1 MPMC Comparative Statics

- Let us consider comparative statics, assuming that the optimal policy has a cutoff structure.
- Let $\mathfrak{F}^{(t)} \equiv \{M_k, 1 \le k \le t : \phi_k = 1\} \cup M_0$ and $\mathcal{A}^{(t)} \equiv \{M_k, 1 \le k \le t : \overline{c}_k \in [l, \overline{a}_k]\} \cup M_0$ denote the sets of feasible and allowable mergers not larger than M_t.
- The optimal cut-offs $(\overline{a}_1, ..., \overline{a}_{\widehat{K}})$ are recursively defined as the smallest solutions to the following set of equations:

$$\begin{aligned}
\underline{\Delta CS}_1 &\equiv \Delta CS(1, \overline{a}_1) = 0, \\
\underline{\Delta CS}_k &\equiv \Delta CS(k, \overline{a}_k) = E_{\mathfrak{F}^{(k-1)}} \Big[\Delta CS \left(M^* \left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)} \right) \right) \mid \\
& \Delta\Pi \left(M^* \left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)} \right) \right) \le \Delta\Pi(k, \overline{a}_k) \Big] \text{ for } 2 \le k \le \widehat{K}.
\end{aligned}$$

1.1 Feasibility Probabilities

- Recall that merger M_k is feasible if $\phi_k = 1$ and infeasible if $\phi_k = 0$. Let $r_k \equiv \Pr(\phi_k = 1)$.

Claim 1 *Consider an increase in the probability of merger M_j's feasibility from r_j to $r_j' > r_j$, assuming that M_j is initially approved with positive probability (i.e., $j \le \hat{K}$). Then, $\underline{\Delta CS}_i{}' = \underline{\Delta CS}_i$ for any weakly smaller merger M_i, $i \le j$, and $\underline{\Delta CS}_i' > \underline{\Delta CS}_i$ for any larger merger M_i, $i > j$, that is approved with positive probability.*

- Idea?
- Let $\mathcal{A}$ denote the optimal approval policy when $\Pr(\phi_j = 1) = r_j$ and $\mathcal{A}'$ the optimal approval policy when $\Pr(\phi_j = 1) = r_j'$.
- From the recursive definition of the cutoffs, it follows immediately that a change in r_j does not affect the cutoffs for any smaller merger M_i, $i < j$, nor the cutoff of merger M_j itself. Hence, $\underline{\Delta CS}_i{}' = \underline{\Delta CS}_i$ for all $i \le j$.
- Consider now the cutoff for merger M_{j+1}. We can rewrite the cutoff condition as

$$\begin{aligned}
\underline{\Delta CS}_{j+1} &= \Pr(\phi_j = 1 \mid \Delta\Pi \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \le \Delta\Pi(j+1, \overline{a}_{j+1})) \\
& \times E_{\mathfrak{F}^{(j)}} \Big[\Delta CS \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \mid \\
& \Delta\Pi \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \le \Delta\Pi(j+1, \overline{a}_{j+1}) \text{ and } \phi_j = 1 \Big] \\
& + \Big[1 - \Pr(\phi_j = 1 \mid \Delta\Pi \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \le \Delta\Pi(j+1, \overline{a}_{j+1}) \Big] \\
& \times E_{\mathfrak{F}^{(j)}} \Big[\Delta CS \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \mid \\
& \Delta\Pi \left(M^* \left(\mathfrak{F}^{(j)}, \mathcal{A}^{(j)} \right) \right) \le \Delta\Pi(j+1, \overline{a}_{j+1}) \text{ and } \phi_j = 0 \Big].
\end{aligned}$$

- Note first that the optimal policy must be such that

$$\begin{aligned} & E_{\mathfrak{F}^{(j)}}\Big[\Delta CS\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \mid M_{j+1}=(j+1,\overline{a}_{j+1}), \\ & \Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(M_{j+1}), \text{ and } \phi_j = 1\Big] \\ > \; & E_{\mathfrak{F}^{(j)}}\Big[\Delta CS\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \mid M_{j+1}=(j+1,\overline{a}_{j+1}), \\ & \Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(M_{j+1}), \text{ and } \phi_j = 0\Big]. \end{aligned}$$

  To see this, consider the case where $\phi_j = 1$ and $\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})$. Two cases can arise: (i) $M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right) \neq M_j$ and (ii) $M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right) = M_j$. In case (i) the outcome is the same as when M_j were not feasible ($\phi_j = 0$). In case (ii), merger M_j will be implemented. If merger M_j were not feasible, we would instead obtain the expected consumer surplus of the next most profitable allowable merger. By the optimality of the approval policy, $\Delta CS(M_j)$ must weakly exceed (and, generically, strictly) the expected consumer surplus of the next most profitable allowable merger.

- Next, note that we can rewrite the conditional probability as

$$\begin{aligned} & \Pr(\phi_j = 1|\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})) \\ = \; & \Pr(\;\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})|\phi_j = 1)r_j \\ & \times\Big\{\Pr(\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})|\phi_j = 1)r_j \\ & +\Pr(\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})|\phi_j = 0)(1-r_j)\Big\}^{-1} \\ = \; & \left\{1+\frac{\Pr(\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})|\phi_j = 0)}{\Pr(\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1})|\phi_j = 1)}\left(\frac{1-r_j}{r_j}\right)\right\}^{-1}. \end{aligned}$$

  Hence, an increase in r_j induces an increase in the conditional probability $\Pr(\phi_j = 1|\Delta\Pi\left(M^*\left(\mathfrak{F}^{(j)},\mathcal{A}^{(j)}\right)\right) \leq \Delta\Pi(j+1,\overline{a}_{j+1}))$.

- But this implies that an increase in r_j induces an increase in the RHS of the cutoff condition for merger M_{j+1}. Hence, $\underline{\Delta CS}'_{j+1} > \underline{\Delta CS}_{j+1}$.

- Consider now the induction hypothesis that $\underline{\Delta CS}'_{k'} > \underline{\Delta CS}_{k'}$ for all $j < k' < k \leq \hat{K}$. In particular, $\underline{\Delta CS}'_{k-1} > \underline{\Delta CS}_{k-1}$. We claim that this implies that $\underline{\Delta CS}'_{k} > \underline{\Delta CS}_{k}$.

- To see this, note that we can decompose the effect of the increase in r_j on the conditional expectation of the next-most profitable merger into two steps:

  1. Increase the feasibility probability from r_j to $r'_j > r_j$, holding fixed the approval policy $\mathcal{A}$.

2. Change the approval policy from $\mathcal{A}$ to $\mathcal{A}'$.

- Consider first step (1). For the same reason as before, the increase in the feasibility probability must raise the conditional expectation

$$E_{\mathfrak{F}^{(k-1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(k-1)},\mathcal{A}^{(k-1)}\right)\right)\mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(k-1)},\mathcal{A}^{(k-1)}\right)\right)\leq \Delta\Pi(k,\overline{a}_k)\right]$$

by the optimality of the approval policy $\mathcal{A}$.

- Consider now step (2). The outcome under the two policies differs only in the event where $M^*\left(\mathfrak{F}^{(k-1)},\mathcal{A}^{(k-1)}\right)\notin\mathcal{A}'$. Let $M_i = M^*\left(\mathfrak{F}^{(k-1)},\mathcal{A}^{(k-1)}\right)$. Under policy $\mathcal{A}$, the outcome in this event is $\Delta CS(M_i)$. Under policy $\mathcal{A}'$ instead, the expected outcome is

$$E_{\mathfrak{F}^{(i-1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(i-1)},\mathcal{A}'^{(i-1)}\right)\right)\mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(i-1)},\mathcal{A}'^{(i-1)}\right)\right)\leq \Delta\Pi(k,\overline{c}_i)\right].$$

But as $M_i\notin\mathcal{A}'$, we must have

$$\begin{aligned} & E_{\mathfrak{F}^{(i-1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(i-1)},\mathcal{A}'^{(i-1)}\right)\right)\mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(i-1)},\mathcal{A}'^{(i-1)}\right)\right)\leq \Delta\Pi(k,\overline{c}_i)\right] \\ > \;& \Delta CS(M_i). \end{aligned}$$

- As the expected consumer surplus increases at each step, we must have $\underline{\Delta CS}'_k > \underline{\Delta CS}_k$.

- This completes the idea of the proof.

- The following limiting result holds even when the optimal approval policy does not have a cutoff structure:

Claim 2 *Consider a sequence of feasibility probabilities $\{r_1^t, r_2^t, ..., r_K^t\}_{t=0}^{\infty}$. If, for every $i\leq k$, $r_i^t\to 0$ as $t\to\infty$, then any merger M_j, $j\leq k+1$,with $\Delta CS(M_j)>0$ will be approved (i.e., $M_j\in\mathcal{A}^t$) for t sufficiently large.*

- The claim implies in particular that $\underline{\Delta CS}_j^t\to 0$ as $t\to\infty$ if $r_i^t\to 0$ for every $i<j$. In the limit as $r_i^t\to 0$ for every merger M_i, $1\leq i\leq K$, any CS-increasing merger will thus be approved as there is no "merger choice" in the limit.

- The idea behind the claim is simple: the only reason to commit not to approve a CS-increasing merger M_j is that the firms may instead propose an alternative merger that, while less profitable, raises CS by more. But such a preferable alternative merger must be a smaller merger. Hence, if the feasibility probabilities of all smaller mergers are sufficiently small, it is optimal to approve the CS-increasing merger M_j as the expected CS-level of the next most profitable merger is sufficiently close to zero.

1.2 Changes in Market Structure

1.2.1 Firm 0's Marginal Cost

- What happens as we change firm 0's marginal cost c_0?

Claim 3 *Consider a reduction in firm 0's marginal cost from c_0 to $c_0' < c_0$. Assuming that bargaining is efficient, this induces a decrease in all post-merger marginal cost cutoffs: $\overline{a}_k' < \overline{a}_k$ for every $1 \leq k \leq \hat{K}$.*

- Idea?

- A change in firm 0's marginal cost does not affect the outcome (consumer surplus, profits) after any merger M_k, $k \geq 1$, but it does affect the pre-merger outcome. In particular, we have $Q^{0\prime} > Q^0$ so that $\gamma \equiv CS^{0\prime} - CS^0 > 0$. Let $\eta \equiv \Pi^{0\prime} - \Pi^0$ denote the induced change in pre-merger aggregate profit. (Whether η is positive or negative depends on how efficient firm 0 is relative to the rest of the industry.)

- For any merger M_k, we thus have $\Delta CS(M_k)' = \Delta CS(M_k) - \gamma$ and $\Delta\Pi(M_k)' = \Delta\Pi(M_k) - \eta$. This implies that the CS-difference and aggregate profit difference between any two mergers M_i and M_j are the same before and after the change in c_0, i.e., $\Delta CS(M_i)' - \Delta CS(M_j)' = \Delta CS(M_i) - \Delta CS(M_j)$ and $\Delta\Pi(M_i)' - \Delta\Pi(M_j)' = \Delta\Pi(M_i) - \Delta\Pi(M_j)$.

- Consider first merger M_1. We have $\Delta CS(1, \overline{a}_1')' = \Delta CS(1, \overline{a}_1') - \gamma = 0$. Hence, $\Delta CS(1, \overline{a}_1') > \Delta CS(1, \overline{a}_1) = 0$, implying that $\overline{a}_1' < \overline{a}_1$.

- Consider now merger M_2. In particular, consider the marginal merger $(2, \overline{a}_2)$. If $\Delta CS(2, \overline{a}_2)' = \Delta CS(2, \overline{a}_2) - \gamma \leq 0$, it follows trivially (from our general characterization of minimum acceptable CS-levels) that $\overline{a}_2' < \overline{a}_2$. Suppose now instead that $\Delta CS(2, \overline{a}_2)' = \Delta CS(2, \overline{a}_2) - \gamma > 0$. Let $(1, \widetilde{a}_1)$ be such that $\Delta\Pi(1, \widetilde{a}_1) = \Delta\Pi(2, \overline{a}_2)$. Thus, $\Delta\Pi(1, \widetilde{a}_1)' = \Delta\Pi(2, \overline{a}_2)'$ so that the set of M_1-mergers that are less profitable than $(2, \overline{a}_2)$ is the same as before. As regards the effects on CS, we distinguish between three cases:

  1. If M_1 is such that $\overline{c}_1 > \overline{a}_1$, the merger will be blocked both before and after the change in c_0. Hence, the change in CS is the same in both cases.
  2. If M_1 is such that $\overline{a}_1 \geq \overline{c}_1 > \overline{a}_1'$, the merger will be approved initially but blocked after the decrease in c_0. Hence, in that case, the initial increase in CS is less than γ, while it is zero after the decrease in c_0.
  3. If M_1 is such that $\overline{c}_1 \leq \overline{a}_1'$, the merger will be approved both before and after the change in c_0. Hence, $\Delta CS(M_1)' = \Delta CS(M_1) - \gamma$.

We thus have

$$\begin{aligned}
& E_{\mathfrak{F}^{(1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}'^{(1)}\right)\right)' \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}'^{(1)}\right)'\right) \leq \Delta\Pi(2, \overline{a}_2)'\right] \\
> \quad & E_{\mathfrak{F}^{(1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}^{(1)}\right)\right) \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}^{(1)}\right)\right) \leq \Delta\Pi(2, \overline{a}_2)\right] - \gamma \\
= \quad & \Delta CS(2, \overline{a}_2) - \gamma \\
= \quad & \Delta CS(2, \overline{a}_2)'.
\end{aligned}$$

Hence, $\overline{a}_2' < \overline{a}_2$.

- Suppose now that $\overline{a}_j' < \overline{a}_j$ for every $j < k \leq \hat{K}$ (Induction Hypothesis). We want to show that this implies that $\overline{a}_k' < \overline{a}_k$. (This holds trivially if $\Delta CS(k, \overline{a}_k)' = \Delta CS(k, \overline{a}_k) - \gamma \leq 0$. Let us thus suppose that $\Delta CS(k, \overline{a}_k)' = \Delta CS(k, \overline{a}_k) - \gamma > 0$.)

- From the argument given above, we know that the set of mergers that are less profitable than $(k, \overline{a}_k)$ is the same before and after the change in c_0. Consider now $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)$, conditional on $\Delta\Pi(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)) \leq \Delta\Pi(k, \overline{a}_k)$. We distinguish between three cases:

  1. If $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right) = M_0$ so that $\Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)) = 0$, then $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)\prime}\right) = M_0$ and thus $\Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)\prime}\right)) = \Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)) = 0$. (This is the case where the next most profitable merger will be blocked both before and after changing c_0.)
  2. If $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right) \neq M_0$ and $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right) \in \mathcal{A}'$, then $\Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)\prime}\right)) = \Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)) - \gamma$. (This is the case where the next most profitable merger is the same under both policies and will be approved both before and after changing c_0.)
  3. If $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right) \neq M_0$ and $M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right) \notin \mathcal{A}'$, then $\Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)\prime}\right)) > \Delta CS(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)) - \gamma$. (This is the case where the next most profitable merger under policy $\mathcal{A}$ would not be approved under policy $\mathcal{A}'$.)

- We thus have

$$\begin{aligned}
& E_{\mathfrak{F}^{(k-1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}'^{(k-1)}\right)\right)' \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}'^{(k-1)}\right)'\right) \leq \Delta\Pi(k, \overline{a}_k)'\right] \\
> \quad & E_{\mathfrak{F}^{(k-1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)\right) \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(k-1)}, \mathcal{A}^{(k-1)}\right)\right) \leq \Delta\Pi(k, \overline{a}_k)\right] - \gamma \\
= \quad & \Delta CS(k, \overline{a}_k) - \gamma \\
= \quad & \Delta CS(k, \overline{a}_k)'.
\end{aligned}$$

Hence, $\overline{a}_k' < \overline{a}_k$.

Claim 4 *Consider a reduction in firm 0's marginal cost from c_0 to $c_0' < c_0$. Assuming that bargaining results in the merger that maximizes the increase in bilateral profit (i.e., the equilibrium of the offer game), this induces a decrease in all post-merger marginal cost cutoffs: $\overline{a}_k' < \overline{a}_k$ for every $1 \leq k \leq \hat{K}$.*

- Idea?

- The key difference to the case of efficient bargaining is that the reduction in c_0 affects different mergers partners differently. Let $\eta_k \equiv [\pi_0^{0\prime} + \pi_k^{0\prime}] - [\pi_0^0 + \pi_k^0]$ denote the induced change in pre-merger joint profit of firms 0 and k. The key observation is that the profit of a more efficient firm falls by a larger amount than that of a less efficient as price falls. That is, η_k is decreasing in k.

- The argument as to why $\overline{a}_1' < \overline{a}_1$ is unaffected by this.

- Consider now the (marginal) merger $M_2 = (2, \overline{a}_2)$. Let $(1, \widetilde{a}_1)$ be such that $\Delta\Pi(1, \widetilde{a}_1) = \Delta\Pi(2, \overline{a}_2)$, and $(1, \widetilde{a}_1')$ be such that $\Delta\Pi(1, \widetilde{a}_1')' = \Delta\Pi(2, \overline{a}_2)'$. We have

$$\begin{aligned}
\Delta\Pi(1, \widetilde{a}_1)' &= \Delta\Pi(1, \widetilde{a}_1) - \eta_1 \\
&< \Delta\Pi(1, \widetilde{a}_1) - \eta_2 \\
&= \Delta\Pi(2, \overline{a}_2) - \eta_2 \\
&= \Delta\Pi(2, \overline{a}_2)' \\
&= \Delta\Pi(1, \widetilde{a}_1')',
\end{aligned}$$

  where the inequality follows from $\eta_1 > \eta_2$. Hence, $\widetilde{a}_1' < \widetilde{a}_1$. That is, before the reduction in c_0, any merger M_1 with $\overline{c}_1 \geq \widetilde{a}_1$ induced a smaller increase in bilateral profit than merger $M_2 = (2, \overline{a}_2)$. After the reduction in c_0, this is still true, but now – in addition – any merger M_1 with $\widetilde{a}_1 > \overline{c}_1 \geq \widetilde{a}_1'$ also induces a smaller increase in bilateral profit than merger $M_2 = (2, \overline{a}_2)$. That is, there are now more and (in an FOSD sense) more efficient mergers M_1 that are less profitable than $M_2 = (2, \overline{a}_2)$. Since the induced CS-increase of merger M_1 is the greater, the lower is $\overline{c}_1$, we thus have again that

$$\begin{aligned}
& E_{\mathfrak{F}^{(1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}'^{(1)}\right)\right)' \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}'^{(1)}\right)'\right) \leq \Delta\Pi(2, \overline{a}_2)'\right] \\
> \; & E_{\mathfrak{F}^{(1)}}\left[\Delta CS\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}^{(1)}\right)\right) \mid \Delta\Pi\left(M^*\left(\mathfrak{F}^{(1)}, \mathcal{A}^{(1)}\right)\right) \leq \Delta\Pi(2, \overline{a}_2)\right] - \gamma \\
= \; & \Delta CS(2, \overline{a}_2) - \gamma \\
= \; & \Delta CS(2, \overline{a}_2)'.
\end{aligned}$$

  Hence, $\overline{a}_2' < \overline{a}_2$.

- Under the induction hypothesis that $\overline{a}_j' < \overline{a}_j$ for every $j < k \leq \hat{K}$, a similar argument can be used to show that $\overline{a}_k' < \overline{a}_k$.