Optimal Regulation in the Presence of Reputation Concerns*

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Abstract

We study a market with free entry and exit of firms that can costly invest in their quality. If the investment is observable, the first best is characterized by high quality and a large output. If the investment is non observable, free entry creates adverse selection, blocking market existence. If buyers learn over time about firms' quality, reputation formation allows firms to recover their investment in expectation and the market exists. However quality and output are low.

We show that, if the government have the same information as the market and can commit to a schedule of taxes and subsidies, it can achieve an allocation arbitrarily close to the first best. Moreover, even if the government has less information than the market, and for example only observes entry, it can still improve welfare by imposing entry costs. However, in this last case there is a trade-off between quality and output that prevents being close to the first-best.

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1 Introduction

The need for government intervention to impose discipline in markets has been frequently discredited by the argument that firms' reputation concerns are enough to self-discipline their behavior. A recent example of this discussion sparked in the aftermath of the 2008 financial crisis. In 1963, the former Federal Reserve chairman, Alan Greenspan wrote "Reputation, in an unregulated economy, is a major competitive tool...Left to their own devices, it is alleged, businessmen would attempt to sell unsafe food and drugs, fraudulent securities, and shoddy buildings...but it is in the self-interest of every businessman to have a reputation for honest dealings and a quality product".[1] Forty five years later, in his remarks before the House of Representatives he declared "Those of us who have looked to the self-interest of lending institutions to protect shareholders equity, myself included, are in a state of shocked disbelief".[2]

In this paper we argue that regulation can in fact leverage on market learning to foster reputation incentives and improve welfare. If the government knows the same as the market, it can achieve an allocation arbitrarily close to the informational unconstrained first best by committing to a scheme of taxes and subsidies that reward firms with good reputations more than what the market is able to without commitment. Furthermore, even if the government knows less than the market, still it can improve welfare by imposing taxes and benefits that move the market forces towards a better exploitation of statistical discrimination across firms that behave differently.

More specifically, we study a market with free entry and exit of firms, where entering firms decide whether to make a costly investment to become high quality or not. High quality firms generate better products in average. When buyers pay the expected utility from consumption, prices increase with reputation, which is the probability buyers assign to the firm being of high quality.

When the investment is observable, the equilibrium provides a first best benchmark in which all entrants invest to be high quality and there is a large output. At the other extreme, when investment is non observable and there is no learning, no entrant invests and the market does not exist. However, if buyers learn about firms' quality over time, for example by experiencing consumption, firms can recover the invest-

[1] "The Assault on Integrity". The Objectivist Newsletter, August 1963

[2] New York Times, "Greenspan Concedes Error on Regulation", October 24th, 2008

ment cost in the future from an expected increase in reputation. In this case there is a market, but characterized by inefficient low quality and low output.

We show that market inefficiency is not generated by the asymmetric information from unobservable investments, but rather because buyers cannot commit to make payments that provide the right incentives. Learning provides the force to restore efficiency, but the market may not be able to fully take advantage of reputation incentives. For example, the market does not provide enough incentives to firms with good reputation, since the evolution of their reputation depends little on performance. We show that a government with access to a commitment technology, such as the faculty to imposing taxes and subsidies, can achieve an allocation arbitrarily close to the first best. This is done by taxing firms with bad reputation and subsidizing firms with good reputation. This payment scheme has the property of introducing stronger incentives at high reputation levels, where they are the most needed, and weaker incentives at lower reputation levels, where they are less needed.

We then show that even when the government knows less than the market, its intervention can still improve welfare. If the government only observes entry, then positive entry costs improve welfare. If additionally the government observes firms' continuation, it can improve welfare further by front-loading costs and back-loading subsidies. Since high quality firms do not exit the market, while low quality firms exit in equilibrium, this policy rewards more in expectation those firms that invest. However, even when this intervention improves welfare, it is still characterized by less quality and less output than the first best benchmark.

Small positive entry costs always increase both quality and output, improving welfare. Contrarily, larger entry costs still increase quality, but at the expense of an output decline. The intuition of this result relies on three properties of firms' expected profits. First, firms' expected profits increase with reputation. Second, the expected profits for high quality firms are higher than for low quality firms, at any reputation level. This is because the reputation of high quality firms is always more likely to increase. Finally, firms' expected profits increase with the aggregate price level in the market, or, which is the same, decline with production.

When entry costs increase, both the firms that invest and the firms that do not invest, require higher expected profits at entry to compensate those costs. Two channels raise expected profits: higher reputation assigned to entrants - in equilibrium higher quality of entrants - and higher aggregate prices - in equilibrium lower output. Any increase in entry costs should be compensated by an increase in initial reputation, or quality of entrants. Initial improvements in initial reputation raises expected profits more for high quality firms than for low quality ones. This is because learning becomes easier and high quality firms gain more from easier learning. This gives room for lower aggregate prices - or higher output - to compensate investment. Additional improvements in initial reputation increase expected profits less for good firms than for bad ones. This is because learning becomes more difficult and bad firms gain relatively more from learning difficulties. Then, only an increase in aggregate prices - or an output reduction - can compensate firms to keep investing.

A technical contribution of the paper is the analytical derivation of the firms' value functions in continuous time when exit is an endogenous choice and firms know their type. We do this for three different processes of signals arrival: Bad news, good news and Brownian motion. This result allows the analytical comparison between entry conditions for high and low quality firms, providing tractability in welfare comparisons across different regulation policies. Furthermore, since in this paper types are not assumed but are the result of investment decisions, we are able to obtain the reputation assigned to entrants and the extent of adverse selection in the market endogenously from entry conditions.

This paper is related to two strands of literature that have not been systematically connected: reputation and regulation. With respect to the reputation strand, Mailath and Samuelson (2001) discusses a reputation model where firms enter at an exogenous reputation to replace those that exogenously die. The work of Tadelis (1999 and 2002) studies the market for names and the endogenous value of reputation as a tradeable asset. Among models with exit of firms, our paper is related to Horner (2002) and Bar-Isaac (2003). However, none of these papers consider a reputation model with both entry and exit decision by firms nor use it to discuss regulation implications. Furthermore our paper does not rely on exogenous types but on initial unobservable choices that determines the extent of adverse selection in the market. From a technical viewpoint, ours is the first paper that fully characterizes value functions with exit decisions in continuous time, when firms know their type and have the option to exit. Related papers of reputation in continuous time are Faingold and Sannikov (2007) and Board and Meyer-ter Vehn (2010) .

With respect to the regulation strand, this paper contributes to Leland (1979), extended later by Shaked and Sutton (1981) and Shapiro (1896), who introduce moral hazard and investment decisions. It also complements von Weizsacker (1980), who discusses how barriers to entry may increase welfare once we consider economies of scale and differentiated products. More recently, Albano and Lizzeri (2001) analyze the efficiency effects of certification intermediaries, but without making reference to reputation concerns and Garcia-Fontes and Hopenhayn (2000) focus on entry restrictions, while we allow for more general regulation possibilities and taxing schemes.

In the next Section we describe the economy. In Section 3 we solve the informational unconstrained first best allocation. In Section 4 we discuss the unregulated market solution. In Section 5 we show the role of regulation in filling the gap between the market solution and the first best. In Section 6 we provide a numerical illustration of regulatory forces. Finally, in Section 7 we make some final remarks.

2 The Economy

Time is continuous and denoted $t \in [0, \infty)$. There is a representative household in this economy. At each moment, this household derives utility from the consumption of two final goods: one which we term the "experience" good and one which we term the "numeraire" good. Let Y_t denote the household's consumption of the experience good and N_t its consumption of the numeraire good at t. The household's utility is given by

$$\int_t e^{-\hat{r}t} \left[\frac{1}{1-\eta} Y_t^{1-\eta} + N_t \right] \tag{1}$$

where $\hat{r}$ is the discount factor and $\eta \in (0, 1)$. At each time t, there is an endowment of 1 unit of the numeraire good. This good is not storable.

The experience good is produced using intermediate goods generated by a continuum of firms, each of which can produce one unit of this intermediate good at each time t at zero cost. The contribution of any single unit of the intermediate good to the production of the experience good at t is stochastic depending on its quality. If a unit of the intermediate good produced at t is "good" (g) it contributes one unit to the production of the experience good at t. If it is "bad" (b) it subtracts from, or destroys, κ units of the experience good produced at t.

Each of the firms producing the intermediate good at t can be one of two types — High quality (H) or Low Quality (L). The probability that the output of a given firm at t is good depends on the firm's type, $Pr(g|H) = \alpha_H > Pr(g|L) = \alpha_L$. We assume that $\frac{\alpha_H}{1-\alpha_H} > \kappa > \frac{\alpha_L}{1-\alpha_L}$ so that the expected contribution to the production of the experience good of a unit of the intermediate good produced by a high quality firm is positive while the expected contribution to the production of the experience good from a unit of the intermediate good produced by a low quality firm is negative. More generally, let ϕ denote the public belief regarding the probability that a given intermediate goods firm is high quality, then the expected contribution to the production of the experience good of a unit of the intermediate good produced by that firm is denoted $y(\phi)$ and is given by $y(\phi) = [\alpha_H\phi + \alpha_L(1-\phi)] + [(1-\alpha_H)\phi + (1-\alpha_L)(1-\phi)](-\kappa)$. Hence $y(\phi)$ is linear in ϕ.

$$y(\phi) = a_1\phi - a_0 \tag{2}$$

where $a_1 = (\alpha_H - \alpha_L)(1+\kappa) > 0$ and $a_0 = \kappa - \alpha_L(1+\kappa) > 0$.

For $\phi \in [0,1]$, let $m_t(\phi)$ denote the measure of units of the intermediate good produced at t by firms that are high quality with probability ϕ. The resource constraint for the experience good is then given by

$$Y_t = \int_\phi y(\phi)m_t(\phi)d\phi \tag{3}$$

Intermediate goods firms can produce one unit of output at zero cost at each time t that they continue in operation. In order to cease production, intermediate goods firms choose an exponential rate of exit $\omega_t^i(\phi) \in [\delta, \bar{\omega}]$, where $\delta > 0$ is an exogenous exit probability and $\bar{\omega}$ is a parameter to keep the evolution of the measure of firms well defined.

At each moment t, new intermediate goods firms can enter and start production. To create a new, high quality, intermediate goods firm, an investment of C units of the numeraire good is required. New low quality intermediate goods firms can be created at zero cost. If we let $m_t^e \in [0, \bar{m}^e]$ denote the measure of new intermediate goods firms created at t and $\phi_t^e \in [0,1]$ the fraction of those new firms that are high quality, hence the reputation assigned to entrants. Then the resource constraint for the numeraire good is given by

$$N_t = 1 - C\phi_t^e m_t^e \tag{4}$$

In what follows, the measure of intermediate goods firms with probability ϕ of being high quality is a state variable that limits production of the experience good. The evolution of this measure will depend on our assumptions regarding the signals available regarding firms' types. We begin by analyzing the optimal allocation in this economy when firms' investment of C, and hence their quality, is publicly observable.

3 Full Information Efficiency Benchmark

Here we assume consumers observe the initial investment, this is, whether firms pays C to become good or not. We describe the optimal allocation in this economy as a planning problem. In this case, the measure of intermediate goods firms $m_t(\phi)$ at each t is described by two numbers, $m_t(1)$ and $m_t(0)$. We assume that an initial stock of high quality firms $m_0(1) > 0$ is given.

At each t, the planner chooses the measure of entering firms $m_t^e \geq 0$, the fraction $\phi_t^e \in [0, 1]$ of those entering firms that invest C to become high quality, the continuation probabilities $\omega_t^H(1)$ and $\omega_t^L(0)$, the production of intermediate goods $m_t(1) \geq 0$ and $m_t(0) \geq 0$ consumption of the experience and numeraire goods Y_t and N_t to maximize welfare (1) subject to the following constraints: (3), (4),

$$dm_t(1) = -\omega_t^H(1)m_t(1) + \phi_t^e m_t^e \tag{5}$$

and

$$dm_t(0) = -\omega_t^L(0)m_t(0) + (1 - \phi_t^e)m_t^e \tag{6}$$

Clearly, since the output of a firm known to be low quality is expected to subtract from production of the experience good ($\pi(0) < 0$), it is optimal to set $\omega_t^L(0) = \bar{\omega}$ and $\phi_t^e = 1$. Likewise, since an existing firm known to be of high quality can contribute $y(1)$ to production of the experience good at zero cost as long as it continues, it is optimal to set $\omega_t^H(1) = \delta$, its minimum value. The optimal choice of entry m_t^e is a dynamic choice.

The cost, in terms of utility, of creating a new firm at t with probability $\phi_t^e = 1$ of being high quality is given by C while the benefit is given by

$$\int_{s \geq 0} e^{-rs} Y_{t+s}^{-\eta} y(1) ds$$

where $r = \hat{r} + \delta$ is the effective discount rate. Hence, an allocation with constant consumption of the experience good at level $Y_t = \bar{Y}$ where

$$y(1)\bar{Y}^{-\eta} = Cr \tag{7}$$

is optimal.

If $y(1)m_0(1)$ is less than this optimal level $\bar{Y}$, the planner creates new high quality firms at the maximum rate possible ($\bar{m}^e$) until this stock of high quality firms is attained. If $y(1)m_0(1)$ exceeds this optimal level, the planner creates no new intermediate goods firms until the stock of existing high quality firms has depreciated down to this level at rate δ. Once this stock of high quality intermediate goods firms $\bar{Y}$ is attained, the planner maintains investment of $y(1)m^e = \delta\bar{Y}$ to maintain this stock at a constant level.

4 Reputation in a Market

As an alternative benchmark, we consider the allocation that emerges in a market in which intermediate goods firms' investment of C is not observable and hence intermediate goods firms' quality is not directly observable.

We assume the experience and intermediate goods are transacted in the spot market at each moment t. The price of the experience good is then its marginal utility $Y_t^{-\eta}$. We also assume that producers of the experience good compete to purchase a unit of intermediate good offered by each intermediate goods firm so that the price that each intermediate goods firm charges for its output reflects the full surplus that this unit of the intermediate good contributes to the output of the experience good. Specifically, the price at t in units of the numeraire good, for a unit of the intermediate good produced by a firm that is believed to be of high quality with probability ϕ is given by its expected value of the marginal product,

$$p_t(\phi) = y(\phi)Y_t^{-\eta} \tag{8}$$

Given these prices, the first best allocation described in the previous section is sustained as an equilibrium. This is because intermediate goods firms obtain a positive price if investing and a negative price if not investing. Hence, new firms always invest. Furthermore the mass of entrants sustain $\bar{Y}$ in steady state. If $Y < \bar{Y}$ additional firms have incentives to enter because the experience good price is high. Contrarily, if $Y > \bar{Y}$ no new firm enter since the experience good price is low to compensate the investment, until firms depreciate at a rate δ and production is $\bar{Y}$.

As an opposite benchmark, in the extreme case with non observable investment and no learning, free entry blocks entirely the existence of a market. In order to recover the initial investment, entering intermediate goods firms that invest should expect discounted profits for at least C. Since investment is non observable and there is no learning, all entering firms have the same expected profits, and they prefer to enter without investing. Forecasting this outcome, the experience good producers are willing to buy only at negative prices ($p(0) < 0$) and in equilibrium no firm enters to produce intermediate goods and hence $Y = 0$.

Instead, we assume that public signals about each intermediate goods firm's quality are revealed over time as long as the firm continues in operation, so that it is possible for agents in the economy to learn over time about their initial investment.

In a stationary equilibrium, consumption aggregates Y_t and N_t are constant, as is entry m_t^e, the fraction ϕ_t^e of entrants who are high quality, and the measure $m_t(\phi)$ of intermediate goods with belief ϕ about their quality producing. We consider stationary equilibria under three alternative assumptions about the structure of signals of quality: *bad news, good news, and Brownian Motion*. In the bad news case, if the firm is of low quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the firm is high quality. In the good news case, the assumption is reversed — if the firm is of high quality, a signal that reveals that quality arrives at rate $\lambda > 0$. No such signal can arrive if the firm is low quality. Finally, in the Brownian Motion case, signals about firm quality arrive continuously. Specifically,

$$dS_t = \mu_i dt + \sigma dZ_t$$

where $i = \{H, L\}$, S_t is a Brownian motion with drifts that depend on the firm's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

4.1 Bad News

In this case $dS_t \in \{0, 1\}$, which means either there is a signal or not at each t. The bad news case is defined by $Pr(dS_t = 1|H) = 0$ and $Pr(dS_t = 1|L) = \lambda$, which means there is a positive Poisson arrival only for low quality firms. When a signal arrives the producer of intermediate goods is revealed to be of low quality and hence the public belief about its quality jumps down to $\phi = 0$. With this reputation, the firm would never be able to sell its output at a non-negative price. Thus, following this event, it is optimal for the firm to cease production and exit as quickly as possible.

It is convenient in this case for us to use the variable $l = (1-\phi)/\phi : [0,1] \to (\infty, 0]$ to summarize the reputation level of an intermediate goods firm. The evolution of l is determined by,

$$\frac{dl_t}{dt} = \left[\frac{Pr(dS_t|L) - Pr(dS_t|H)}{Pr(dS_t|H)}\right] l_t$$

When bad news arrive (i.e., $dS_t = 1$)

$$\frac{dl_t}{dt} = \left[\frac{\lambda - 0}{0}\right] l_t = \infty$$

and reputation jumps immediately to $l = \infty$. Since $\phi = \frac{1}{1+l}$, this means reputation drops down immediately to $\phi = 0$.

While there are no news (i.e., $dS_t = 0$), reputation increases. After t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_0 = .. = S_t = 0|L)}{Pr(S_0 = .. = S_t = 0|H)}\right] l_0 = \left[\frac{(1-\lambda)^t}{1}\right] l_0 = (1-\lambda)^t l_0$$

which means l_t is decreasing (reputation is increasing) over time at a rate $1 - \lambda$.

While there are no news, the evolution of reputation for firms with high and low quality is the same. After bad news, a firm exits and obtains zero thereafter. Then, the value functions for both types only differ in their discount factor. More formally, a value function for a low quality firm with reputation l, for a general $\pi(l(\phi))$, is

$$V_L(l) = \int_{s=0}^{\infty} e^{-(r+\lambda)s}\pi((1-\lambda)^s l)ds$$

and the value function for a high quality firm with reputation l is

$$V_H(l) = \int_{s=0}^{\infty} e^{-rs}\pi((1-\lambda)^s l)ds$$

Solving explicitly the integrals for the case of linear payoffs, no marginal costs and $Y = 1$, $\pi(\phi) = a_1\phi - a_0$ (hence $\pi(l(\phi)) = \frac{a_1}{1+l} - a_0$),

$$V_L(l) = \frac{1}{r+\lambda}\left[a_1 . {}_2F_1\left(1, m_{r+\lambda}; m_{r+\lambda}+1; -l\right) - a_0\right] \tag{9}$$

$$V_H(l) = \frac{1}{r}\left[a_1 . {}_2F_1\left(1, m_r; m_r+1; -l\right) - a_0\right] \tag{10}$$

where ${}_2F_1(1, m; m+1, -l)$ is an hypergeometric function,

$$m_r = -\frac{r}{ln(1-\lambda)} > 0 \qquad \texttt{and} \qquad m_{r+\lambda} = -\frac{r+\lambda}{ln(1-\lambda)} > m_r > 0$$

The hypergeometric function has well defined properties when $m > 0$. In particular, it is monotonically increasing in ϕ from 0 to 1.

$${}_2F_1(1, m; m+1, -l(\phi)) : [0, 1] \to [0, 1]$$

Now we denote $V_i(\phi) = V_i(l(\phi))$ for all ϕ and $i \in \{L, H\}$. Since $\lim_{\phi\to 0} V_L(\phi) = -\frac{a_0}{r+\lambda} < 0$, there is a $\phi = \bar{\phi}$ such that $V_L(\bar{\phi}) = 0$. This point will be relevant in the discussion of equilibrium. For the moment we just discuss the properties of value functions in the range $[\bar{\phi}, 1]$.

$$V_L(\phi) \;:\; [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r+\lambda}]$$
$$V_H(\phi) \;:\; [\bar{\phi}, 1] \to [V_H(\bar{\phi}), \frac{a_1 - a_0}{r}]$$

where $V_H(\bar{\phi}) = \frac{1}{r}\left[a_1 \cdot {}_2F_1\left(1, m_r; m_r+1; -\frac{1-\bar{\phi}}{\bar{\phi}}\right) - a_0\right] > 0.$

Three properties of these value functions are,

- $V_H(\phi) > V_L(\phi)$ for all ϕ. This is because ${}_2F_1(1, m; m+1, -l) >_2 F_1(1, m'; m'+1, -l)$ for all l if $m < m'$. In our case, $m_{r+\lambda} > m_r$, $V_H(l(\phi)) > V_L(l(\phi))$ for all l.
- $\frac{\partial \frac{V_L(\phi)}{V_H(\phi)}}{\partial \phi} > 0$, so the ratio $\frac{V_L(\phi)}{V_H(\phi)}$ is increasing in ϕ.
- $\frac{\partial(V_H(\phi)-V_L(\phi))}{\partial \phi} > 0$, so the difference $V_H(\phi) - V_L(\phi)$ is increasing in ϕ.

4.2 Good News

In this case $Pr(dS_t = 1|H) = \lambda$ and $Pr(dS_t = 1|L) = 0$. When a signal arrives the intermediate goods firm is revealed to be of high quality and hence the public belief ϕ regarding this firm jumps up to $\phi = 1$. After good news the firm does not lose its reputation.

Again, we use the variable $l = (1-\phi)/\phi$. When good news arrive (i.e., $dS_t = 1$)

$$\frac{dl}{dt} = \left[\frac{0-\lambda}{\lambda}\right] l_t = -l_t$$

and reputation jumps immediately to $l = 0$, or $\phi = 1$.

While there are no news (i.e., $dS_t = 0$), reputation decreases. After t periods of no news, accumulating the change in reputation

$$l_t = \left[\frac{Pr(S_0 = .. = S_t = 0|L)}{Pr(S_0 = .. = S_t = 0|H)}\right] l_0 = \left[\frac{1}{(1-\lambda)^t}\right] l_0 = (1-\lambda)^{-t} l_0$$

which means l_t is increasing (reputation is decreasing) over time at a rate $\frac{1}{1-\lambda}$.

Denoting $\pi(l(1))$ the payoffs for a firm with $\phi = 1$, the value function for a firm that generates good news is,

$$V(l(1)) = \frac{\pi(l(1))}{r}$$

There is a key difference between good news and bad news. Under bad news, reputation only increases, which means exit never occurs, unless a bad signal is revealed.

Under good news, reputation continuously decrease and firms that hit $\bar{\phi}$ will exit with probability λ such that ϕ is never below $\bar{\phi}$.

Formally, a value function for a low quality firm with l is

$$V_L(l) = \int_{s=0}^{\infty} e^{-rs} \max\left\{\pi((1-\lambda)^{-s} l), 0\right\} ds$$

Since reputation is always decreasing, the point at which low quality firms exit is determined by $\pi(l(\bar{\phi})) = \pi(\bar{l}) = 0$. Then,

$$V_L(l) = \int_{s=0}^{T(l)} e^{-rs} \pi((1-\lambda)^{-s} l) ds \tag{11}$$

where $T(l)$ is the time required for l to increase up to $\bar{l}$. Since the evolution of reputation is deterministic $\bar{l} = (1-\lambda)^{-T(l)} l$. Then,

$$T(l) = \frac{log(l) - log(\bar{l})}{log(1-\lambda)} \tag{12}$$

Similarly, the value function for a high quality firm with l is

$$V_H(l) = \int_{s=0}^{T(l)} e^{-(r+\lambda)s} \left[\pi((1-\lambda)^{-s} l) + \lambda \frac{\pi(l(1))}{r}\right] ds + \int_{s=T(l)}^{\infty} e^{-(r+\lambda)s} \lambda \frac{\pi(l(1))}{r} ds \tag{13}$$

In the case of linear payoffs, no marginal costs and $Y = 1$, exit reputation is given by $\pi(\bar{l}) = \frac{a_1}{1+\bar{l}} - a_0 = 0$. Then $\bar{l} = \frac{a_1 - a_0}{a_0}$ and $T(l)$ is given by equation (12). Value functions are,

$$\begin{aligned} V_L(l) \quad = \quad & \frac{1}{r}\left[a_1\left(1 - {}_{\cdot 2}F_1\left(1, m_r; m_r+1; -\frac{1}{l}\right)\right) - a_0\right] \\ & -\frac{e^{-rT(l)}}{r}\left[a_1\left(1 - {}_{\cdot 2}F_1\left(1, m_r; m_r+1; -\frac{a_0}{a_1 - a_0}\right)\right) - a_0\right] \end{aligned} \tag{14}$$

$$
\begin{aligned}
V_H(l) &= \frac{1}{r+\lambda}\left[a_1\left(1 - {}_2F_1\left(1, m_{r+\lambda}; m_{r+\lambda}+1; -\frac{1}{l}\right)\right) - a_0 + \lambda\frac{a_1 - a_0}{r}\right] \\
&\quad -\frac{e^{-(r+\lambda)T(l)}}{r+\lambda}\left[a_1\left(1 - {}_2F_1\left(1, m_{r+\lambda}; m_{r+\lambda}+1; -\frac{a_0}{a_1 - a_0}\right)\right) - a_0\right] \quad (15)
\end{aligned}
$$

Now we denote $V_i(\phi) = V_i(l(\phi))$ for all ϕ and $i \in \{L, H\}$. Since $T(l(1)) = \infty$, using the previously discussed properties of the hypergeometric functions,

$$
\begin{aligned}
V_L(\phi) &: [\bar{\phi}, 1] \to [0, \frac{a_1 - a_0}{r}] \\
V_H(\phi) &: [\bar{\phi}, 1] \to [\frac{\lambda}{r+\lambda}\frac{a_1 - a_0}{r}, \frac{a_1 - a_0}{r}]
\end{aligned}
$$

Three properties of these value functions are:

- $V_H(\phi) > V_L(\phi)$ from the properties of hypergeometric functions discussed above.
- $\frac{\partial(V_L(\phi)/V_H(\phi))}{\partial\phi} > 0$, so the ratio $\frac{V_L(\phi)}{V_H(\phi)}$ is increasing in ϕ.
- $\frac{\partial(V_H(\phi) - V_L(\phi))}{\partial\phi} < 0$, so the difference $V_H(\phi) - V_L(\phi)$ is decreasing in ϕ.

4.3 Brownian Motion

Assume now the signal process follows a Brownian motion

$$dS_t = \mu_i dt + \sigma dZ_t$$

where $i = \{H, L\}$, drifts depend on the firm's type $\mu_H > \mu_L$ and the noise σ is the same for both types.

The following Proposition shows reputation, both for high and low quality firms, also follows a Brownian motion. The proof is in the Appendix.

Proposition 1 *The reputation process high quality firms expect is a submartingale*

$$d\phi_t^H = \frac{\lambda^2(\phi_t)}{\phi_t}dt + \lambda(\phi_t)dZ_t \quad (16)$$

and the reputation process low quality firms expect is a supermartingale

$$d\phi_t^L = -\frac{\lambda^2(\phi_t)}{(1-\phi_t)}dt + \lambda(\phi_t)dZ_t \tag{17}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ *and* $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ *is the signal to noise ratio.*

Four clear properties arise from inspecting equations (16) and (17). First, high quality firms expect a positive drift in their evolution of reputation while low quality firms expect a negative drift. Second, when reputation ϕ_t is either 0 or 1, drifts and volatilities are zero, which means at those points reputation do not change, both for high and low quality firms. Third, reputation varies more at intermediate levels of ϕ_t, and volatilities are larger. Finally, the drift for high quality firms is higher than for low quality firms for bad reputations and lower for good reputations, since ϕ_t is in the denominator of the drift for high quality firms, while $(1-\phi_t)$ is in the denominator of the drift for low quality firms.

We can now write the ordinary differential equations that characterize the value functions for high and low quality firms. The discussion about the determination of these ODEs is in the Appendix.

$$r\rho V_L(\phi) = \rho\pi(\phi) - \phi^2(1-\phi)V_L'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_L''(\phi) \tag{18}$$

$$r\rho V_H(\phi) = \rho\pi(\phi) + \phi(1-\phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_H''(\phi) \tag{19}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}$$

Solving these ODEs, we can obtain the value functions for high and low quality firms. The discussion about the determination of these value functions is in the Appendix.

$$V_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\,d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\} \tag{20}$$

$$V_L(l) = K\left\{\int_0^1 \theta^{\gamma-1}\pi(\theta l)\,d\theta - \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\,d\theta\right\} \tag{21}$$

where $\theta = l'/l$,

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \texttt{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}$$

In reduced form

$$\begin{aligned} V_L(l) &= K[B_L(l) - A_L(l)] \ \texttt{and} \\ V_H(l) &= K[B_H(l) - A_H(l)] \end{aligned}$$

where

$$B_L(l) = \int_0^1 \theta^{\gamma-1}\pi(\theta l)\, d\theta \qquad \texttt{and} \qquad A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\, d\theta$$

$$B_H(l) = \int_0^1 \theta^{\gamma-2}\pi(\theta l)\, d\theta \qquad \texttt{and} \qquad A_H(l) = \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\, d\theta - \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}$$

The derivatives of these value functions are (the proof is also in the Appendix),

$$lV_L'(l) = K[-\gamma B_L(l) + (1-\gamma)A_L(l)] \ \texttt{and} \tag{22}$$

$$lV_H'(l) = K[(1-\gamma)B_H(l) - \gamma A_H(l)] \tag{23}$$

Boundary conditions (value matching and smooth-pasting for low and high types) must be satisfied at $\bar{l}$. These conditions jointly determine $\bar{l}$, χ and ψ:[3]

$$V_L(\bar{l}) = V_L'(\bar{l}) = V_H'(\bar{l}) = 0$$

[3]Value matching and smooth pasting conditions for low quality firms arises from optimal exiting decisions and the smooth pasting condition for high quality firms arises from a belief process that is reflecting at $\bar{\phi}$

Hence,

$$\begin{aligned} V_L(\bar{l}) &= 0 \Rightarrow \int_{\chi/\bar{l}}^{1} \theta^{-\gamma}\pi\left(\theta\bar{l}\right)d\theta = \int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right)d\theta \\ \bar{l}V_L'(\bar{l}) &= 0 \Rightarrow (1-\gamma)\int_{\chi/\bar{l}}^{1} \theta^{-\gamma}\pi\left(\theta\bar{l}\right)d\theta = \gamma\int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right)d\theta \end{aligned}$$

Combining the two conditions, we find the equation that pins down $\bar{l}$:

$$\int_0^1 \theta^{\gamma-1}\pi\left(\theta\bar{l}\right)d\theta = 0 \tag{24}$$

and the equation that pins down χ

$$\int_{\chi/\bar{l}}^{1} \theta^{-\gamma}\pi\left(\theta\bar{l}\right)d\theta = 0 \tag{25}$$

Finally, the condition that pins down ψ is

$$\begin{aligned} (1-\gamma)B_H(\bar{l}) &= \gamma A_H(\bar{l}) \\ (1-\gamma)\int_0^1 \theta^{\gamma-2}\pi(\theta\bar{l})d\theta &= \gamma\left[\int_{\psi/\bar{l}}^{1} \theta^{-\gamma-1}\pi(\theta\bar{l})d\theta - \frac{\pi(0)}{\gamma}\left(\frac{\psi}{\bar{l}}\right)^{-\gamma}\right] \end{aligned} \tag{26}$$

Hence, we completely characterized value functions and the reputation at which low quality firms exit. These results are important to discuss the reputational equilibrium in the market, with and without government intervention.

4.4 Equilibrium Without Government Intervention

In this section we analyze the market solution in a stationary equilibrium without commitment, when profits are linear in ϕ and scale with $Y^{-\eta}$. This is

$$\pi(\phi) = p(\phi) = y(\phi)Y^{-\eta} = (a_1\phi - a_0)Y^{-\eta}$$

Since in the stationary equilibrium $\Pi_i(\phi) = V_i(\phi)Y^{-\eta}$, it is possible to separate the effects of reputation ϕ and the effects of production Y in the discounted expected profits. Then, Y and ϕ^e are determined by the following free entry conditions.

$$\begin{aligned} V_H(\phi^e)Y^{-\eta} &\geq C \\ V_L(\phi^e)Y^{-\eta} &\geq 0 \end{aligned} \tag{27}$$

It is clear that both conditions bind. If the free entry condition for high type firms is not binding, $\phi^e = 1$. Since $V_H(1) = V_L(1) = \frac{y(1)}{r}$, the condition for low type firms is not binding either. Hence $\phi^e = 1$ is not an equilibrium. If the free entry condition for low type firms is not binding, $\phi^e = 0$. However, since $y(0) < 0$ there is no entry and $\phi^e = 0$ is not an equilibrium either. Hence the free entry conditions are characterized by both equations in 27 binding and $\phi^e \in (0, 1)$.

We obtain the average quality of entrants ϕ^e from the entry condition for the low type. By construction $\phi^e = \bar{\phi}$, since $V_L(\bar{\phi}) = 0$. Then,

$$\phi^e = \bar{\phi} = \frac{1}{1 + \bar{l}}$$

where $\bar{l}$ is defined in equation (24) for the Brownian motion case. Hence, without regulation, new firms enter with the lowest possible reputation sustainable in the market, this is the reputation at which low quality firms are willing to exit.

The stationary consumption of the experience good in the market economy Y_M, is determined by the difference between free entry conditions,

$$Y_M = \left[\frac{V_H(\bar{\phi}) - V_L(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} \tag{28}$$

By comparison, the optimum stationary consumption of the experience good in the full information benchmark $\bar{Y}$ from equation (7) is,

$$\bar{Y} = \left[\frac{y(1)}{rC}\right]^{\frac{1}{\eta}} = \left[\frac{V_H(1)}{C}\right]^{\frac{1}{\eta}} > Y_M \tag{29}$$

The market solution is clearly suboptimal when compared to the unconstrained benchmark, with less quality of entrants ($\phi^e = \bar{\phi} < 1$) and less production of the experience good ($Y_M < \bar{Y}$).

5 Optimal Regulation

Now assume there is a government that can impose taxes or give subsidies to each intermediate goods firm. The main point is the government can commit to a payment scheme while experience good producers cannot. We explore optimal taxation under different information structures. First we assume the government observes each firm's revenues, hence reputation, and charge taxes based on those revenues in order to distort the reputation gains. Then, we assume the government knows less than the market. In particular the government only observes entry and then it cannot do much more than charging a tax upon entry. We focus on the Brownian motion case, but briefly describe the bad news and good news cases as well.

5.1 The Government Knows the Same as the Market

First, we assume the government has the same information than the market. In particular it can observe the reputation of a firm. It is not necessary it observes the evolution of the public signal, since it can infer the reputation, for example, by observing firms' revenues. We show the government can achieve an allocation arbitrarily close to the full information benchmark by committing to a scheme of payments that distort revenues via taxes and subsidies, paying less than the market for bad reputation firms and more to good reputation firms. To abstract from budget balance considerations we assume the resources to pay subsidies are obtained by lump sum taxes and that taxes to firms go back to consumers as lump sum subsidies.

We propose the government charges entry costs F and commits to a payments scheme (after taxes and subsidies) (again for $Y = 1$, since they just scale by Y), $\pi(x) = c - d(x - \hat{x})$, where $x = log(\frac{1-\phi}{\phi}) : [0,1] \rightarrow (\infty, -\infty)$ and c, d and $\hat{x}$ are parameters optimally determined to achieve an allocation arbitrarily close to first best.[4]

Transforming the ordinary differential equations as a function of x we obtain,

$$
\begin{aligned}
r\rho V_L(x) &= \rho\pi(x) + \frac{1}{2}V_L'(x) + \frac{1}{2}V_L''(x) \\
r\rho V_H(x) &= \rho\pi(x) - \frac{1}{2}V_H'(x) + \frac{1}{2}V_H''(x)
\end{aligned}
$$

[4]The reason we do not just assume $\pi(x) = c - dx$ will be clear later, but specifically it allows to make c and d only a function of $\bar{Y}$

We can obtain the homogeneous solutions from

$$\begin{aligned}\frac{1}{2}V_L''(x)+\frac{1}{2}V_L'(x)-r\rho V_L(x) &= 0\\ \frac{1}{2}V_H''(x)-\frac{1}{2}V_H'(x)-r\rho V_H(x) &= 0\end{aligned}$$

For low quality firms: $\nu_1^L=-\frac{1}{2}-\sqrt{\frac{1}{4}+2r\rho}<0$ and $\nu_2^L=-\frac{1}{2}+\sqrt{\frac{1}{4}+2r\rho}>0$.

For high quality firms: $\nu_1^H=\frac{1}{2}-\sqrt{\frac{1}{4}+2r\rho}<0$ and $\nu_2^H=\frac{1}{2}+\sqrt{\frac{1}{4}+2r\rho}>0$.

Conjecturing $V_L(x)=C-D(x-\hat{x})$, then $V_L'(x)=-D$ and $V_L''(x)=0$. The particular solution for the low type can be obtained from rewriting the ODE as,

$$-\frac{1}{2}D-r\rho[C-D(x-\hat{x})]+\rho[c-d(x-\hat{x})]=0$$

Equating the coefficient of x separately to zero (since the equation must hold as an identity in x), we obtain that $rD=d$ and $rC=c-\frac{d}{2r\rho}$. Similarly for $V_H(x)$. Adding the particular and homogenous solutions we obtain the value functions as,

$$\begin{aligned}rV_L(x) &= c-\frac{d}{2r\rho}-d(x-\hat{x})+K_1^Le^{\nu_1^Lx}+K_2^Le^{\nu_2^Lx}\\ rV_H(x) &= c+\frac{d}{2r\rho}-d(x-\hat{x})+K_1^He^{\nu_1^Hx}+K_2^He^{\nu_2^Hx}\end{aligned}$$

Assuming the government cannot pay more than $\hat{\pi}$, $\lim_{x\to-\infty}V_i(x)=\bar{\pi}$ and $K_1^L=K_1^H=0$. Then we relabel K_2^i as K_i and $\nu_2^i=\nu_i$. K_2^L is determined from smooth pasting condition for the low quality firm ($V_L'(\bar{x})=0$),

$$K_L=\frac{d}{\nu_Le^{\nu_L\bar{x}}}$$

The exiting point $\bar{x}$ is determined by value matching $V_L(\bar{x})=0$

$$c-\frac{d}{2r\rho}-d(\bar{x}-\hat{x})+\frac{d}{\nu_L}=0$$

or

$$\bar{x}=\frac{c}{d}-\frac{1}{2r\rho}+\frac{1}{\nu_L}+\hat{x} \tag{30}$$

Finally, K_H is determined by the smooth pasting condition for the high quality firm ($V_H'(\bar{x}) = 0$).

$$K_H = \frac{d}{\nu_H e^{\nu_H \bar{x}}}$$

Now we can describe an optimal taxation the government can impose to achieve an allocation arbitrarily close to first best,

Proposition 2 *A government can achieve an allocation arbitrarily close to first best $\phi^e = 1 - \epsilon$, with $\epsilon \to 0$, and $\bar{Y}$ by committing to a payment scheme after taxes which is a function of reputation. A particular payment scheme that achieves this result is $\pi(x) = c - d(x - \hat{x})$ where $x = log\left(\frac{1-\phi}{\phi}\right)$ and $\hat{x} = x^e = log\left(\frac{\epsilon}{1-\epsilon}\right)$.*

The regulator has access to three instruments (c, d and F) to achieve three targets, $x^e \to -\infty$ (or $\phi^e \to 1$), $\bar{x} \to \infty$ (or $\bar{\phi} \to 0$) and $Y = \bar{Y}$. If the government achieves the target that almost all firms entering invest, then it is optimal it targets that almost none of them exit. We assume $\hat{x} = x^e$ and a payoff function

$$\pi(x) = \begin{cases} c - d(x - x^e) & \text{for} \quad x \geq \frac{c-\hat{\pi}}{d} + x^e \\ \hat{\pi} & \text{for} \quad x < \frac{c-\hat{\pi}}{d} + x^e \end{cases}$$

where $\hat{\pi}$ is the maximum subsidy the government can pay. For given targets x^e and $\bar{x}$ we can determine the optimal ratio c/d from equation 30. This ratio together with the ratio of entry conditions determine optimal entry costs F

$$\frac{2r\rho[c + \frac{d}{\nu_L}e^{\nu_L(x^e - \bar{x})}] - d}{2r\rho[c + \frac{d}{\nu_H}e^{\nu_H(x^e - \bar{x})}] + d} = \frac{F}{C + F}$$

Finally, a target $\bar{Y}$ together with c/d and F from above, jointly determine c (hence d) from the low quality firms' entry condition,

$$\bar{Y}^{-\eta}\left[\frac{c}{r} + \frac{d}{r\nu_L}e^{\nu_L(x^e - \bar{x})} - \frac{d}{r2r\rho}\right] = F$$

5.1.1 Bad and Good News

In the case of bad news, we assume the government has two instruments - entry costs F and a per moment subsidy f_B - to target $\phi^e = 1$ and $\bar{Y}$. From entry conditions,

$$\frac{V_L(1|f_B)}{V_H(1|f_B)} = \frac{r}{r+\lambda} = \frac{F}{C+F} \tag{31}$$

$$\bar{Y}^{-\eta} V_L(1|f_B) = F \tag{32}$$

From the first condition $F = \frac{r}{\lambda}C$. From the second condition $\frac{rC}{a_1 - a_0}\left[\frac{a_1 - a_0 + f_B}{r+\lambda}\right] = \frac{r}{\lambda}C$, Hence

$$f_B = \frac{r}{\lambda}(a_1 - a_0)$$

In the case of good news, the government can be arbitrarily close to the first best, but not achieve it perfectly. The reason is that, once a low quality firm has a perfect reputation, there is no possibility to distinguish it from a high quality firm even after observing good news. We also assume the regulator has two instruments - entry costs F and a reward for having a perfect reputation (i.e., $\phi = 1$), f_G - to target a $\hat{\phi}^e$ arbitrarily close to 1 and $\bar{Y}$. From entry conditions, F is optimally derived from targeting $\hat{\phi}^e$.

$$\bar{Y}^{-\eta} V_L(\hat{\phi}^e) = F$$

and f_G is determined by the ratio

$$\frac{V_L(\hat{\phi}^e)}{V_H(\hat{\phi}^e|f_G)} = \frac{F}{C+F}$$

5.2 The Government Knows Less than the Market

5.2.1 The regulator only observes firms' entry

First, let's assume the regulator can only observe the entry of intermediate good firms. In a sense the government has much less information about firms than the market. Still it can improve welfare from the market situation just imposing the right entry costs F. Recall firms' payoffs in the absence of per moment taxes and subsidies are linear in reputation $\pi(\phi) = (a_1\phi - a_0)Y^{-\eta}$. In this case, entry conditions are,

$$\begin{aligned} V_H(\phi^e)Y^{-\eta} &= C + F \\ V_L(\phi^e)Y^{-\eta} &= F \end{aligned} \tag{33}$$

The next Proposition shows three important entry properties. First, higher entry costs imply a higher quality of entrants. Second, this relation is unique. Finally, higher entry costs imply the reputation assigned to entrants ϕ^e is higher than the lowest reputation operating in the market $\bar{\phi}$.

Proposition 3 *If firms' payoffs are linear in reputation ϕ, entry costs F uniquely determines the reputation assigned to entrants, ϕ^e. Furthermore, an increase in F monotonically increases ϕ^e.*

This result relies on an increasing ratio of value functions $V_L(\phi)/V_H(\phi)$. This is relevant because the ratio of entry condition in steady state is characterized by this ratio and determines the F required to obtain a given quality of entrants ϕ^e in equilibrium. Now analyzing each single entry condition we can study the impact of entry costs on the market size Y. The proof is in the Appendix.

Proposition 4 *There are optimal entry costs $F^* > 0$ that maximize total output of the experience good Y.*

Proof Take the difference between entry conditions (33), such that,

$$\left(V_H(\phi^e) - V_L(\phi^e)\right) Y^{-\eta} = C.$$

Taking derivatives with respect to ϕ^e

$$\frac{\partial Y}{\partial \phi^e} = \frac{\left[\frac{\partial V_H(\phi^e)}{\partial \phi^e} - \frac{\partial V_L(\phi^e)}{\partial \phi^e}\right] Y}{\eta \left[V_H(\phi^e) - V_L(\phi^e)\right]}$$

The maximum output of the experience good is determined by the following condition

$$\frac{\partial V_H(\phi^e)}{\partial \phi^e} = \frac{\partial V_L(\phi^e)}{\partial \phi^e} \tag{34}$$

Using the derivatives of the value functions, this condition implies

$$\gamma[B_L(l) - A_H(l)] = (\gamma - 1)[B_H(l) - A_L(l)]$$

which is fulfilled for some $\phi \in (\bar{\phi}, 1)$ when firms' payoffs are linear. Recall that both derivatives are zero at $l(\bar{\phi})$ (by construction of smooth pasting conditions) and at $l(1)$ (this is straightforward to check by considering that ${}_2F_1(1, m; m+1, 0) = 1$ for all m). Also by construction both derivatives are positive for all $\phi \in [\bar{\phi}, 1]$.

Q.E.D.

By imposing F, the government can achieve a higher welfare than in absence of entry costs, not only because $\phi^e > \bar{\phi}$ but also because

$$\bar{Y} > Y_F = \left[\frac{V_H(\phi^e) - V_L(\phi^e)}{C}\right]^{\frac{1}{\eta}} > Y_M \tag{35}$$

since $V_H(\phi^e) - V_L(\phi^e)$ is the maximum achievable by optimization. However, regulation in this case is not even close to achieve the first best $\bar{Y}$ in equation (29). This is because an increase in F beyond the point at which $\frac{\partial V_H(\phi^e)}{\partial \phi^e} = \frac{\partial V_L(\phi^e)}{\partial \phi^e}$ implies a reduction in $V_H(\phi^e) - V_L(\phi^e)$ and hence in Y.

Even when the government has information restrictions when compared to the market, it is still able to increase efficiency by fostering the incentives coming from reputation concerns. Specifically, entry costs F help the market in achieving an initial quality of entrants at an intermediate level where learning happens faster. This intervention helps statistical discrimination, and high quality firms have the chance to distinguish themselves from low quality firms at a faster rate. Hence the market allows for more production and a lower price to compensate firms to invest in quality.

When the initial quality of entrants is very high, the reputation assigned to entrants is also very high. This introduces high incentives for firms to enter without investing. The way to compensate firms to really invest in quality is a higher price for the experience goods and hence a lower production of them in equilibrium. This is the main reason the effectiveness of entry costs to improve welfare is limited.

Even when F^* maximizes Y it does not maximize welfare. Increasing entry costs from 0 to F^* allows an increase both in quality ϕ^e and production Y, having a clear positive impact on welfare. Increasing entry costs above F^* keeps increasing quality but at the cost of reducing production. This creates a trade-off that determine an optimum $F^{**} > F^*$ that maximizes welfare.

Proposition 5 *There are optimal entry costs $F^{**} > F^*$ that maximizes welfare, which are characterized by the following condition,*

$$(V_H'(\phi^{e**}) - V_L'(\phi^{e**})) = -\frac{\eta\delta C^{\frac{1}{\eta}} a_0 g'(\phi^{e**})[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{1-\frac{1}{\eta}}}{[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{\frac{1}{\eta}-1} - \delta g(\phi^{e**})} < 0$$

where $g(\phi^e) = \frac{(\delta+\varrho)\phi^e}{(\delta+\varrho)\phi^e+\delta(1-\phi^e)}$, $g'(\phi^e) = \frac{\delta}{(\delta+\varrho)\phi^e} g^2(\phi^e) > 0$ and $\varrho = \omega^L(\bar{\phi})m(0,\bar{\phi})$ is the endogenous exiting probability of low quality firms. This implies $\phi^{e**} \in (\phi^{e*}, 1)$.

The proof is in the Appendix. At the one hand, increasing quality reduces the mass of firms required to produce a given Y and hence liberates numeraire good for consumption. At the other hand, a reduction in equilibrium Y reduces welfare directly.

5.2.2 Bad and Good News

When the regulator only observes entry, it can use F to improve both quality and production in the case of bad news (since the difference between value functions increase with reputation), but not in the case of good news (since that difference is decreasing with reputation).

More specifically, in the case of bad news, it is possible to find an entry cost such that

$$\frac{V_L(1)}{V_H(1)} = \frac{r}{r+\lambda} = \frac{F^*}{C+F^*}$$

This implies an optimal $F^* = \frac{r}{\lambda}C$ allows the regulator to achieve $\phi^e = 1$. In this case Y_F is determined by the entry condition for the low type. The first best and the market solution are identically defined as above,

$$\bar{Y} = \left[\frac{a_1 - a_0}{C}\right]^{\frac{1}{\eta}} > Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}} > Y_M = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}}$$

Contrarily, under good news entry costs can only decrease production. More specifically,

$$\bar{Y} = \left[\frac{a_1 - a_0}{C}\right]^{\frac{1}{\eta}} > Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}}$$

Even when, in absence of entry costs, the market achieves the same production than in the bad news case with entry costs, the numeraire good for consumption is lower because the average quality of intermediate goods firms is lower. Hence, when the government only observes entry it can achieves a higher welfare when signals are characterized by bad news than when signals are based on good news.

5.2.3 The regulator also observes firm's age

Now we assume the government can also observe firms' age and then it can impose taxes and/or subsidies to continuing firms. We show the regulator can impose in expectation higher entry costs to low quality firms than to high quality firms by front loading costs and back loading subsidies. For example, charging entry costs and paying a large subsidy if the firm achieves a certain age. This subsidy is higher in expectation to high quality firms, which only exit exogenously at a rate δ than to low quality firms, which also exit endogenously when their reputation reaches $\bar{\phi}$, what happens with a strictly positive probability. Given any entry cost F and subtracting the expected present value of subsidies, there is a gap between the effective entry costs, such that $F_H < F_L$. Formally, entry conditions are

$$\begin{aligned} V_H(\phi^e)Y^{-\eta} &= C + F_H \\ V_L(\phi^e)Y^{-\eta} &= F_L \end{aligned} \tag{36}$$

Since the maximum production is still obtained by condition 34, the regulator optimally chooses a combination of $F_H < F_L$ to achieve the same ϕ^e as in the case in which only entry is observed. In this case production is given by

$$Y_C = \left[\frac{V_H(\phi^e) - V_L(\phi^e)}{C + F_H - F_L}\right]^{\frac{1}{\eta}} > Y_F \tag{37}$$

Starting from a common entry cost F, a higher welfare is achievable by imposing

$F_H < F_L < F$ to reach the same ϕ^e (maintaining the ratio) for a certain C. The obvious result is that the more information a government can access, the more it can approach to the full information benchmark, achieving it having the same information the market has but additional access to a commitment technology.

6 Numerical Exercises

In this section we provide a numerical illustration of value functions and optimal regulation under different types of news (Brownian motion, bad news and good news), and different information available to the government.

6.1 Brownian Motion

We assume the following parameters $r = \hat{r} + \delta = 0.1$, $\sigma = 0.2$, $\mu_H - \mu_L = 0.2$ (hence $\rho = 0.1$), $\alpha_H = 0.9$, $\alpha_L = 0.7$ and $\kappa = 3$ (hence $a_1 = 0.8$ and $a_0 = 0.2$). We also assume $C = 1$ and $\eta = 0.5$.

The first best is characterized by $\phi^e = 1$ and

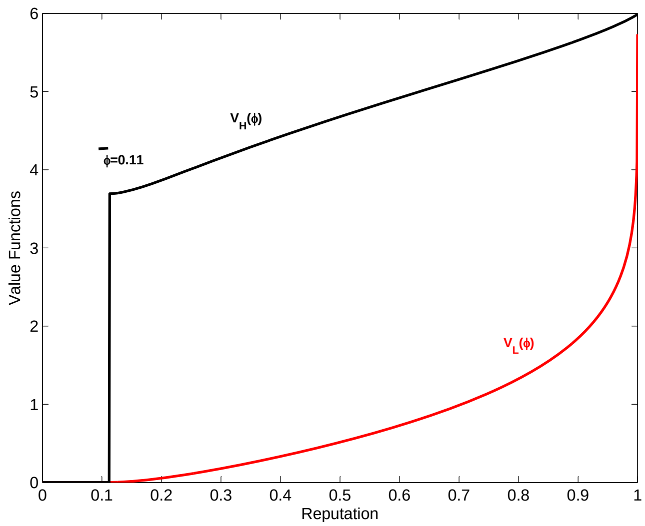
$$\bar{Y} = \left[\frac{a_1 - a_0}{rC}\right]^{\frac{1}{\eta}} = 6^2 = 36$$

The market result is characterized by the value functions in Figure 1. In this case $\phi^e = \bar{\phi} = 0.11$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{rC}\right]^{\frac{1}{\eta}} = 3.7^2 = 14$$

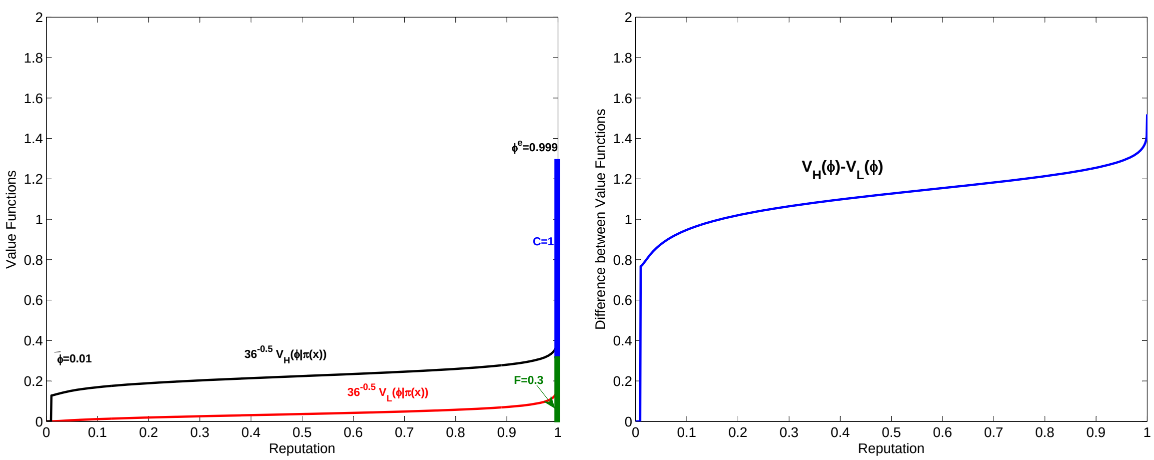
First we assume the government observes revenues, and then it can infer each firm's reputation. We also assume the government can impose entry costs F and commit to a scheme of payments as a function of reputation. The government can reach the first best $\bar{Y} = 36$, almost all entering firms investing (here we target $\phi^e = 0.999$) and in expectation almost no firm exiting (here we target $\bar{\phi} = 0.01$). Optimal regulation in this case is characterized by payments $\pi(x) = c - d(x - x^e)$, where $x = log\left(\frac{1-\phi}{\phi}\right)$, $x^e = log\left(\frac{1-0.999}{0.999}\right)$, $c = 0.16$ and $d = 0.02$. This payment scheme implies that firms are

Figure 1: Market Value Functions



taxed until they reach a reputation $\phi = 1 - 1^{e-15}$, above which they are subsidized. The entry costs needed to achieve this allocation are $F = 0.29$.

Figure 2: Value Functions under Optimal Regulation

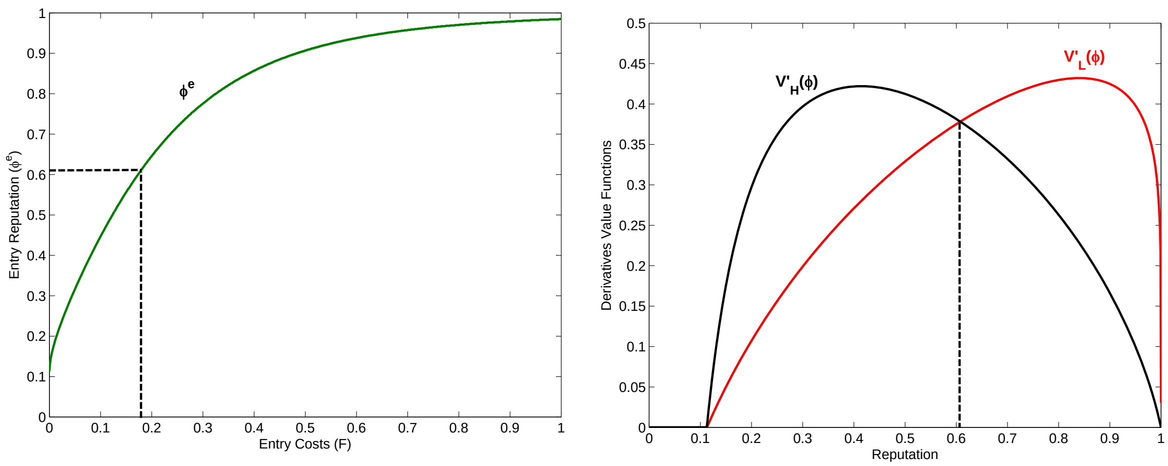


This policy is able to implement the full information benchmark because its payments schedule generates value functions that do not approach each other for high reputation levels (as shown in Figure 2), allowing firms to compensate investment without relying in a production decline.

Now assume the government can only observe entry of intermediate goods firms. By increasing F, the regulator can both increase quality and production. The first panel of Figure 3 shows how ϕ^e increases with F (coming from an increasing ratio $\frac{V_L(\phi)}{V_H(\phi)}$). Entry costs F^* are determined to achieve the point at which the derivatives of value functions are the same (condition 34). The second panel of Figure 3 also shows these derivatives. Production is maximized by imposing $F^* = 0.18$ (this is, 18% of the cost to become a good firm) to reach $\phi^e = 0.61$. In this case,

$$Y_F = \left[\frac{V_H(0.61) - V_L(0.61)}{C}\right]^{\frac{1}{\eta}} = 4.2^2 = 18$$

Figure 3: Optimal Entry Costs F



Now assume the government can also observe continuation. There is always a combination of entry costs and subsidy schemes as functions of age to reduce the effective entry cost for high quality firms to $F_H = 0$. Since the optimal reputation entry remains at $\phi^e = 0.61$, then this combination should be also determined to generate an effective entry cost to low type firms F_L such that the ratio of value functions $\frac{V_L(0.61)}{V_H(0.61)} = 0.15$ is implemented. This is given by $F_L = \frac{V_L(0.61)}{V_H(0.61)}C = 0.15$. This policy implies the consumption of the aggregate good increases even further to,

$$Y_C = \left[\frac{V_H(\phi^e) - V_L(\phi^e)}{C + F_H - F_L}\right]^{\frac{1}{\eta}} = \left(\frac{4.2}{0.85}\right)^2 = 24$$

6.2 Bad News

Here we show the optimal regulation when the signals are characterized by bad news. We assume the same technology and preferences parameters as above and $\lambda = 0.1$ for the signal structure. The first best is also the same, characterized by $\phi^e = 1$ and $\bar{Y} = 0.36$. The market result is characterized by the value functions in Figure 4, $\phi^e = \bar{\phi} = 0.16$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} = 0.9^2 = 0.8$$

Assume the government can observe the reputation of the firm. It can distort payments such that $F = \frac{r}{\lambda}C = 1$ (to achieve $\phi^e = 1$) and $f_B = \frac{r}{\lambda}(a_1 - a_0) = 0.6$ per moment, such that,

$$\bar{Y} = \left[\frac{\frac{\lambda}{r+\lambda}\left(1 + \frac{r}{\lambda}(a_1 - a_0)\right)}{C}\right]^{\frac{1}{\eta}} = 6^2 = 36$$

Assume the government can only observe entry. As discussed, $F = \frac{r}{\lambda}C = 1$, such that $\phi^e = 1$ and hence

$$Y_F = \left[\frac{\frac{\lambda}{r+\lambda}(a_1 - a_0)}{C}\right]^{\frac{1}{\eta}} = 3^2 = 9$$

6.3 Good News

We assume the same parameters as in the bad news case. In this case the market result is characterized by the value functions in Figure 5, $\phi^e = \bar{\phi} = 0.25$ and

$$Y_M = \left[\frac{V_H(\bar{\phi})}{C}\right]^{\frac{1}{\eta}} = 3^2 = 9$$

Assume the government can observe the reputation of the firm. It can distort payments to obtain an allocation arbitrarily close to the first best. We assume a target of $\hat{\phi}^e = 0.99$ and $\bar{Y}$. Evaluated at $\bar{Y}$, the entry condition for low quality firms determines $F = \frac{1}{6}5.6 = 0.93$. The reward for generating good signals f_G that implements $\hat{\phi}^e = 0.99$ at $F = 0.93$ is $f_G = 17.4$.

When the signal structure is characterized by good news, the difference in value functions is always decreasing with reputation. In this case entry costs always decrease production. Then $F^* = 0$. Even when production is the same as under bad news with entry costs, welfare is lower because quality is lower and then it is more costly to achieve the same production in terms of the numeraire good.

Figure 4: Value Functions - Bad News

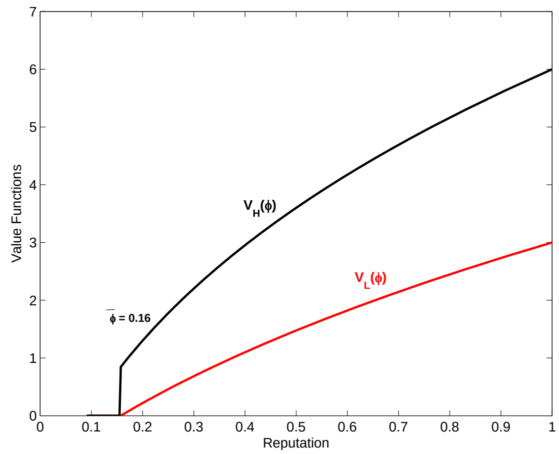
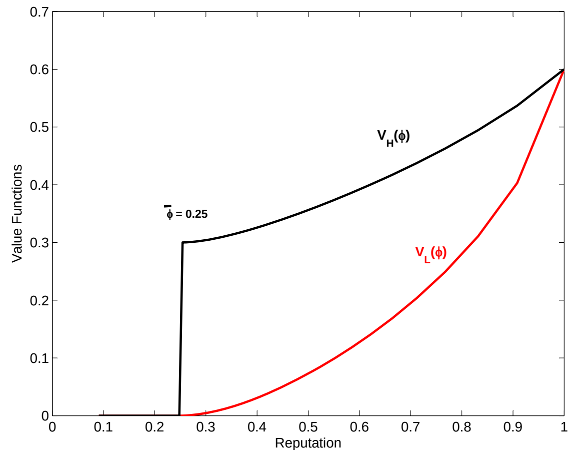


Figure 5: Value Functions-Good News



7 Conclusions

We argue there may be room for regulation in markets with fully functioning reputation incentives but limited commitment. In this case we show learning is under-exploited by the market in providing incentives. If the government has access to the same information as the market, but additionally it has access to a commitment technology, it can construct a scheme of payments based on reputation that foster reputation incentives for firms to invest in quality and achieve the unconstrained first best. Moreover, if the government has less information than the market, still can achieve efficiency by setting positive entry costs.

Entry costs are typically criticized for reducing production and the market size. The main logic is clearly exposed in Hopenhayn (1992): Higher entry costs should be compensated by higher prices, and from entry conditions this can be achieved by less total output. This argument has been widely used by the economic literature - from supporting trade liberalization to explaining TFP differences across countries - and by international organisms in proposing policy reforms to underdeveloped countries. Still, as shown by Djankov et al. (2002), there is a heavy regulation of entry of start up firms around the world, under the main justification of discouraging the entry of bad firms. In this paper we provide a unifying framework to study the trade off that entry costs create between production and quality. Interestingly we show there is a range of entry costs that increase both quality and total output.

From a technical viewpoint we contribute in providing analytical solutions in continuous time for a model of reputation with free entry and exit of firms that know their type and initial decisions. This explicit solution allows a complete welfare comparison across different regulation policies.

An important next step in understanding optimal regulation in the presence of reputation concerns is considering moral hazard problems in each period. This will extend the set of policies the government can use to maximize the incentives that learning provides to markets. This paper suggests moral hazard may not be a problem in itself when there is learning, but a lack of commitment to take full advantage from reputation concerns.

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A Appendix

A.1 Proof Proposition 1

The activities of the two types of firms induce two different probability measures over the paths of the signal S_t. Reputation evolves following the equation. Fix a prior ϕ^e.

$$\phi_t = \frac{\phi^e Pr(S_t|H)}{\phi^e Pr(S_t|H) + (1-\phi^e)Pr(S_t|L)}$$

or

$$\phi_t = \frac{\phi^e \xi_t}{\phi^e \xi_t + (1-\phi^e)} \tag{38}$$

where ξ_t is the ratio between the likelihood that a path $S_s : s \in [0,t]$ arises from type H and the likelihood that it arises from type L. As in Faingold and Sannikov (2007), from Girsanov's Theorem, this ratio follows a Brownian motion characterized by $\mu_\xi = 0$ and $\sigma_\xi = \xi_t \zeta$,

$$d\xi_t = \xi_t \zeta dZ_s^L \tag{39}$$

where $\zeta = \frac{\mu_H - \mu_L}{\sigma}$ and $dZ_s^L = \frac{dS_t - \mu_L dt}{\sigma}$ is a Brownian motion under the probability measure generated by type L.[5]

By Ito's formula,

$$d\phi = [\phi' \mu_\xi + \frac{1}{2}\phi'' \sigma_\xi^2]dt + \phi' \sigma_\xi dZ_s^L$$

$$d\phi_t = -\frac{1}{2}\frac{2\phi^{e2}(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^3}\xi_t^2 \zeta^2 dt + \frac{\phi^e(1-\phi^e)}{(\phi^e \xi_t + (1-\phi^e))^2}\xi_t \zeta dZ_s^L$$

and from equation (38) we can express it in terms of ϕ_t rather than ϕ^e

$$d\phi_t = -\phi_t^2(1-\phi_t)\zeta^2 dt + \phi_t(1-\phi_t)\zeta dZ_s^L$$

$$d\phi_t = \phi_t(1-\phi_t)\zeta[dZ_s^L - \phi_t \zeta dt]$$

replacing by the definition of dZ_s^L,

$$d\phi_t = \lambda(\phi_t) dZ_t^\phi \tag{40}$$

where $\lambda(\phi_t) = \phi_t(1-\phi_t)\zeta$ and $dZ_t^\phi = \frac{1}{\sigma}[dS_t - (\phi_t \mu_H + (1-\phi_t)\mu_L)dt]$.

Conversely, suppose that ϕ_t is a process that solves equation (40). Define ξ_t using

[5] It is also possible to solve the problem defining $\xi_t = \frac{Pr(S_t|L)}{Pr(S_t|H)}$ such that $\phi_t = \frac{\phi^e}{\phi^e + (1-\phi^e)\xi_t}$, where $d\xi_t = \xi_t \zeta dZ_s^H$

equation (38),

$$d\xi_t = -\frac{1-\phi^e}{\phi^e}\frac{\phi_t}{1-\phi_t}$$

By applying Ito's formula again, ξ_t satisfies equation (39). This implies ξ_t is the ratio between the likelihood that a path $S_s : s \in [0, t]$ arises from type H and the likelihood it arises from type L. Hence ϕ_t is determined by Bayes rule.

Finally, consider that, for different types will have different paths, that in expectation will move their reputation. Replacing dS_t^i in dZ_t^ϕ in equation (40) for the two different types of firms, deliver equations (16) and (17).

A.2 Ordinary Differential Equations when signals follow a Brownian motion

Here we obtain the differential equation that characterizes the continuation values for each type.

Proposition 6 *Define Ψ the space of progressively measurable processes ψ_t for all $t \geq 0$ with $E[\int_0^T \psi_t^2 dt] < \infty$ for all $0 < T < \infty$. A bounded process W_t^i for all $t \geq 0$ is the continuation value for type $i = \{H, L\}$ if and only if, for some process ψ_t^i in Ψ we have,*

$$dW_t^i = [rW_t^i - r\pi(\phi_t)]dt + \psi_t^i dZ_t \tag{41}$$

Proof The flow continuation value W_t^i for type i is the expected payoff at time t,

$$W_t^i = rE_t^i\left[\int_t^\infty e^{-r(s-t)}\pi(\phi_s)ds\right]$$

Denote U_t^i the discounted sum of payoffs for type i conditional on the public information available at time t,

$$U_t^i = rE_t^i\left[\int_0^\infty e^{-rs}\pi(\phi)ds\right] = \int_0^t e^{-rs}r\pi(\phi_s)ds + W_t^i \tag{42}$$

Since U_t^i is a martingale, by the Martingale Representation Theorem, there exist a process ψ_t^i in Ψ such that,

$$dU_t^i = e^{-rt}\psi_t^i dZ_t \tag{43}$$

Differentiating (42) with respect to time

$$dU_t^i = e^{-rt}r\pi(\phi_t)dt - re^{-rt}W_t^i dt + e^{-rt}dW_t^i \tag{44}$$

Combining (43) and (44), we can obtain (41). Q.E.D.

In a Markovian equilibrium, we know $W_t^i = V_i(\phi_t)$. Since this continuation value depends on the reputation, which follows a Brownian motion, using Ito's Lemma,

$$dV_i(\phi) = \left[\mu_{i,\phi} V_i'(\phi) + \frac{1}{2}\sigma_\phi^2 V_i''(\phi)\right] dt + \sigma_\phi V_i'(\phi) dZ \tag{45}$$

where $\mu_{H,\phi} = \frac{\lambda^2(\phi)}{\phi}$, $\mu_{L,\phi} = -\frac{\lambda^2(\phi)}{(1-\phi)}$ and $\sigma_\phi = \lambda(\phi)$ from Proposition 1.

Matching drifts of equations (41) and (45) for each type i, yields the linear second order differential equation that characterizes continuation values $V_H(\phi)$ and $V_L(\phi)$,

$$\frac{1}{2}\lambda^2(\phi)V_L''(\phi) - \frac{\lambda^2(\phi)}{(1-\phi)}V_L'(\phi) - rV_L(\phi) + \pi(\phi) = 0 \tag{46}$$

and

$$\frac{1}{2}\lambda^2(\phi)V_H''(\phi) + \frac{\lambda^2(\phi)}{\phi}V_H'(\phi) - rV_H(\phi) + \pi(\phi) = 0 \tag{47}$$

A.3 Value functions when signals follow a Brownian motion

A.3.1 Solving the ODE's

The second order differential equations can be rewritten as

$$\begin{aligned} r\rho V_L(\phi) &= \rho\pi(\phi) - \phi^2(1-\phi)V_L'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_L''(\phi) \\ r\rho V_H(\phi) &= \rho\pi(\phi) + \phi(1-\phi)^2 V_H'(\phi) + \frac{1}{2}\phi^2(1-\phi)^2 V_H''(\phi) \end{aligned}$$

where

$$\rho = \frac{\sigma^2}{(\mu_H - \mu_L)^2}$$

Changing variables to $l = (1-\phi)/\phi$ these ODE's can be written as

$$\begin{aligned} r\rho V_L(l) &= \rho\pi(l) + lV_L'(l) + \frac{1}{2}l^2 V_L''(l) \\ r\rho V_H(l) &= \rho\pi(l) + \frac{1}{2}l^2 V_H''(l) \end{aligned}$$

a) Solving for $V_L(l)$: We conjecture a solution of the form:

$$V_L(l) = K\left[l^{-\gamma}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - l^{\gamma-1}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

for some parameters γ and K. With this, we have

$$V_L'(l) = K\left[(-\gamma)\,l^{-\gamma-1}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)\,l^{\gamma-2}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right]$$

$$\begin{aligned} V_L''(l) &= K\left[(-\gamma)(-\gamma-1)\,l^{-\gamma-2}\int_{\chi_1}^{l} l'^{\gamma}\frac{\pi(l')}{l'}dl' - (\gamma-1)(\gamma-2)\,l^{\gamma-3}\int_{\chi_2}^{l} l'^{1-\gamma}\frac{\pi(l')}{l'}dl'\right] \\ & K(1-2\gamma)\frac{\pi(l)}{l^2} \end{aligned}$$

$$lV_L'(l) + \frac{1}{2}l^2V_L''(l) = \frac{\gamma(\gamma-1)}{2}V_L(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)$$

which solves the ODE when $2r\rho = \gamma(\gamma-1)$ and $K(1-2\gamma) = -2\rho$, or

$$\gamma = \frac{1}{2} + \sqrt{\frac{1}{4} + 2r\rho} \qquad \text{and} \qquad K = \frac{\rho}{\sqrt{\frac{1}{4} + 2r\rho}}$$

Recall $\gamma(\rho) : [0,\infty] \to [1,\infty]$ and $K(\rho) > 0$. The parameters χ_1 and χ_2 will be determined later from boundary conditions.

b) Solving for $V_H(l)$: Define: $\Delta_H(l) = \pi(0) - V_H(l)$, $\bar{\pi}(l) = \pi(0) - \pi(l)$. Notice $\bar{\pi}(l)$ is increasing in l.

Rewriting the ODE for the high type as

$$\rho\Delta_H(l) = \rho\bar{\pi}(l) + \frac{1}{2}l^2\Delta_H''(l)$$

Proceeding as above we conjecture a solution of the form:

$$\Delta_H(l) = K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right]$$

for the same parameters γ and K defined previously. With this, we have

$$
\begin{aligned}
\Delta_H'(l) &= K\left[(1-\gamma)l^{-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma l^{\gamma-1}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
\Delta_H''(l) &= K\left[-\gamma(1-\gamma)l^{-\gamma-1}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + \gamma(\gamma-1)l^{\gamma-2}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\quad + K(1-2\gamma)\frac{\bar{\pi}(l)}{l^2} \\
\frac{1}{2}l^2\Delta_H''(l) &= \frac{\gamma(\gamma-1)}{2}\Delta_H(l) + K\left(\frac{1-2\gamma}{2}\right)\pi(l)
\end{aligned}
$$

that fulfill the ODE by construction with the parameters γ and K defined above. The parameters ψ_1 and ψ_2 will be determined later also from boundary conditions.

A.3.2 Dealing with the boundary conditions at $l = 0$

Notice that we need $\lim_{l\to 0} V_L(l) = \lim_{l\to 0}\pi(l) = \pi(0)$, and $\lim_{l\to 0}\Delta_H(l) = \lim_{l\to 0}\bar{\pi}(l) = \lim_{l\to 0}\pi(l) - \pi(0) = 0$. The two limiting properties hold if and only if $\chi_1 = 0$ and $\psi_1 = 0$ (we then relabel $\chi_2 = \chi$ and $\psi_2 = \psi$).

We will proceed with the proof for the high type. The proof for the low type is related. Using Lipschitz continuity of $\bar{\pi}(l)$, assuming $\bar{\pi}(l) \leq \Lambda l$, and $\psi_2 \leq \infty$:

$$
\begin{aligned}
\Delta_H(l) &= K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}\frac{\bar{\pi}(l')}{l'}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}\frac{\bar{\pi}(l')}{l'}dl'\right] \\
&\leq \Lambda K\left[l^{1-\gamma}\int_{\psi_1}^{l} l'^{\gamma-1}dl' + l^{\gamma}\int_{l}^{\psi_2} l'^{-\gamma}dl'\right] \\
&= \Lambda K\left[l^{1-\gamma}\left(\frac{l^\gamma}{\gamma} - \frac{\psi_1^\gamma}{\gamma}\right) + l^\gamma\left(\frac{\psi_2^{1-\gamma}}{1-\gamma} - \frac{l^{1-\gamma}}{1-\gamma}\right)\right] \\
&= \Lambda K\left[l\left(\frac{1}{\gamma} - \frac{1}{1-\gamma}\right)\right] \\
&= \Lambda l
\end{aligned}
$$

if and only if $\psi_1 = 0$ and assuming $\psi_2 = \infty$. Hence, $\lim_{l\to 0}\Delta_H(l) = 0$ if and only if $\psi_1 = 0$. A similar analysis delivers $\lim_{l\to 0} V_L(l) = \pi(0)$ if and only if $\chi_1 = 0$

A.3.3 Simplifying Value Functions

Changing variables inside the integrals: $\theta = l'/l$, so $ld\theta = dl'$ and the limits of integration. We start from obtaining $V_H(l)$.

$$\Delta_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\bar{\pi}(\theta l)\,d\theta + \int_1^{\psi/l} \theta^{-\gamma-1}\bar{\pi}(\theta l)\,d\theta\right\}$$

Since $\bar{\pi}(\theta l) = \pi(0) - \pi(\theta l)$ and $V_H(l) = \pi(0) - \Delta_H(l)$

$$\begin{aligned} V_H(l) &= \pi(0)\left(1 - K\int_0^1 \theta^{\gamma-2}d\theta - K\int_1^{\psi/l}\theta^{-\gamma-1}d\theta\right) \\ &\quad + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\,d\theta\right\} \\ V_H(l) &= \pi(0)\left[\frac{K}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right] + K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta + \int_1^{\psi/l}\theta^{-\gamma-1}\pi(\theta l)\,d\theta\right\} \end{aligned}$$

Hence

$$V_H(l) = K\left\{\int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta - \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\,d\theta + \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}\right\} \tag{48}$$

Similarly, the low type's value function can be written as

$$V_L(l) = K\left\{\int_0^1 \theta^{\gamma-1}\pi(\theta l)\,d\theta - \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\,d\theta\right\} \tag{49}$$

In reduced form

$$V_L(l) = K[B_L(l) - A_L(l)] \quad \text{and} \tag{50}$$

$$V_H(l) = K[B_H(l) - A_H(l)] \tag{51}$$

where

$$B_L(l) = \int_0^1 \theta^{\gamma-1}\pi(\theta l)\,d\theta \qquad \text{and} \qquad A_L(l) = \int_{\chi/l}^1 \theta^{-\gamma}\pi(\theta l)\,d\theta$$

$$B_H(l) = \int_0^1 \theta^{\gamma-2}\pi(\theta l)\,d\theta \qquad \text{and} \qquad A_H(l) = \int_{\psi/l}^1 \theta^{-\gamma-1}\pi(\theta l)\,d\theta - \frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}$$

A.3.4 Derivatives

Taking derivatives of $V_L(l)$ components and multiplying by l,

$$l\frac{\partial A_L(l)}{\partial l}=\int_{\chi/l}^{1}\theta^{-\gamma}\pi'\left(\theta l\right)\theta l d\theta-\left(\frac{\chi}{l}\right)^{-\gamma}\pi(\chi)\left(-\frac{\chi}{l^2}\right)l$$

Integrating the first term by parts

$$\begin{aligned}\int_{\chi/l}^{1}\theta^{1-\gamma}\pi'\left(\theta l\right)l d\theta &= \theta^{1-\gamma}\pi(\theta l)|_{\chi/l}^{1}-\int_{\chi/l}^{1}(1-\gamma)\theta^{-\gamma}\pi(\theta l)d\theta\\ &= \pi(l)-\left(\frac{\chi}{l}\right)^{1-\gamma}\pi(\chi)-(1-\gamma)\int_{\chi/l}^{1}\theta^{-\gamma}\pi(\theta l)d\theta\end{aligned}$$

Then

$$l\frac{\partial A_L(l)}{\partial l}=\pi(l)-(1-\gamma)\int_{\chi/l}^{1}\theta^{-\gamma}\pi(\theta l)d\theta=\pi(l)-(1-\gamma)A_L(l)$$

Similarly

$$\begin{aligned}l\frac{\partial A_H(l)}{\partial l} &= \pi(l)+\gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}\pi(\theta l)d\theta-\frac{\pi(0)}{\gamma}(-\gamma)\left(\frac{\psi}{l}\right)^{-\gamma-1}\left(-\frac{\psi}{l^2}\right)l\\ &= \pi(l)+\gamma\int_{\psi/l}^{1}\theta^{-\gamma-1}\pi(\theta l)d\theta-\gamma\frac{\pi(0)}{\gamma}\left(\frac{\psi}{l}\right)^{-\gamma}=\pi(l)+\gamma A_H(l)\end{aligned}$$

$$l\frac{\partial B_L(l)}{\partial l}=\pi(l)-\gamma\int_{0}^{1}\theta^{\gamma-1}\pi(\theta l)d\theta=\pi(l)-\gamma B_L(l)$$

$$l\frac{\partial B_H(l)}{\partial l}=\pi(l)-(\gamma-1)\int_{0}^{1}\theta^{\gamma-2}\pi(\theta l)d\theta=\pi(l)-(\gamma-1)B_H(l)$$

The derivatives can then be simplified as follows,

$$lV_L'(l) = K[-\gamma B_L(l)+(1-\gamma)A_L(l)] \quad \text{and} \tag{52}$$

$$lV_H'(l) = K[(1-\gamma)B_H(l)-\gamma A_H(l)] \tag{53}$$

A.4 Proof of Proposition 3

From entry conditions 33,

$$\frac{V_L(\phi^e)}{V_H(\phi^e)}=\frac{F}{C+F}, \tag{54}$$

which is independent of Y. Then, to prove the Proposition it is enough to prove the ratio $\frac{V_L(\phi^e)}{V_H(\phi^e)}$ is an increasing function of ϕ^e, or which is the same $\frac{V_L(l_0)}{V_H(l_0)}$ is an decreasing function of l_0, that maps from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

First, we define the domain and image of the function. The lower reputation in the market at moment 0 is $\bar{l}$, where $V_L(\bar{l}) = 0$ and $V_H(\bar{l}) = 0 > 0$. We also know that $V_L(1) = V_H(1) > 0$. Finally, $0 < V_L(l) < V_H(l)$ for all other $l_0 \in [0, \bar{l})$. This implies $\frac{V_L(l_0)}{V_H(l_0)}$ is a mapping from $l_0 = [0, \bar{l}]$ to $[1, 0]$.

We show the ratio $\frac{V_L(l)}{V_H(l)}$ is monotonically decreasing in $l \in [0, \bar{l}]$. This is the case if

$$\frac{lV_L'(l)}{V_L(l)} < \frac{lV_H'(l)}{V_H(l)}$$

$$\frac{-\gamma B_L(l) - (\gamma - 1)A_L(l)}{B_L(l) - A_L(l)} < \frac{-(\gamma - 1)B_H(l) - \gamma A_H(l)}{B_H(l) - A_H(l)}$$

$$\frac{B_L(l) - A_L(l)}{B_H(l) - A_H(l)} < \frac{\gamma B_L(l) + (\gamma - 1)A_L(l)}{(\gamma - 1)B_H(l) + \gamma A_H(l)}$$

After some algebra, dropping the argument l, this condition implies,

$$B_H\left[(B_L - A_L) + (2\gamma - 1)A_L\right] > A_H\left[2\gamma(B_L - A_L) + (2\gamma - 1)A_L\right]$$

or

$$B_H\left[\left(1 - \gamma\frac{A_H}{B_H}\right)(B_L - A_L) + (2\gamma - 1)A_L\right] > A_H\left[\gamma(B_L - A_L) + (2\gamma - 1)A_L\right] \quad (55)$$

We show the left hand side of (55) is positive and the right hand side of (55) is negative for all $l \in [0, \bar{l}]$, hence the condition is always satisfied and the ratio of value functions decreasing in that range.

1. $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

   First, we develop the integrals $B_L(l)$ and $B_H(l)$.
   Recall the profit function is linear in ϕ, ($y(\phi) = a_1\phi - a_0$) and $\phi = \frac{1}{1+l}$, For $Y = 1$,

   $$\pi(\theta l) = \frac{a_1}{1 + \theta l} - a_0$$

   and consider the general solution to the following integral (see Abramowitz and Stegun (1972)),

$$\int \theta^m \left(\frac{a_1}{1+\theta l} - a_0\right) d\theta = a_1 \theta^{m+1} \Phi(-\theta l, 1, m+1) - \frac{\theta^{m+1}}{m+1} a_0$$

where $\Phi(-\theta l, 1, m+1)$ is a Hurwitz Lerch zeta-function.

Applying this result to B_L,

$$B_L(l) = \int_0^1 \theta^{\gamma-1} \left(\frac{a_1}{1+\theta l} - a_0\right) d\theta = \left[a_1 \theta^{\gamma} \Phi(-\theta l, 1, \gamma) - \frac{\theta^{\gamma}}{\gamma} a_0\right]_0^1$$

$$B_L(l) = a_1 \Phi(-l, 1, \gamma) - \frac{a_0}{\gamma}$$

and similarly,

$$B_H = a_1 \Phi(-l, 1, \gamma - 1) - \frac{a_0}{\gamma - 1}$$

Our strategy is to prove first $B_L(l) > 0$ for all $l \in [0, \bar{l}]$ an then to prove $B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$.

Important properties of Herwitz Lerch zeta functions for the parameters we are considering ($\gamma \in [1, 2]$) are (see Laurincikas and Garunkstis (2003)):

- $\Phi(0, 1, \gamma) = \frac{1}{\gamma}$
- $\frac{\partial \Phi(-l,1,\gamma)}{\partial l} = \frac{1}{l}\left[\frac{1}{l+1} - \gamma \Phi(-l, 1, \gamma)\right] < 0$
- $(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \gamma \Phi(-l, 1, \gamma)$

By construction, $B_L(\bar{l}) = 0$, hence $\Phi(\bar{l}, 1, \gamma) = \frac{a_0}{\gamma a_1}$. Given the properties above

$$B_L(l) : [0, \bar{l}] \rightarrow [\frac{a_1 - a_0}{\gamma}, 0]$$

Furthermore, $B_L(l)$ is monotonically decreasing in the range

$B_H(l) > B_L(l)$ for all $l \in [0, \bar{l}]$ if

$$\gamma(\gamma - 1)[\Phi(-l, 1, \gamma - 1) - \Phi(-l, 1, \gamma)] > \frac{a_0}{a_1}$$

Considering the third property above,

$$(\gamma - 1)\Phi(-l, 1, \gamma - 1) > \Phi(-l, 1, \gamma) + (\gamma - 1)\Phi(-l, 1, \gamma) > \frac{a_0}{\gamma a_1} + (\gamma - 1)\Phi(-l, 1, \gamma)$$

and hence, $B_H(l) > 0$ for all $l \in [0, \bar{l}]$

2. $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

We develop the integral $A_L(l)$ and $A_H(l)$ following the steps above.

$$A_L(l) = \int_{\chi/l}^{1} \theta^{-\gamma} \left(\frac{a_1}{1+\theta l} - a_0 \right) d\theta = \left[a_1 \theta^{1-\gamma} \Phi(-\theta l, 1, 1-\gamma) - \frac{\theta^{1-\gamma}}{1-\gamma} a_0 \right]_{\chi/l}^{1}$$

$$A_L(l) = a_1 \left[\Phi(-l, 1, 1-\gamma) - (\chi/l)^{1-\gamma} \Phi(-\chi, 1, 1-\gamma) \right] + \frac{a_0}{\gamma - 1} \left(1 - (\chi/l)^{1-\gamma} \right)$$

and,

$$A_H(l) = a_1 \left[\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma) \right] + \frac{a_0}{\gamma} - \frac{a_1}{\gamma} (\psi/l)^{-\gamma}$$

Consider $A_H(0) = A_H(\psi) = -\frac{a_1 - a_0}{\gamma} < 0$. We show that, if the function grows, the maximum is still negative. This is, we prove that $A_H(\hat{l}) < 0$ where $\hat{l} = argmax A_H(l)$ (hence $\frac{\partial A_H(l)}{\partial l}|_{l=\hat{l}} = 0$).

$$\frac{\partial A_H(l)}{\partial l} = \frac{a_1}{l} \left[\left(\frac{1}{1+l} + \gamma \Phi(-l, 1, -\gamma) \right) - \gamma (l/\psi)^{\gamma} \Phi(-\psi, 1, -\gamma) \right] - \frac{a_1}{l} (l/\psi)^{\gamma}$$

The condition satisfied at $l \frac{\partial A_H(l)}{\partial l} = 0$ is,

$$\left[\Phi(-l, 1, -\gamma) - (\psi/l)^{-\gamma} \Phi(-\psi, 1, -\gamma) \right] = \frac{1}{\gamma} (l/\psi)^{\gamma} - \frac{1}{1+l}$$

Evaluating $A_H(\hat{l})$ considering that condition,

$$a_1 \left[\frac{1}{\gamma} (l/\psi)^{\gamma} - \frac{1}{1+l} \right] + \frac{a_0}{\gamma - 1} \left(1 - (\chi/l)^{1-\gamma} \right) < 0$$

since

$$\gamma a_1 \frac{1}{1+l} > a_0$$

Hence, $A_H(l) < 0$ for all $l \in [0, \bar{l}]$

Finally, just for completeness, $A_L(0) = -\frac{a_1 - a_0}{\gamma - 1} < 0$, $A_L(\chi) = 0$ because $\chi/l = 1$ and $A_L(\bar{l}) = 0$ by construction. It can be further shown that $A_L(l) < 0$ for all $l \in (0, \chi)$ and $A_L(l) > 0$ for all $l \in (\chi, \bar{l})$.

3. $\gamma(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l) > 0$ for all $l \in [0, \bar{l}]$

   Recall $\gamma(B_L - A_L) + (2\gamma - 1)A_L = \gamma B_L + (\gamma - 1)A_L = -\frac{lV_L'(l)}{K}$.

   By construction $\gamma B_L + (\gamma - 1)A_L = 0$ at $l = 0$ and $l = \bar{l}$.

   For $l \in (\chi, \bar{l})$, since $A_L(l) \geq 0$ and $B_L(l) > 0$, $\gamma B_L + (\gamma - 1)A_L > 0$. In particular, at $\bar{l}$, $A_L(\chi) = 0$ and $\gamma B_L(\chi) > 0$.

   As shown above, for $l \in [0, \chi]$, $B_L(l) > 0$ monotonically increasing and $A_L(l) < 0$ monotonically increasing. This implies $\gamma B_L + (\gamma - 1)A_L$ goes monotonically from 0 at $l = 0$ to $\gamma B_L(\bar{l}) > 0$, and hence positive in the whole range.

4. $\left[\left(1 - \gamma \frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$

   First, recall $(\gamma - 1)B_H + \gamma A_H = -\frac{lV_H'(l)}{K}$. Hence, as in the point above, $(\gamma - 1)B_H + \gamma A_H = 0$ at $l = 0$ and $l = \bar{l}$ by construction, which we can rewrite as $1 - \gamma\frac{A_H(0)}{B_H(0)} = 1 - \gamma\frac{A_H(\bar{l})}{B_H(\bar{l})} = \gamma$. Hence at these two extreme points, the term in the left hand side is 0, the same as the one in the right hand side.

   More generally $(\gamma - 1)B_H + \gamma A_H > 0$ (and then $1 < 1 - \gamma\frac{A_H(l)}{B_H(l)} < \gamma$). Since $A_L(\chi) = 0$, $\left(1 - \gamma\frac{A_H(l)}{B_H(l)}\right) B_L(l) > 0$. By the same monotonicity arguments above, $\left[\left(1 - \gamma \frac{A_H(l)}{B_H(l)}\right)(B_L(l) - A_L(l)) + (2\gamma - 1)A_L(l)\right] > 0$ for all $l \in [0, \bar{l}]$.

A.5 Proof of Proposition 5

First, we describe the stationary distribution of firms, this is the steady state mass of high quality firms $m(1)$ and of low quality firms $m(0)$. Firms H leave the market with probability δ. Firms L leave the market with probability $\delta + \varrho$, where $\varrho = \omega^L(\bar{\phi})m(0, \bar{\phi})$ is the endogenous exiting probability. For example, under bad and good news $\varrho = \lambda$.

From the evolution of firms H and L,

$$\begin{aligned}(\delta + \varrho)(m - m(1)) + (1 - \phi^e)m^e &= 0 \\ \delta m(1) + \phi^e m^e &= 0\end{aligned}$$

Then

$$\begin{aligned}m(1) &= \frac{(\delta + \varrho)\phi^e}{(\delta + \varrho)\phi^e + \delta(1 - \phi^e)} m = g(\phi^e)m \\ m^e &= \frac{\delta}{\phi^e} m(1)\end{aligned}$$

Since $Y(\phi^e, m) = (a_1 g(\phi^e) - a_0)m$

$$\max_{m,\phi^e} \frac{1}{1-\eta} Y(\phi^e, m)^{1-\eta} + 1 - C\phi^e m^e + \varphi[\Pi_H(\phi^e) - \Pi_L(\phi^e) - C]$$

First order conditions are

$$\begin{aligned}
Y^{-\eta} a_1 g'(\phi^e) m - C\delta g'(\phi^e) m + \varphi \left[\frac{\partial \Pi_H(\phi^e)}{\partial \phi^e} - \frac{\partial \Pi_L(\phi^e)}{\partial \phi^e} \right] &= 0 \\
Y^{-\eta}(a_1 g(\phi^e) - a_0) - C\delta g(\phi^e) + \varphi \left[\frac{\partial \Pi_H(\phi^e)}{\partial m} - \frac{\partial \Pi_L(\phi^e)}{\partial m} \right] &= 0 \\
\Pi_H(\phi^e) - \Pi_L(\phi^e) &= C
\end{aligned}$$

where $g'(\phi^e) = \frac{\delta}{(\delta+\varrho)\phi^e} g^2(\phi^e) > 0$

Taking derivatives

$$\begin{aligned}
\frac{\partial \Pi_H(\phi^e)}{\partial \phi^e} - \frac{\partial \Pi_L(\phi^e)}{\partial \phi^e} &= -\eta \frac{a_1 g'(\phi^e)}{a_1 g(\phi^e) - a_0} C + Y^{-\eta}(V'_H - V'_L) \\
\frac{\partial \Pi_H(\phi^e)}{\partial m} - \frac{\partial \Pi_L(\phi^e)}{\partial m} &= -\eta \frac{C}{m} \\
\varphi &= \frac{Y^{1-\eta} - C\delta g(\phi^e) m}{\eta C}
\end{aligned}$$

Evaluate the first order conditions at $V'_H - V'_L = 0$

$$C\delta m g'(\phi^e) \left[\frac{a_1 g(\phi^e)}{a_1 g(\phi^e) - a_0} - 1 \right] > 0$$

This means the quality of entrants that maximizes welfare ϕ^{e**} is greater than the one that maximizes production ϕ^{e*}. From Proposition 3, this implies $F^{**} > F^*$. We can further characterize the optimal quality of entrants to maximize welfare in terms of the difference between value functions.

$$(V'_H(\phi^{e**}) - V'_L(\phi^{e**})) = -\frac{\eta \delta C^{\frac{1}{\eta}} a_0 g'(\phi^{e**}) [V_H(\phi^{e**}) - V_L(\phi^{e**})]^{1-\frac{1}{\eta}}}{[V_H(\phi^{e**}) - V_L(\phi^{e**})]^{\frac{1}{\eta}-1} - \delta g(\phi^{e**})} < 0$$

which implies $\phi^{e**} \in (\phi^{e*}, 1)$. Naturally $F^{**} = F^*$ is optimum only if $a_0 = 0$, $\delta = 0$ or $\eta = 0$.