Golden Rule of capital on the mere basis of the variability of longevity: a chance must be given to both the complete life view and the

instantaneous, intensity view. This is the approach adopted throughout this paper.

In order to characterize the Golden Rule of capital accumulation, we use a two-period OLG model with physical capital based on Chakraborty (2004), where the probability of survival to the second period of life depends positively on some longevity-enhancing health expenditures. We then explore the definition of the Golden Rule capital level in that context.

The rest of this note is organized as follows. Section 2 presents the model. Section 3 derives the Golden Rule of capital accumulation in an economy with endogenous longevity. Section 4 explores an alternative definition of the Golden Rule, named the Lifetime Golden Rule. Section 5 concludes.

2 The model

Let us consider an OLG model with the same population structure and technology as in the model studied by Chakraborty (2004). Time is discrete and goes from 0 to infinity; households live at best for two periods.

Demography The size of the cohort born at t is L_t . It grows over time at a constant, exogenous rate n ($n > -1$):

$$L_{t+1} = (1 + n)L_t \quad (1)$$

All agents of a cohort t live the first period of life for sure, but only a proportion π_{t+1} of that cohort will enjoy a second period of life. Hence, life expectancy at birth for the cohort born at time t is equal to $1 + \pi_{t+1}$. The proportion of survivors π_{t+1} ($0 < \pi_{t+1} < 1$) depends positively on the amount of health expenditures per worker h_t :

$$\pi_{t+1} = \pi(h_t) \quad (2)$$

with $\pi(h) \in [0, 1]$ for all $h > 0$, and $\pi'(h) > 0$. We also assume that $\lim_{h_t \rightarrow \infty} \pi(h_t) = 1$, which gives an upper bound to the life expectancy $1 + \pi$.

As in Chakraborty, first-period agents supply their labor inelastically, while second-period agents are retired.

Technology Firms at time t produce some output Y_t according to the following production function: $Y_t = F(K_t, L_t)$ where Y_t denotes the total output, K_t the total capital stock, and L_t denotes the labour force. $F(\cdot)$ is a positively-valued production function, increasing, and strictly concave with respect to capital. Capital depreciates at a constant rate δ ($0 \leq \delta \leq 1$). Under constant returns to scale, production can be rewritten as:

$$y_t = f(k_t) \quad (3)$$

where y_t denotes the output per worker, and k_t the capital stock per worker, while $f(\cdot) = F(k, 1)$ is the production function in its intensive form. Under the above assumptions

on $F(\cdot)$, we have, for all $k > 0$, $f(k) > 0$, $f'(k) > 0$ and $f''(k) < 0$. (de la Croix and Michel, 2002)

The marginal productivity of capital is equal to $f'(k)$. The marginal productivity of labour is given by the function

$$\omega(k) = f(k) - kf'(k)$$

It can be shown that the marginal productivity of labor $\omega(k)$ satisfies $\omega(k) \geq 0$ and $\omega'(k) = -kf''(k) > 0$.

Health Spending As in Chakraborty (2004), health spending are here purely longevity-enhancing spending, which do not contribute to the quality of life-periods, but only to the quantity of life. Total health spending are: $H_t = L_t h_t$. Just like the two consumption levels of young and old individuals, health spending are not decided by the “planner” who determines the level of capital. We still assume that health spending are possibly a function of capital per worker through the function: $h_t = h(k_t)$. We consider three alternative assumptions concerning this function.

A1 Health spending per worker are constant: $h_t = \bar{h}$.

A2 Health spending per worker are a constant fraction τ of the marginal productivity of labour:

$$h_t = \tau\omega(k_t) \tag{4}$$

This is the assumption made by Chakraborty (2004).

A3 Health spending per worker are a constant fraction θ of output per worker:

$$h_t = \theta f(k_t) \tag{5}$$

3 The Golden Rule

Consider a stationary environment in which the variables k , h and π are constant over time and all the aggregate variables, production Y_t , consumption C_t , investment I_t , health spending H_t , and capital K_t grow at the constant rate n . Let us derive the level of capital per worker k maximizing steady-state consumption per head. The feasibility constraint imposes that investment I_t is equal to production $F(K_t, L_t)$ minus consumption C_t minus health spending H_t : $I_t = K_{t+1} - (1 - \delta)K_t = F(K_t, L_t) - C_t - H_t$ so that total consumption C_t is equal to: $C_t = F(K_t, L_t) - K_{t+1} - H_t + (1 - \delta)K_t$. Thus, consumption per worker, equal to $C_t/L_t = c_t = c$, can be written as:

$$c = f(k) - k(\delta + n) - h(k) \tag{6}$$

Consumption per head $C_t/(L_t + \pi_t L_{t-1})$ where $\pi_t = \pi(h(k)) = \pi$ is related to consumption per worker through the following identity:

$$\frac{C_t}{L_t + \pi L_{t-1}} = \frac{c_t}{\left(1 + \frac{\pi}{1+n}\right)} = \frac{1+n}{1+n+\pi} c_t$$

given that $L_t = (1+n)L_{t-1}$. Consumption per head corresponds to consumption per worker c_t , multiplied by $(1+n)/(1+n+\pi)$. Note that this latter factor depends on π , and, thus, under **A2** and **A3**, on capital per worker. Hence, contrary to what prevails in standard OLG models with exogenous longevity, the capital level maximizing consumption per head does not necessarily coincide with the one maximizing consumption per worker.

It follows that consumption per head at the steady-state, denoted by $\phi(k)$, can be written as:

$$\phi(k) = [f(k) - k(\delta + n) - h(k)] \frac{1+n}{1+n+\pi(h(k))} \quad (7)$$

In order to discuss the conditions necessary and sufficient for the existence of a Golden Rule capital level, let us first differentiate consumption per head $\phi(k)$ with respect to capital:

$$\phi'(k) = [f'(k) - (\delta + n) - h'(k)] - \frac{[f(k) - k(\delta + n) - h(k)] \pi'(h(k)) h'(k)}{1+n+\pi(h(k))} \quad (8)$$

where $h'(k) = 0$ under **A1**, $h'(k) = \tau\omega'(k) = -\tau kf''(k) > 0$ under **A2**, and $h'(k) = \theta f'(k) > 0$ under **A3**.

As de la Croix and Michel (2002) argued, the expression $\phi'(k) = 0$ defines an *interior* Golden Rule capital level only if $\phi(k)$ is neither always decreasing in k (implying that the capital level maximizing $\phi(k)$ is 0), nor always increasing in k (implying that the capital maximizing $\phi(k)$ is infinite). The interiority of the solution requires the following condition, which guarantees that $\phi'(k)$ is positive when k tends to 0, but negative when it tends to $+\infty$.

Proposition 1 *Assume that $\{n, \delta, f(k), \pi(h)\}$ satisfy:*

$$\begin{aligned} \lim_{k \rightarrow 0^+} f'(k) &> \delta + n + \lim_{k \rightarrow 0^+} \left(\frac{\pi'(h(k)) [f(k) - h(k)]}{1+n+\pi(h(k))} + 1 \right) h'(k) \\ \lim_{k \rightarrow +\infty} f'(k) &< \delta + n \end{aligned}$$

Then, there exists a capital per worker k_{GR} maximizing consumption per head in \mathbb{R}_+ . Such a level satisfies $\phi'(k_{\text{GR}}) = 0$:

$$f'(k_{\text{GR}}) = \delta + n + h'(k_{\text{GR}}) \left(1 + \frac{\pi'(h(k_{\text{GR}}))(f(k_{\text{GR}}) - k_{\text{GR}}(\delta + n) - h(k_{\text{GR}}))}{1+n+\pi(h(k_{\text{GR}}))} \right) \quad (9)$$

Proof. The conditions $\lim_{k \rightarrow 0^+} \phi'(k) > 0$ and $\lim_{k \rightarrow +\infty} \phi'(k) < 0$ are sufficient to obtain an interior maximum. The first limit can be written as:

$$\lim_{k \rightarrow 0^+} \phi'(k) = \lim_{k \rightarrow 0^+} \left[f'(k) - (\delta + n) - h'(k) - \frac{\pi'(h(k)) h'(k) [f(k) - h(k)]}{1+n+\pi(h(k))} \right]$$

The condition $\lim_{k \rightarrow 0+} \phi'(k) > 0$ can be rewritten as:

$$\lim_{k \rightarrow 0+} f'(k) > \delta + n + \lim_{k \rightarrow 0+} \left(\frac{\pi'(h(k)) [f(k) - h(k)]}{1 + n + \pi(h(k))} + 1 \right) h'(k)$$

which is the condition in the Proposition. Regarding the second condition, we have:

$$\begin{aligned} \lim_{k \rightarrow \infty} \phi'(k) &= \lim_{k \rightarrow \infty} \left[f'(k) - (\delta + n) - h'(k) - \frac{\pi'(h(k)) h'(k) [f(k) - h(k)]}{1 + n + \pi(h(k))} \right] \\ \lim_{k \rightarrow \infty} \phi'(k) &= \lim_{k \rightarrow \infty} \left[f'(k) - (\delta + n) - h'(k) \left(1 + \frac{\pi'(h(k)) [f(k) - h(k)]}{1 + n + \pi(h(k))} \right) \right] \end{aligned}$$

Given that $h'(k) \geq 0$, the condition $\lim_{k \rightarrow \infty} \phi'(k) < 0$ is always true when $\lim_{k \rightarrow \infty} f'(k) - (\delta + n) < 0$.

Hence, under the two conditions of the Proposition, the function $\phi(k)$ reaches a maximum between 0 and $+\infty$. Given that the function $\phi(k)$ is continuous, this maximum k_{GR} satisfies $\phi'(k_{\text{GR}}) = 0$. ■

To better understand the Proposition it is useful to look at its implications for the three cases **A1-A3**. With the case in which health spending are constant, the condition of the proposition would collapse to

$$\lim_{k \rightarrow +\infty} f'(k) < \delta + n < \lim_{k \rightarrow 0+} f'(k)$$

which is assumption A5 in de la Croix and Michel (2002). Moreover, equation (8) would simplify into:

$$\phi'(k) = f'(k) - (\delta + n)$$

and the Golden Rule \bar{k}_{GR} satisfies the usual condition

$$f'(\bar{k}_{\text{GR}}) = \delta + n \tag{10}$$

The Golden rule capital level with exogenous longevity is independent from the postulated level of the probability of survival π . However, it is important to stress that the level of consumption per head for a given capital level is not independent from the level of π . Although it is for the same level of capital per worker that steady-state consumption per head is maximized, the level of the consumption profile is higher the lower π is. The intuition behind this is that π , by increasing the population size, reduces consumption per head per period of life for a given level of k . Thus, although π does not affect the Golden Rule capital level, it does influence the level of consumption per head under each capital level.

Under **A2** or **A3**, the interiority of the Golden Rule capital level requires a stronger condition regarding the level of $\lim_{k \rightarrow 0+} f'(k)$. The intuition behind the additional term in the condition is that the interiority of the Golden Rule requires also, in the context of endogenous health spending and longevity, that a small increase of capital in the neighbourhood of 0 does not lead to an explosion of the population through a rise of the survival probability, in which case the optimal capital level would be zero. In other words, $\pi'(h(0+))$ should be small enough.

Corollary 1 *The Golden Rule capital level under **A2** or **A3** is lower than its level under **A1**.*

Proof. Compare (9) to (10). Under **A2** or **A3** the right hand side of (9) is larger than the right hand side of (10) for any $k_{\text{GR}} > 0$ because $h' > 0$ and because net production $f(k_{\text{GR}}) - k_{\text{GR}}(\delta + n) - h(k_{\text{GR}})$ is positive thanks to the limit conditions in Proposition 1. This implies that the marginal productivity of capital must here be strictly larger than its level under **A1**. Thus, it follows from $f''(k) < 0$ for all $k > 0$ that the Golden Rule capital level must be smaller under **A2** or **A3** than under **A1**. ■

The intuition behind that result goes as follows. In a Chakraborty-type economy, raising capital per worker tends also to increase, through $h(k)$, the proportion π of survivors in the cohort, and, thus, the population size (unlike what happens in economies where longevity is exogenous). Hence, under that additional effect, the level of k maximizing steady-state consumption per head must be inferior to its level under exogenous longevity.

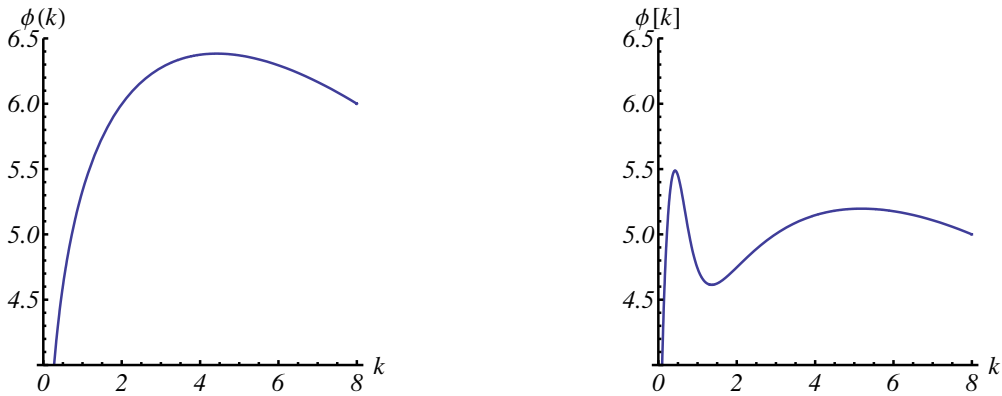
Finally, it should be stressed that the condition stated in Proposition 1 is not necessarily satisfied by a *unique* capital level. Clearly, several capital levels may satisfy that condition, and lead to local maxima of steady-state consumption, unlike what prevails in the standard model, where steady-state consumption exhibits only a single maximum in capital per worker. More precisely, only in case **A1**, we have $\phi''(k) = f''(k)$, which is negative, so that $\phi(k)$ is concave for all values of k . Under **A2** or **A3**, the second order derivatives $\phi''(k)$ depends on the second order derivatives of the survival function π . Moreover, under **A2**, $\phi''(k)$ also depends on the third-order derivative of the production function via ω'' , for which we have no reasonable assumption to impose. It is therefore not necessarily negative, contrary to what prevails when health spending are exogenous.

An example illustrates this point. Take a Cobb-Douglas production function $f(k) = 10k^{1/3}$. Assume **A2** with a tax rate $\tau = 0.15$, full depreciation of capital $\delta = 1$ and constant population $n = 0$. The Golden Rule is $\bar{k}_{\text{GR}} = 6.086$. Consider two different cases for the survival function $\pi(h)$. One concave, $\pi(h) = h/(1+h)$, as in Chakraborty (2004), and one logistic $\pi(h) = 0.0001/(0.0001 - \exp(-10h))$. The left panel of Figure 1 shows that net production is globally concave in the first example and the first order condition $\phi'(k_{\text{GR}}) = 0$ gives a global maximum with $k_{\text{GR}} = 4.42$. In the right panel, net production is concave-convex-concave. We have two local maxima, 0.428 and 5.182, and the one with the smallest k is the global maximum.

4 The Lifetime Golden Rule

When interpreting the above results, one may argue that the maximization of consumption per head per period of life is not an adequate goal, in the sense that it only captures the intensity of life's goodness (i.e. in per period terms), but not the goodness of life as a whole. More precisely, it can be argued that the standard Golden Rule ceases to

Figure 1: Net production $\phi(k)$ with a concave survival function $\pi(h)$ (left panel) and with a logistic one (right panel)



be an appropriate goal once longevity becomes a variable, that is, once there appears some trade-offs between consumption and longevity. The introduction of longevity, by making lifetime welfare dependent on two - rather than one - variables, would thus make the standard Golden Rule - focusing on a single dimension of welfare - inadequate.

Although that criticism of the relevancy of the Golden Rule in the context of endogenous longevity is certainly appealing, it is far from clear that this suffices to make the standard Golden Rule irrelevant. Paying an exclusive attention to the intensity of life's goodness may still be defensible, even in a context where longevity is variable. This defensibility comes from the singular nature of longevity as a dimension of welfare. It is only through the passage of time that longevity, unlike consumption, takes its value. But a social planner may want to maximize the level of welfare *per period lived*, and such an objective, which does not take longevity into account, does not seem implausible at all. This would consist of an 'intensity view' of welfare, in contrast with (more standard) 'complete view' of welfare. It is not obvious that such an intensity view of welfare can be *a priori* regarded as more or less plausible than the complete view.

Having stressed this, it remains true that the intuitive appeal of the complete view of welfare invites the development of an alternative Golden Rule concept, which would incorporate longevity achievements. This is the task of the present section.

Under the assumption of endogenous lifetime, a plausible - possibly more adequate - goal may be the maximization not of consumption per head, but of *expected lifetime* consumption per head. Such an objective has the virtue to take longevity into account, but without having to rely on assumptions on preferences. Let us now derive the capital level maximizing expected lifetime consumption per head, defined as the consumption per head multiplied by life expectancy:

$$\frac{C_t}{L_t + \pi_t L_{t-1}} (1 + \pi_t) = \frac{(1+n)(1+\pi_t)}{1+n+\pi_t} c_t \quad (11)$$

Expected lifetime consumption per head at the steady-state, denoted by $\psi(k)$, is:

$$\psi(k) \equiv c \frac{(1+n)(1+\pi)}{1+n+\pi} = [f(k) - k(\delta+n) - h(k)] \frac{(1+n)(1+\pi)}{1+n+\pi} \quad (12)$$

The existence of a (finite positive) level of k maximizing $\psi(k)$ would be guaranteed under the conditions insuring that $\psi'(k)$ is positive for low capital levels but negative for high ones. Those conditions coincide with the ones implying $\phi'(k) > 0$ for k tending towards 0 and $\phi'(k) < 0$ for k tending towards $+\infty$.

Proposition 2 *Under the conditions of Proposition 1, there exists a capital per worker maximizing the expected lifetime consumption per head at the steady-state. That Lifetime Golden Rule capital level, denoted by k_{LGR} , is such that:*

$$f'(k_{\text{LGR}}) = \delta + n + h'(k_{\text{LGR}}) \left(1 + \frac{-n\pi'(h(k_{\text{LGR}}))(f(k_{\text{LGR}}) - k_{\text{LGR}}(\delta+n) - h(k_{\text{LGR}}))}{(1+\pi(h(k_{\text{LGR}})))(1+n+\pi(h(k_{\text{LGR}})))} \right). \quad (13)$$

Corollary 2 *Under A1 we have that $k_{\text{LGR}} = k_{\text{GR}}$.*

Under A2 or A3 we have that $k_{\text{LGR}} > k_{\text{GR}}$.

Proof. Under **A1**, $h'(k) = 0$, (13) is equivalent to (10) which implies that $k_{\text{LGR}} = \bar{k}_{\text{GR}}$. Under **A2** or **A3**, different cases should be distinguished, depending on the sign of n . Under $n = 0$, the RHS of (13) is smaller than the RHS of (9), so that the Lifetime Golden Rule capital level must exceed the Golden Rule capital level k_{GR} . Moreover, given that the RHS of (13) is $\delta + h'(h) > \delta$, it follows that the Lifetime Golden Rule capital is here lower than under exogenous health spending.

Under $n > 0$, the RHS of (13) is now smaller than $\delta + n + h'(k)$, from which one can see that k_{LGR} must necessarily exceed the Golden Rule capital level k_{GR} . However, k_{LGR} may or may not exceed the Golden Rule capital under exogenous longevity \bar{k}_{GR} .

A similar reasoning could be applied to the case $-1 < n < 0$. The difference between the RHS of (13) and the one of ((9) is a factor $-n/(1+\pi)$. That factor is, in the case $-1 < n < 0$, positive but lower than 1. Hence we also have $k_{\text{LGR}} > k_{\text{GR}}$ in this case. ■

Hence, when we take care of lifetime consumption rather than instantaneous consumption, the optimal stock of capital is larger.

5 Concluding remarks

Endogenizing health spending and longevity does not leave the Golden Rule of capital accumulation unchanged. Clearly, if the goal is the maximization of steady-state consumption per head, the Golden Rule capital level is inferior to its level under exogenous longevity, as raising k increases the population size through a higher survival to the second period. Hence, the endogeneity of longevity tends here to qualify the extent of underaccumulation of capital, as the ‘target’ of capital becomes lower.

Given that one may not be fully satisfied with the goal of maximization of consumption per head (as this leaves longevity aside), we also characterize the Lifetime Golden Rule capital level, which maximizes the expected lifetime consumption per head. The Lifetime Golden Rule capital level is superior to the standard Golden Rule capital level under endogenous longevity. Hence, shifting to the goal of expected lifetime consumption maximization reinforces - rather than weakens - the likelihood of underaccumulation of capital in comparison with the goal of maximization of consumption per period.

Finally, it should be stressed that this study does not rely on assumptions on preferences. That independence from preferences can be regarded as either a weakness or as a strength of the present study. True, the best social objective consists ideally of the Golden Age, i.e. the capital per worker maximizing steady-state *lifetime utility*. However, its definition is not trivial, as this requires to deal with some necessary assumptions on the utility of death, unlike what was needed in the study of the Golden Rule and Lifetime Golden Rule capital levels. In the light of the difficulty to fix some level to the utility of death, avoiding assumptions of that kind by focusing on consumption may well be a virtue.

In sum, this note constitutes only a first step in the normative analysis of the relation between capital accumulation and survival conditions. The examination of the Golden Age under various assumptions on the utility of death is on our research agenda.

6 References

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