

Optimal retirement benefit guarantees

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Abstract

Retirement benefit guarantees can ensure a minimum standard of living in retirement. I propose a normative framework to study the optimal design of such guarantees. The model features a standard life-cycle setting, in which individual agents' choices can have negative external effects on public finances, whenever their retirement consumption drops below a minimum level. Within this framework, I show that two alternative forms of intervention are optimal. The first policy mandates that agents use part of their accumulated assets to purchase a claim providing a fixed income stream for the duration of their life. The second policy mandates the purchase of an appropriately structured portfolio insurance policy.

Keywords: Continuous time optimization, Life Cycle savings and portfolio choice, Ricardian Equivalence, Borrowing constraints, Optimal contracts

JELClassification: C6, D6, D9, E2, E6, G1

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1 Introduction

Mostly motivated by the aging of the population, several countries around the world have adopted new approaches in the ways their citizens prepare for retirement. Countries with substantially different economic structures and histories like Australia, Chile, Mexico and Sweden have partially replaced unfunded “pay as you go” retirement systems with funded retirement systems featuring private accounts.¹ Even in many countries where such changes have not occurred (such as the US), there has been intense political debate on proposals to introduce individual accounts into the social security system. Simultaneously, in the private sector, defined contribution plans have increased in popularity and importance as opposed to defined benefit plans.

A common concern, acknowledged even by proponents of these trends, is that such changes imply an increased importance of financial markets, exposing retirement income to market risk. For instance, a downturn in the stock market could create pressures to provide direct or indirect transfers to the affected retirees, increasing distortionary taxes.² Indeed, the recent financial crisis is a reminder of how quickly a downturn in a major market (such as the housing market) can have adverse effects on public finances.

Because of these concerns, it is very common for countries to complement the shift towards private accounts and defined contribution plans with various measures to ensure a minimum standard of living in retirement. Such measures include minimum return guarantees, minimum retirement incomes, phased (as opposed to lump sum) withdrawals upon entering retirement, and mandates to use part of the accumulated balances to purchase a fixed annuity and ensure a minimum defined benefit. For instance, a recent Government Accountability Office report³ investigates such regulations in the UK, Switzerland and the Netherlands and documents that these countries use some combination of the above mea-

¹Mitchell and Lachance (2003) report that more than 20 countries have established individual accounts.

²For instance Shoffner et al. (2005) note in a Social Security Report that a common fear about individuals that have ran out of assets is that “...Such individuals might then qualify for, and as a result place a greater burden on, means-tested antipoverty programs.”

³Bovbjerg (2009)

asures. The idea to use incentives or mandates, so that part of the accumulated balances in defined contribution plans be taken in the form of a defined-benefit annuity, is also the topic of a current policy debate in the US.⁴

The pervasive use of measures to ensure a minimum standard of living in retirement has led to various studies that evaluate the costs and benefits of specific (and sometimes ad-hoc) policy interventions adopted in certain countries.⁵ Less emphasis has been placed on developing an integrated theoretical framework to derive the optimal government policies that would ensure a minimum standard of living in a fully funded retirement system. The present paper takes a first step in that direction by using methods developed in the last two decades in financial economics.

The proposed framework is in the tradition started by Ramsey (1927). A benevolent, rational government aims to maximize social welfare. Agents in the society maximize their individual welfare, which does not coincide with social welfare. The reason for the discrepancy is that an agent's consumption in retirement can have negative, *external* effects. This occurs when retirement consumption drops below a given minimum level and triggers "bailout" transfers financed by distortionary taxes.⁶ To avoid such negative external effects, the government adopts policies to ensure that a retiree's consumption does not fall below the specified minimum level.

The allowed government policies are governmental transfers from and to the agent. They can be chosen subject to two constraints:

The first constraint is informational. The government does not observe the agent's assets or consumption, but it does observe returns on financial markets. This informational

⁴See e.g. "The Obama administration is weighing how the government can encourage workers to turn their savings into guaranteed income streams following a collapse in retiree accounts when the stock market plunged." by Theo Francis in <http://www.bloomberg.com/apps/news?pid=20603037&sid=aHFCE999fWR0>

⁵For some examples see e.g. Feldstein (2005b), Feldstein (2005a), Feldstein and Rangelova (2001), Fuster et al. (2008), Mitchell and Lachance (2003), Constantinides et al. (2002), and the numerous contributions in the special NBER volume edited by Campbell and Feldstein (2001).

⁶Inside the model, such negative and external effects arise, when agents find it optimal to falsely claim that they experienced adverse idiosyncratic shocks, as a result overburdening the welfare system, which is financed by distortionary taxes on workers.

constraint leads to a “hidden action” problem, analogous to the problems considered in the voluminous literature on moral hazard. The fact that the agent cannot observe (and hence cannot dictate) the agent’s consumption, savings and portfolio choices implies that the government’s policies need to induce the agent to choose policies that are consistent with the goal of a minimum standard of living.

The second constraint is a full-financing constraint. The net present value of the transfers provided to the agent should be equal to the present value of the taxes raised by the agent. This second constraint is motivated by current policy discussions in the US, which focus on regulatory provisions for fully-funded private accounts and pensions. The broader issue of the advantages and disadvantages of full funding - as opposed to “pay as you go” - is outside the scope of this paper, and the reader is referred to the large literature that discusses this issue.⁷ A practical implication of the full-financing constraint is that all of the policies considered in the paper can be implemented by having agents purchase appropriate financial products provided by the private sector.

Besides their intuitive and practical appeal, the above two constraints also help to make the theoretical results of the paper most surprising for the following reason: Fundamental results in finance and macroeconomics imply that *in the absence of frictions*, only the net present value of an agent’s resources guides her consumption choices. Hence, if agents can choose their consumption freely, and the transfers received by the agent are financed by herself, then no government intervention can succeed in affecting the agent’s consumption choices. The agent will simply “undo” the effects of the transfers by altering her portfolio and her savings plans. In the literature this insight is known as “Ricardian Equivalence”.⁸

To overcome this hurdle, I assume a friction, which leads to the failure of Ricardian Equivalence. Specifically, I assume that agents cannot borrow against future governmental transfers. (Such constraints can be easily enforced in courts by forbidding securitization of

⁷For some exemplary recent contributions, see e.g., Krueger and Kubler (2006) and Ball and Mankiw (2007) for two alternative views on the issue.

⁸Barro (1974) and Abel (1987) contain a modern treatment of this idea that is originally due to D. Ricardo. Similar results are obtained in Bodie et al. (1992), who show that an agent’s consumption is proportional to an agent’s total wealth, which includes the net present value of all sorts of income etc..

such payments). Because of the resulting borrowing constraint, the government can affect the agent's consumption choices and it becomes possible to discuss "optimal" transfer processes, i.e. transfers that maximize an agent's utility while ensuring that her retirement consumption does not drop below a given level.

The main finding is that there can be multiple optimal governmental interventions. Within the context of the baseline model, one such optimal intervention is to require new retirees to use part of their accumulated assets upon entering retirement to purchase a fixed income stream for the duration of their life, while leaving the rest of the assets at their disposal. The level of that fixed income stream is explicitly derived and shown to be a multiple of the minimum level of consumption that the government is aiming to enforce. Another optimal intervention implied by the model takes the form of "portfolio insurance". The government (or some insurance company) sets certain incentive compatible "guidelines" as to how the consumer is expected to consume, save and allocate her assets. Based on these guidelines, the government infers the agent's asset evolution, and makes transfers once the value of the agent's portfolio threatens to become zero.

A surprising feature of the analysis is that these two policies imply the same welfare for the agent, but in general the initial payment required to finance the first policy is larger than the respective payment for the second policy. I show that this result is driven by the fact that the insights of Ricardian equivalence continue to apply in states of the world where the borrowing constraint is not binding. From a practical perspective, this implies that pricing retirement benefit guarantees as contingent claims (as is routinely done in the literature) may be informative for determining premiums, but may be misleading for welfare comparisons.

Methodologically the paper relates to the finance literature on optimal portfolio choice in the presence of constraints, and in particular to the literature that uses convex duality / dynamic Lagrange multiplier methods to solve such problems.⁹ The typical approach in this literature is to take the income process of the agent as given (see e.g. He and Pages (1993))

⁹The monograph by Karatzas and Shreve (1998) contains a thorough treatment of such methods. A sample of general-equilibrium finance applications of this approach include Cuoco (1997), Basak and Cuoco (1998), and Gallmeyer and Hollifield (2008) amongst many others.

and derive the Lagrange multipliers associated with the borrowing constraint. The new methodological aspect of the present paper is that the convex duality approach is applied in a “backwards” fashion. The government first solves for the best possible Lagrange multiplier process that is associated with the borrowing constraint, and then searches for a transfer process that is associated with these Lagrange multipliers. This new methodology could prove useful in a variety of “hidden-action” setups, where the principal designs compensation schemes that exploit constraints faced by the agent. Finally, the paper relates to a literature in dynamic public finance¹⁰. That literature considers optimal insurance and contract design problems predominantly in setups of “hidden information” and idiosyncratic shocks. The present paper differs from that literature in that it deals with a “hidden action” problem in the presence of aggregate shocks.

The paper is structured as follows. Section 2 sets up the model. Section 3 introduces a government with the task of keeping the agent’s consumption above a minimum level by usage of appropriate taxes and transfers. Section 4 considers the agent’s reaction to the presence of such intervention. Section 5 derives an upper bound to welfare no matter which set of admissible taxes/transfers is utilized. Section 6 illustrates two distinct ways of attaining that upper bound, which are hence optimal. Section 7 discusses pre-retirement implications. Section 8 provides a justification for the assumed negative externality that arises when an agent’s consumption drops below the minimum level. Section 9 discusses the implications of closing the model in general equilibrium. Section 10 concludes. All proofs are in the appendix.

2 The model

2.1 Agents, preferences, and endowments

The baseline model is very similar to the small open economy version of Blanchard (1985) and the life-cycle model of Farhi and Panageas (2007). Accordingly, the investment opportunity

¹⁰See e.g. Cole and Kocherlakota (2001), Golosov and Tsyvinski (2006) amongst many others.

set (interest rate, equity premium etc.) is taken as given. Section 9 shows that the main conclusions of this baseline model remain valid in a closed, general-equilibrium economy.

All agents are identical. The typical agent faces a probability of death q per unit of time dt , and newly born agents also arrive at the same rate. All agents have constant relative risk aversion γ , and a constant discount rate ρ . To expedite the exposition and shorten proofs, I concentrate on the empirically relevant case $\gamma > 1$.¹¹

Life has two phases. A “work” phase and a “retirement phase”. During the work phase agents enjoy a constant level of leisure $l < 1$ and obtain an income stream equal to Y . Once they retire, leisure irreversibly jumps to $l = 1$ and they receive no more labor income.

Agents’ instantaneous utility is given by the standard specification used in the macroeconomics literature, namely¹² $u(c_t, l_t) \equiv l_t^\alpha \frac{c_t^{1-\gamma}}{1-\gamma}$. To ensure that $u_l > 0$, I assume that $\alpha < 0$. Letting τ denote the time of retirement, the agent maximizes

$$F_{tb} = E_{tb} \int_{tb}^{\tau} e^{-(\rho+q)(t-t^b)} \frac{(l^\alpha) (c_t)^{1-\gamma}}{1-\gamma} dt + \int_{\tau}^{\infty} e^{-(\rho+q)(t-t^b)} \frac{(c_t)^{1-\gamma}}{1-\gamma} dt, \quad (1)$$

I allow τ to be either a fixed time or optimally chosen by the agent, as part of her optimization problem. Section 7 discusses the implications of these two alternative assumptions.

2.2 Investment opportunity set

Agents can invest in the money market, where they receive a constant strictly positive interest rate $r > 0$. In addition they can invest in a risky security with a price per share that evolves as

$$\frac{dP_t}{P_t} = \mu dt + \sigma dB_t,$$

¹¹With a few additional technical assumptions the results can be extended to $\gamma < 1$ at the cost of lengthier proofs.

¹²This specification is identical to the utility specification used in Farhi and Panageas (2007) up to a re-definition of the parameters. The reason for the popularity of this specification in macroeconomics is that it leads to a stationary choice of hours in general equilibrium models, consistent with the data.

where $\mu > r$ and $\sigma > 0$ are given constants and B_t is a one-dimensional Brownian motion on a complete probability space (Ω, F, P) .¹³ The realization of this Brownian motion is the only source of uncertainty in this economy. The extension to multiple assets is straightforward and is left out.

As is well understood, dynamic trading in the stock and the bond leads to a dynamically complete market. (See e.g. Duffie (2001) or Karatzas and Shreve (1998)). As Karatzas and Shreve (1998) show, the assumptions of a constant interest rate and risk premium imply the existence of a unique stochastic discount factor H_t , so that the time- t price of any claim that delivers dividends equal to D_u , for $u \geq t$ is given by¹⁴

$$E_t \int_t^\infty \frac{H_u}{H_t} D_u du,$$

and H_t is given by:

$$H(t) \equiv \exp \left\{ - \int_0^t \kappa dB_s - rt - \frac{1}{2} \kappa^2 t \right\}, \quad \kappa \equiv \frac{\mu - r}{\sigma}. \quad (2)$$

The agent can also enter into “annuity-style” contracts with a competitive life insurance company as in Blanchard (1985). Specifically, these contracts specify the following cash-flows: The insurance company offers an income stream of p per unit of time dt , in exchange for receiving one dollar if the agent dies over the next interval dt . Competition between insurance companies implies that $p = q$.

2.3 Portfolio and wealth processes

Throughout life, an agent chooses a portfolio process π_t and a consumption process c_t . The portfolio process π_t is the *dollar amount* invested in the risky asset (the “stock market”) at time t . The rest, $W_t - \pi_t$, is invested in the money market. Since the key insights do not

¹³ $F = \{F_t\}$ denotes the P -augmentation of the filtration generated by B_t .

¹⁴From a macroeconomic perspective, one can also think of H_t as the marginal utility of consumption of the world-representative agent.

depend on the presence of bequest motives, I simplify matters and assume that the agent has no bequest motives. As a result, the agent has an incentive to enter Blanchard-style annuity contracts for the full amount of her financial wealth. This results in an income stream of qW_t per unit of time dt while she is alive. In exchange, the entire remaining wealth of the agent gets transferred to the insurance company when the agent dies. Accordingly, the wealth process of a retired agent evolves as

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt, \quad (3)$$

and the wealth process of a working agent is given by:

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt + Y dt - c_t dt.$$

An additional requirement is that wealth must remain non-negative throughout:

$$W_t \geq 0 \text{ for all } t. \quad (4)$$

This constraint excludes un-collateralized borrowing.

2.4 Externalities when consumption falls below a minimum standard of living

As already mentioned in the introductory section, societies typically opt to introduce regulatory measures to ensure that retirees' consumption does not fall below a minimum standard.

To capture the reasons for such interventions in a simple way and expedite the presentation of the main results, it is easiest to start by assuming that every time an individual's consumption falls below a level ξ in retirement, this drop has a *negative externality* on the rest of society. The source of this externality is revisited in section 8, which shows how this externality can arise endogenously in a society that provides transfers to individuals with insufficient assets through distortionary taxation on other individuals.

Section 8 also shows that in the presence of such external effects, there is a wedge between an individual's and the central planner's objectives, since the central planner wants to ensure that agents choose consumption plans that satisfy

$$c_t \geq \xi \text{ for all } t > t_b + \tau \tag{5}$$

From now and until section 8, I simply assume that the central planner aims to impose constraint (5) on the agents. Based on this assumption, I derive the implications of this constraint for the provision of retirement benefits. For brevity, I will henceforth refer to the central planner as the government.

3 Introducing a role for the government

To achieve the goal of imposing constraint (5) on the agent's choices, the government can use transfers to modify the agent's behavior so that her consumption plans satisfy equation (5).

To make matters realistic, the government's information set is limited. The government can observe an agent's income and the realized returns on the stock market, but not the agent's assets or her consumption.

Based on that information set, the government needs to structure transfers to the individual so as to ensure that constraint (5) holds. To keep with the assumption that the retirement system is fully funded, such transfers are financed by the agent upon entering retirement.

To obtain these optimal transfers it is most useful to use backward induction and split the problem into a "post-retirement" part (which is solved first) and a "pre-retirement" part, which is solved subsequently. In the post-retirement part the government determines the optimal transfer process that maximizes the agent's retirement utility subject to (5), and the appropriate incentive compatibility constraints, assuming that these transfers are financed with a lump sum tax upon entering retirement. This is done in sections 3.1 - 6.

The pre-retirement part is discussed in section 7.

3.1 The post-retirement problem

It is now possible to provide a mathematical formulation to the government's post-retirement problem. Because of the time-invariance of the problem, I henceforth simplify notation and normalize the time of retirement $t^b + \tau$ to be equal to zero. I also normalize the value of the stochastic discount factor at retirement to be equal to $H_0 = 1$.¹⁵

Problem 1 *The government's objective is to determine an admissible cumulative non-decreasing transfer process G_t and an initial tax D_0 so as to maximize:*

$$\Omega \equiv \max_{G_t, D_0} E_0 \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \quad (6)$$

subject to

$$c_t \geq \xi \text{ for all } t > 0 \quad (7)$$

$$D_0 = E_0 \int_0^\infty e^{-qt} H_t dG_t \quad (8)$$

and subject to the constraint that c_t solves the agent's optimization problem given G_t

$$c_t = \arg \max_{\langle c_t, \pi_t \rangle} E_0 \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt \quad (9)$$

s.t.:

$$dW_t = qW_t dt + \pi_t \{\mu dt + \sigma dB_t\} + \{W_t - \pi_t\} r dt - c_t dt + dG_t \quad (10)$$

$$W_{0+} = W_0 - D_0 \quad (11)$$

$$W_t \geq 0 \text{ for all } t > 0 \quad (12)$$

¹⁵This latter normalization is without loss of generality since all quantities of interest depend on the ratio of the stochastic discount factor between two points in time, rather than its level.

Equation (6) states that the government aims to maximize the agent's welfare, subject to the additional requirement that the agent's consumption not fall below the minimum level ξ that would trigger the assumed negative externalities on the rest of society.

Equations (8) and (11) state that the cost of providing the transfer process G_t to the consumer should be self-financed by a tax D_0 . Parenthetically, this "self-financing" requirement implies that the government is not required to implement the provision of transfers to consumers. It can simply specify the optimal process G_t that each consumer should purchase and leave it to competitive financial companies to price and provide these transfers.

Finally, equations (9)-(12) capture the incentive compatibility requirement. Equation (9) states that the optimal process c_t cannot be mandated by the government (since the government observes neither the consumption nor the assets of the agent). Instead, the optimal consumption process is chosen optimally by the consumer, who is faced with the budget dynamics of equation (10). These dynamics are identical to the ones in equation (3), except for the presence of the transfers dG_t , and the fact that the consumer needs to finance these transfers by paying the amount D_0 upon entering retirement. Accordingly, an instant after entering retirement, her wealth W_{0+} is equal to the funds she has accumulated in the pre-retirement phase (W_0) net of the lump sum payment D_0 .

The final requirement that constrains a consumer's choices is the borrowing constraint (12). This constraint plays a central role in the analysis. Without this constraint, it would be impossible for the government to find a set of taxes and transfers that would induce the agent to choose a consumption path that satisfies (7). The reason is due to a well understood result in Public Finance, known as Ricardian Equivalence: Since the market is dynamically complete *in the absence of the constraint* (12), a consumer's feasible consumption plans are only constrained by the consumer's intertemporal budget constraint, i.e. the requirement that the net present value of her consumption be equal to the wealth she has accumulated. Since the net present value of government transfers is equal to the lump sum tax D_0 , the consumer's intertemporal budget constraint is unaffected by the government intervention, no matter what process G_t the government chooses. Accordingly, the government-provided

tax-financed transfers cannot affect the consumer's plans. Agents can continue to consume as they would in the absence of government intervention and only modify their portfolios so as to undo the effects of the transfers.

The presence of a borrowing constraint such as (12), however, makes taxes and transfers non-neutral. The reason is that a borrowing constraint implies stronger restrictions than a simple intertemporal budget constraint on the agent's feasible consumption choices. Hence, by a judicious choice of an initial tax and subsequent transfers, the government can affect the agent's consumption. Parenthetically, the borrowing constraint (12) is realistic and easy to implement in practice. It suffices that the government instruct courts not to enforce agreements that would let lenders seize future government transfers as collateral for loans.

Because of the central role played by the borrowing constraint (12), the next section reviews some known results related to the implications of the constraint (12) for optimal consumption processes. Subsequent sections use these results to solve problem 1.

4 The agent's consumption choices in the presence of government intervention and borrowing constraints

Suppose that at the time of retirement (time 0) the government collects an amount D_0 and then promises an admissible cumulative transfer process G_t . It is natural to ask how the agent's consumption choices will be affected by this intervention in the presence of the constraint (12).

To gain some intuition, it is useful to start by assuming that there is no uncertainty ($\sigma = 0$), so that $\mu = r$, the agent's dynamic budget constraint is given by $dW_t = (q + r)W_t dt - c_t dt + dG_t$, and the stochastic discount factor is deterministic ($H_t = e^{-rt}$). The deterministic

dynamics of W_t, H_t imply that the constraint $W_t \geq 0$ amounts to the requirement¹⁶

$$\int_0^t c_s e^{-qs} H_s \leq W_0 - D_0 + \int_0^t e^{-qs} H_s dG_s \text{ for all } t \geq 0. \quad (13)$$

Applying the Lagrangian method, an agent's problem can be converted into an unconstrained problem by attaching Lagrange multipliers $\lambda, \zeta_t \geq 0$ to obtain

$$\begin{aligned} \mathcal{L} = & \int_0^\infty e^{-(\rho+q)t} \frac{c_t^{1-\gamma}}{1-\gamma} dt + \lambda \left[W_0 - D_0 + \int_0^\infty e^{-qt} H_t (dG_t - c_t dt) \right] \\ & + \left\{ \int_0^\infty \zeta_t \left(W_0 - D_0 + \int_0^t e^{-qs} H_s (dG_s - c_s ds) \right) dt \right\} \end{aligned} \quad (14)$$

Applying integration by parts to the second line of (14) and imposing the transversality condition $\lim_{t \rightarrow \infty} e^{-qt} H_t W_t = 0$ gives

$$\mathcal{L} = \int_0^\infty e^{-(\rho+q)t} \left(\frac{c_t^{1-\gamma}}{1-\gamma} - e^{\rho t} \lambda X_t H_t c_t \right) dt + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s + \lambda [W_0 - D_0], \quad (15)$$

where

$$X_t \equiv 1 - \int_0^t \frac{\zeta_s}{\lambda} ds.$$

Maximizing \mathcal{L} over c_t amounts to simply maximizing the expression inside round brackets in equation (15), which gives

$$c_t = (\lambda e^{\rho t} H_t X_t)^{-\frac{1}{\gamma}}. \quad (16)$$

If all $\zeta_s = 0$ (i.e. when the borrowing constraint $W_t \geq 0$ is not binding) then $X_t = 1$, and equation (16) amounts to the familiar result that an agent's marginal utility of consumption ($e^{-\rho t} c_t^{-\gamma}$) be proportional to the stochastic discount factor H_t .

However, when the borrowing constraint is binding, then consumption is affected by the

¹⁶To derive this equation, note that in the deterministic case $d(e^{-qt} H_t W_t) = e^{-qt} H_t (dG_t - c_t dt)$. Integrating the left and right hand side of this equation and imposing the requirement $W_t \geq 0$ leads to (13).

presence of the decreasing process X_t , which reflects the cumulative effect of the Lagrange multipliers associated with the borrowing constraint. By construction X_t is a process that is non-increasing and starts at $X_0 = 1$.

To fully determine the solution to the consumer's problem, one needs to determine the Lagrange multipliers λ, ζ_s . He and Pages (1993) show that this amounts to first maximizing \mathcal{L} over c_t (given arbitrary λ, X_t) and then minimizing the resulting expression over λ, X_t . Specifically, He and Pages (1993) show the following Proposition, which holds also in the presence of uncertainty:¹⁷

Proposition 1 *Let \mathcal{D} be the set of non-increasing, non-negative and progressively measurable processes that start at $X(0) = 1$. Then, the value function $V(W_0)$ of an agent can be expressed as:*

$$V(W_0) = \min_{\lambda > 0, X_s \in \mathcal{D}} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) + \lambda (W_0 - D_0) \right] \quad (17)$$

Let X_t^*, λ^* denote the process X_t and the constant λ that minimize the above expression. Then the optimal consumption process c_t^* for a consumer faced with the borrowing constraint (12) is given by (16) evaluated at $\lambda = \lambda^*, X_t = X_t^*$. Moreover, the process X_t^* decreases only when the associated wealth process (W_t) falls to zero and is otherwise constant, i.e.:

$$\int_0^\infty W_t dX_t^* = 0 \quad (18)$$

Finally, the resulting wealth process for any $t > 0$ satisfies:

$$W_t = \frac{E_t \left(\int_t^\infty e^{-q(s-t)} X_s^* H_s c_s^* ds \right)}{X_t^* H_t} - \frac{E_t \left(\int_t^\infty e^{-q(s-t)} X_s^* H_s dG_s \right)}{X_t^* H_t} \quad (19)$$

¹⁷Marcet and Marimon (1998) show a similar result in the context of recursive contracts.

5 Government transfers and their welfare effects: an upper bound

Proposition 1 gives an intuitive way to summarize the effects of the incentive compatibility requirement (equations [9]-[12]).

It asserts that every government transfer process G_t will be associated with a constant $\lambda^*(G_t)$ and a Lagrange multiplier process $X_t^*(G_t)$. Given this correspondence between a choice of G_t and the resulting pair (λ^*, X_t^*) , there is a straightforward way to obtain an upper bound to the value function of problem 1. In particular consider the following problem:

Problem 2 *Maximize:*

$$J(W_0) \equiv \max_{c_t, X_t \in \mathcal{D}, \lambda > 0} E_0 \int_0^\infty e^{-(\rho+q)s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \quad (20)$$

subject to:

$$E_0 \left(\int_0^\infty e^{-qs} H_s c_s ds \right) \leq W_0 \quad (21)$$

$$c_t \geq \xi \quad (22)$$

$$c_t = (\lambda e^{\rho t} H_t X_t)^{-\frac{1}{\gamma}} \quad (23)$$

Problem 2 is the problem of a government that can choose directly the consumption of the agent, subject to the intertemporal budget constraint (21), the constraint on the minimum consumption level (equation [22]), and the additional requirement that any chosen consumption process should have a representation in the form of equation (23) for some X_t . In effect, problem 2 allows the government to choose *directly* the Lagrange multipliers (λ, X_t) without being concerned whether there exists any pair of taxes and transfers (D_0, G_t) that would render these Lagrange multipliers as shadow values of the consumer's optimization problem (9).

Figure 1 gives an intuitive argument to show that the optimized value J to Problem 2

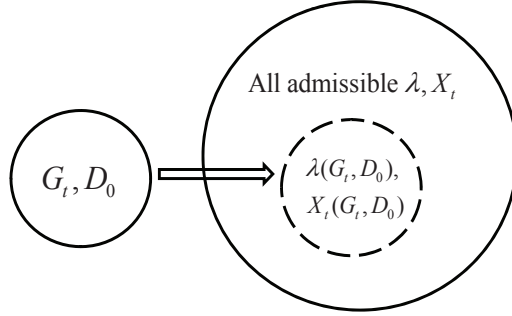


Figure 1: An illustration of Lemma 1. The admissible choices of problem 1 map into a subset of the admissible choices of problem 2.

provides an upper bound to the value function Ω of problem 1. Indeed, any admissible consumption process of problem 1 needs to satisfy¹⁸ equations (21) and (22). Moreover, Proposition 1 asserts that there always exists some pair of λ, X_t such that any admissible consumption process of problem 1 can be expressed in the form of equation (23). Therefore, any G_t, D_0 maps into a subset of pairs (X_t, λ) allowed by Problem 2, and the value function of problem 2 must therefore provide an upper bound to problem 1. The following Lemma provides a formal proof.

Lemma 1 *Let \mathcal{G} be the class of all transfer processes G_t that enforce (7) and satisfy (8). Furthermore, let $V(W_0)$ be given as in equation (17). Then the value functions of problems 1 and 2 satisfy :*

$$V(W_0) \leq J(W_0) \tag{24}$$

The remainder of this section derives an explicit solution to problem 2, while the next section shows that there exist transfer processes G_t^* that are optimal, because they make

¹⁸The consumer's dynamic budget constraint (10) implies the intertemporal budget constraint

$$W_0 - D_0 + \int_0^\infty e^{-qt} H_t (dG_t - c_t dt) \geq 0.$$

Combining the intertemporal budget constraint with condition (8) implies (21).

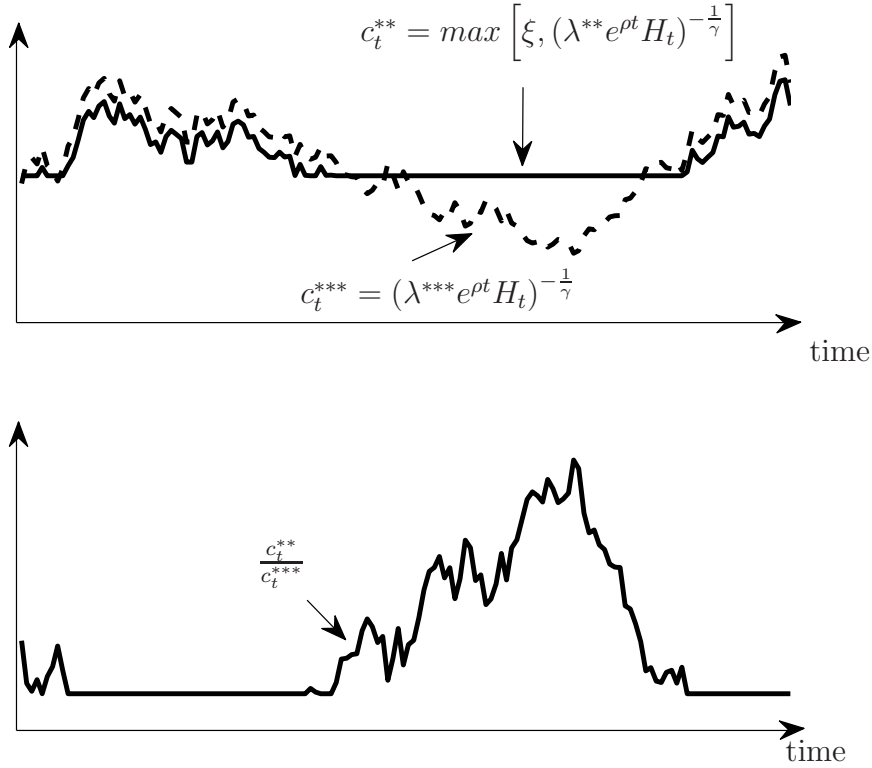


Figure 2: Implications of the constraints in problem 2

equation (24) hold with equality.

As a first step towards solving problem 2 it is useful to ask whether constraints (21), (22), and (23) will bind at an optimum. The top panel of figure 1 gives an optimal consumption path for a random realization of H_t assuming that one maximizes (20) subject only to the intertemporal budget constraint (21). The resulting solution is $c_t^{***} = (\lambda^{***} e^{\rho t} H_t)^{-\frac{1}{\gamma}}$ and it corresponds to what the consumer would choose, if left alone. Because H_t is log-normal, so is c_t and accordingly $c_t < \xi$ with positive probability. Imposing the constraint $c_t \geq \xi$ (but not the constraint [23]) leads to the optimal consumption path $c_t^{**} = \max \left[\xi, (\lambda^{**} e^{\rho t} H_t)^{-\frac{1}{\gamma}} \right]$.¹⁹ The solution c_t^{**} is what the government would choose, if it could directly observe and

¹⁹Clearly, $(\lambda^{**})^{-\frac{1}{\gamma}} < (\lambda^{***})^{-\frac{1}{\gamma}}$, otherwise it would be impossible that both c_t^{***} and c_t^{**} satisfy (21).

mandate the agent's consumption and portfolio choices.

However, the government cannot directly observe these choices. Instead, it needs to induce the agent to choose consumption paths that satisfy $c_t \geq \xi$, by exploiting binding borrowing constraints. This is captured by equation (23). The bottom panel of Figure 2 shows that this incentive compatibility requirement is in general binding. Indeed, equation (23) implies that any admissible consumption process should satisfy the property that the ratio $c_t/c_t^{***} = \left(\frac{\lambda}{\lambda^{***}}\right)^{-\frac{1}{\gamma}} X_t^{-\frac{1}{\gamma}}$ should be a non-decreasing process (since X_t is non-increasing). Clearly, the ratio c_t^{**}/c_t^{***} has decreasing sections and therefore c_t^{**} cannot satisfy (23). Therefore, $J(W_0)$ (and accordingly the value function Ω in problem 1) will in general be lower than what the government could attain if it observed and mandated consumption.

The next proposition determines the solution of problem 2:

Proposition 2 *Let the constants ϕ, K be defined as²⁰*

$$\phi \equiv \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) + \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2(\rho + q)\kappa^2}}{\kappa^2} > 1, \quad (25)$$

$$K \equiv \frac{\gamma}{\frac{\gamma-1}{\gamma}\frac{\kappa^2}{2} + \gamma(r+q) + (\rho-r)}, \quad (26)$$

and assume that

$$W_0 \geq \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K \xi. \quad (27)$$

²⁰To see why $\phi > 1$, notice that ϕ solves the quadratic equation

$$\frac{\kappa^2}{2}\phi^2 + \left(\rho - r - \frac{\kappa^2}{2}\right)\phi - (\rho + q) = 0$$

Evaluating the left hand side of this equation at $\phi = 1$ gives:

$$-(r+q) < 0$$

Hence the larger of the two roots of the quadratic equation is larger than 1.

Additionally, for any $\lambda > 0$, let the process X_t^* be given by

$$X_t^*(\lambda) \equiv \min \left[1, \frac{\xi^{-\gamma}/\lambda}{\max_{0 \leq s \leq t} (e^{\rho s} H_s)} \right]. \quad (28)$$

Then the value function of problem (2) is given by

$$J(W_0) = \min_{\lambda \geq 0} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right] \quad (29)$$

$$= \min_{\lambda \geq 0} \left[-\frac{K \xi^{1-\gamma}}{\gamma \phi (\phi - 1)} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^\phi + K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}} + \lambda W_0 \right]. \quad (30)$$

Letting λ^* be the scalar that minimizes (30), the optimal triplet that solves problem (2) is given by λ^* , $X_t^* = X_t(\lambda^*)$, and $c_t^* = (\lambda^* e^{\rho t} H_t X_t^*)^{-\frac{1}{\gamma}}$.

Proposition 2 provides the optimal solution and the value function of problem 2, assuming that the agent enters retirement with a level of assets that are no smaller than the lower bound of equation (27). Assumption (27) will be maintained henceforth and discussed in further detail in section 7.

6 Optimal Transfer Processes

This section illustrates two optimal distinct processes G_t^* that attain the upper bound $V(W_0; G_t^*) = J(W_0)$.

6.1 A constant income stream

The simplest form of government transfer process is a constant income stream: The government collects a lump sum tax of $D_0 = \frac{y_0}{r+q}$ and in exchange it delivers a constant stream of y_0 until the agent dies. Surprisingly, this policy is optimal, as long as y_0 is chosen judiciously. The following proposition gives a closed form solution for y_0 .

Proposition 3 *Let y_0 be given by*

$$y_0 \equiv (r + q) K \xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right), \quad (31)$$

where K is given in (26) and ϕ is given in (25). The policy of collecting $D_0 = \frac{y_0}{r+q}$ and providing transfers equal to y_0 until the agent dies, attains the upper bound $V(W_0; G_t = y_0) = J(W_0)$ and is therefore optimal.

An interesting feature of the optimal policy in proposition 3 is contained in the following Lemma

Lemma 2 *The optimal policy of proposition 3 has the property*

$$\frac{y_0}{\xi} > 1$$

Lemma 2 shows that if the government wants to ensure a minimum consumption of one dollar, it needs to deliver more than one dollar in guaranteed income. This result is driven by the fact that agents cannot be excluded from markets, and the presence of a constant income guarantee incentivizes them to use some of the constant income to take risks in the stock market. Therefore the government needs to set $y_0 > \xi$ to ensure that $c_t \geq \xi$.

6.2 Portfolio Insurance

Providing agents with a constant income is not the unique optimal way to attain the upper bound in Proposition 2. The approach presented in this section also succeeds in attaining the same upper bound. To describe this approach, let λ^* be the scalar that minimizes (30). Then define the government's transfer process as:

$$dG_t = - \left(\frac{1}{\gamma} + \phi - 1 \right) K \xi \frac{dX_t^*}{X_t^*} \quad (32)$$

where $X_t^*(\lambda^*)$ is the process defined in (28).

This section shows the following two results:

a) The process (32) attains the upper bound of Proposition 2.

b) The process (32) has an intuitive economic interpretation as a type of minimum return guarantee (portfolio insurance) on the agent's optimal portfolio of stocks and bonds.

The following proposition formalizes the first claim and provides results that are useful towards establishing the second claim.

Proposition 4 *Let λ^* be the scalar that minimizes (30) and $X_t^*(\lambda^*)$ be the process that is given in (28). Consider an agent who anticipates transfers given by (32) and is faced with an initial tax of D_0 , where D_0 satisfies (8). Then*

a) *her value function will coincide with the upper bound given in (30)*

b) *Letting*

$$Z_t \equiv \lambda^* e^{\rho t} H_t X_t^*, \quad (33)$$

the agent will invest

$$\pi_t = \frac{\kappa}{\sigma} K \xi \left[(\phi - 1) \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + \frac{1}{\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{-\frac{1}{\gamma}} \right] \quad (34)$$

dollars in the stock market and consume

$$c_t = Z_t^{-\frac{1}{\gamma}}, \quad (35)$$

while the agent's optimal wealth process W_t is given by

$$W_t = -K \xi \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + K Z_t^{-\frac{1}{\gamma}}. \quad (36)$$

c) *The initial tax D_0 associated with (32) is given by*

$$D_0 = K \xi^{\frac{1}{\gamma} + \phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi-1} \quad (37)$$

The portfolio policy (34) will aid in the interpretation of (32) as a form of portfolio insurance. To obtain some intuition on the nature of (32), consider first the following puzzling feature of the optimal portfolio policy: As $c_t \rightarrow \xi$, equation (35) implies that $Z_t \rightarrow \xi^{-\gamma}$ and (36) implies that $W_t \rightarrow 0$. However, the portfolio of the agent becomes

$$\lim_{Z_t \rightarrow \xi^{-\gamma}} \pi_t = \left(\frac{1}{\gamma} + \phi - 1 \right) K \xi \frac{\kappa}{\sigma} > 0. \quad (38)$$

Because the agent's financial wealth approaches zero as $Z_t \rightarrow \xi^{-\gamma}$, but her stock position doesn't, a further negative return on the stock market would lead to a negative financial asset position in the absence of any transfers. This means that the transfers given by (32) act as a minimum return guarantee, which ensures that the agent receives just enough funds to sustain her financial wealth at zero and keep her consumption at ξ .

It is useful here to clarify that these transfers do not require that the government actually observe the path of the agent's assets or her consumption. By the definition of X_t^* in equation (28), the government only needs to know the evolution of the stochastic discount factor H_t , which can be inferred from the path of the stock market²¹ and the assets of the agent at the time of retirement,²² which can be inferred from the consumer's optimal retirement decision as section 7 shows. A simple way of thinking about the transfer process G_t in (32) is that the government and the agent have a joint understanding of how the consumer will consume and invest in the presence of the transfers given by (32). Based on its (correct) understanding of the consumer's optimal policies, the government can infer the agent's wealth and make just enough transfers when needed, so as to keep the agent's wealth above 0 and her optimal consumption above ξ .

²¹Note that $\log(H_t) - \log(H_0) = -(r + 0.5\kappa^2)t - \kappa(B_t - B_0) = -(r + 0.5\kappa^2)t - \frac{\kappa}{\sigma}(B_t - B_0) = -(r + 0.5\kappa^2)t - \frac{\kappa}{\sigma}[\log P_t - \log P_0 - (\mu - 0.5\sigma^2)t] = \frac{\kappa}{\sigma}(P_t - P_0) + \left(\frac{\kappa}{\sigma}(\mu - 0.5\sigma^2) - (r + 0.5\kappa^2)\right)t$.

²²The level of assets at retirement determine λ^* by equation (30).

6.3 Comparing the two policies

Given that both policies attain the upper bound of equation (30), this means that they imply the same value function for the agent, and hence are equivalent from a welfare perspective²³.

However, the two policies do differ. They make transfers of different magnitudes in different states of the world. The initial payments that they imply are also different. Indeed, the initial payment associated with the constant income policy is:

$$D_0^{const.} = \frac{y_0}{r+q} = K\xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right), \quad (39)$$

whereas by equation (37), the initial payment of the portfolio insurance policy is:

$$D_0^{p.i.} = K\xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right) \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi-1} \quad (40)$$

Since $c_0 \geq \xi$ and²⁴ $c_0^{-\gamma} = \lambda^*$, it follows that $\lambda^*/\xi^{-\gamma} \leq 1$ and accordingly $\frac{D_0^{p.i.}}{D_0^{const.}} \leq 1$. Hence the “portfolio insurance” policy implies an initial payment that cannot be larger than the initial payment of the “constant income” policy. This is intuitive, since the constant income policy delivers the same transfers in all states of the world, including states of the world where the borrowing constraint doesn’t bind. By contrast, the “portfolio insurance” policy delivers payments only when the borrowing constraint binds.

However, when $c_0 = \xi$ (or alternatively $W_{0+} = 0$) the two policies imply the same initial payment. Hence the initial payment of the two policies differs only when the borrowing constraint is *not* binding, but is identical when the borrowing constraint *does* bind. This is the reason why the two policies imply different initial payments, but are identical from a welfare perspective. The additional resources delivered by the constant income policy are delivered in states of the world where the borrowing constraint is not binding and hence can be “undone” by agents’ portfolio choice, consistent with the Ricardian Equivalence theorem.

²³The derivations in the appendix also show that they imply exactly the same consumption process “path by path”.

²⁴Recall that $H_0 = X_0^* = 1$.

The above discussion illustrates that simply comparing the costs of retirement benefit guarantees does not provide sufficient information for welfare comparisons.²⁵

7 Minimum level of assets and implications for pre-retirement savings

A maintained assumption of the analysis so far was that the agent's assets upon entering retirement were above the minimum level of equation (27). As the next Proposition shows, this assumption is not only sufficient, but it is also necessary for the existence of transfer processes that can induce a consumption process that satisfies $c_t \geq \xi$.

Proposition 5 *An admissible transfer process G_t that can induce $c_t \geq \xi$ exists if and only if $W_0 \geq W^{\min}$.*

Proposition 5 has implications for the agent's pre-retirement problem. Specifically, the feasibility of enforcing the constraint $c_t \geq \xi$ post-retirement is equivalent to requiring that the agent arrives in retirement with assets that are at least as large as implied by condition (27).

If the government can enforce lump-sum taxation in retirement (say through direct punishments),²⁷ the results in Farhi and Panageas (2007) imply that an agent who can choose her retirement time optimally will retire only once her assets reach some level \bar{W} , which will be at least as large as $\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K \xi$. Hence, no government intervention is required pre-retirement.

However, in many cases it may be more realistic to assume that agents have no choice as to their retirement age (say because productivity deteriorates), and the government cannot

²⁵As a final remark, a source of difference between the two policies is informational. The constant income policy does not require exact knowledge of the level of an agent's assets at retirement, whereas the portfolio insurance policy does. However, even if assets were unobservable at retirement, an agent who could hide, but not over-report her assets, would have an incentive to report her assets truthfully. The reason is that $D_0^{p,i}$ is decreasing²⁶ in W_0 . Since a larger $D_0^{p,i}$ "tightens" the agent's borrowing constraints at retirement, an agent has an incentive to report as large a value of W_0 as she can. Assuming that the agent can hide, but not over-report her assets, she would voluntarily report the actual value of W_0 .

²⁷Mathematically, this would imply that $\Omega(W_0) = -\infty$ whenever W_0 is smaller than W^{\min} .

enforce lump sum taxation. In that case, some measure to ensure a mandatory level of pre-retirement savings is required. For instance, if the retirement date τ is a fixed time, and the government's only pre-retirement instrument is levying a constant fraction of income, the government would have to collect a fraction χ of the agent's income and place it in a riskless asset on the agent's behalf. Because of the constraint (12) this would be sufficient to ensure that the agent has enough assets at retirement to pay D_0 . The minimum level of χ that will enforce condition (27) is given by requiring that the net present value of labor tax payment $\int_{t_b}^{t_b+\tau} e^{-(r+q)(t-t_b)} \chi Y dt$ equals the net present value of minimum required assets $e^{-(r+q)\tau} W^{\min}$. This implies that the minimum level of χ is given by

$$\chi = \frac{(r+q)e^{-(r+q)\tau}}{1-e^{-(r+q)\tau}} \frac{K\xi}{Y} \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right). \quad (41)$$

An intuitive argument also shows that it is not optimal to collect more than χ . Given that the agent is faced with borrowing constraints prior to retirement, any government intervention that reduces income today and returns it in the form of a lump sum payment upon entering retirement will reduce the agent's ability to smooth consumption. The following proposition states this formally:

Proposition 6 *The optimal pre-retirement mandatory savings rate that will ensure (27) is given by (41).*

8 Alternative justifications for the constraint $c_t \geq \xi$

The underlying assumption behind problem 1 is that there is a difference between the government's objective and the individual agent's objective. The difference stems from the assumed negative external effects that arise, when an agent's consumption falls below the level ξ . This section revisits the reasons behind the assumed wedge between the government's and the agent's objectives.

8.1 The constraint $c_t \geq \xi$ in the presence of a welfare system

One potential reason for ensuring that agents can self-finance a minimum standard of living in retirement is to deter them from over-burdening the welfare system, which is financed through distortionary taxation.

To substantiate this claim, I enrich the model to give a reason for the presence of a welfare system, along with a stylized model of such a system. Specifically, assume that the cohort of agents that are born at time t_b retire at some fixed date $t_b + \tau$. Until then, agents are identical in every respect. Upon entering retirement, however, a small fraction θ of agents experiences an *unobservable* and idiosyncratic shock that results in a negative income stream of \bar{Y} for the rest of their lives. (The remaining fraction of the agents remain identical to the agents described in the paper so far). The idiosyncratic shock is catastrophic, in the sense that no agent could self-insure against that shock by accumulating savings

$$\int_{t_b}^{t_b+\tau} e^{-(q+r)(t-t_b)} Y dt < \int_{t_b+\tau}^{\infty} e^{-(q+r)(t-t_b)} \bar{Y} dt \quad (42)$$

Equation (42) states that even if an agent saved all her wages, the resulting present value would still be smaller than the present value of the negative shock \bar{Y} .

Because of these catastrophic idiosyncratic shocks, the government can raise the welfare of the time t_b -cohort of agents by creating a “welfare” system, which works as follows. Any agent who enters retirement can request transfers ($dN_t \geq 0$) from the government. Requesting such transfers requires time and effort (filing paperwork, staying in lines etc.). Specifically, an agent incurs a utility reduction of $\xi^{-\gamma}$ units per unit of transfer that she receives. Accordingly, an agent maximizes

$$\int_{t_b+\tau}^{\infty} e^{-(\rho+q)(t-t_b-\tau)} \frac{c_t^{1-\gamma}}{1-\gamma} dt - \int_{t_b+\tau}^{\infty} e^{-(\rho+q)(t-t_b-\tau)} \xi^{-\gamma} dN_t.$$

Given the above assumptions, any agent (whether she has experienced an idiosyncratic shock or not) can always guarantee herself a consumption process above ξ by requesting

welfare transfers. To see this, let $V(W_t)$ denote an agent's value function and suppose that the agent considers some deviation from her optimal plan, whereby she requests a small amount $\varepsilon > 0$ from the government. Since $V(W_t)$ is the value function, such deviations should not be optimal, so that $V(W_t) \geq V(W_t + \varepsilon) - \xi^{-\gamma}\varepsilon$ or $\frac{V(W_t) - V(W_t + \varepsilon)}{\varepsilon} \geq -\xi^{-\gamma}$. Since this inequality must hold for any ε , letting $\varepsilon \rightarrow 0$ gives $V_W \leq \xi^{-\gamma}$. Additionally, an agent's optimal consumption decisions must satisfy the first order condition $c_t^{-\gamma} = V_W$. Therefore, it must be the case that $c_t \geq \xi$. Alternatively put, an agent can always guarantee herself a level of consumption above ξ by using the welfare system.

A final assumption is that the welfare system is financed by distortionary taxation. Specifically, the government collects a labor tax equal to $\bar{\omega}Y$, during the work years of the agents, so as to finance any welfare payments later on. However, taxation is distortionary, i.e, even though agents pay $\bar{\omega}Y$ in taxes, only a fraction $(1 - \delta)\bar{\omega}Y$ reaches the government. The constant δ captures the work-disincentives and the associated deadweight costs resulting from distortionary labor taxes. (See e.g. Barro (1979) for a seminal treatment). The resulting budget constraint of the government is given by

$$\int_{t_b}^{t_b + \tau} e^{-(\rho+q)(t-t_b-\tau)} (1 - \delta)\bar{\omega}Y = \theta \int_{t_b + \tau}^{\infty} e^{-qt} \left(\frac{H_t}{H_{t_b}} \right) dN_t,$$

assuming that only the θ -fraction of agents who suffer the idiosyncratic shock ever request transfers. Because distortionary taxation and the utility costs of requesting welfare transfers are both deadweight costs, the ex-ante welfare of the time- t_b cohort of agents is maximized if transfers are only requested from agents who experience idiosyncratic shocks. Obviously, since the idiosyncratic shock is private information, "separation" of the two types of agents can only occur if the agents modeled in section 2 (i.e. the agents who do not experience idiosyncratic shocks) find it optimal to not use the welfare system. In that sense the constraint (5) is a standard incentive compatibility constraint, which ensures that only agents who experience idiosyncratic shocks find it optimal to access the welfare system, but not the agents who don't experience such shocks.²⁸

²⁸The above reasoning is valid for sufficiently small θ . As $\theta \rightarrow 0$, the ex-ante social welfare of the time-

8.2 Behavioral justifications

Problem 1 is also consistent with a behavioral interpretation. Several authors in behavioral economics model the inability of an agent to commit as a principal-agent problem. The principal is taken as the “prudent”, time-zero “self”, who has a different objective than the subsequent “reckless selves” who are making decisions.²⁹ For instance, if one were to interpret ξ as inelastic retirement expenditures associated with aging (say medical costs), then the “prudent” self would like to impose the constraint $c_t \geq \xi$ on the choices made by the subsequent “reckless” selves, who will simply ignore this constraint. In such a behavioral interpretation of the problem, the government’s choice of post-retirement transfers maximizes the welfare of the prudent “self”, by exploiting borrowing constraints on the “reckless” selves.

9 Arbitrary stochastic discount factors and multiple assets and sources of uncertainty

The assumption of a small open economy facilitated the analysis by rendering the stochastic discount factor exogenous to the model. Another simplifying assumption is that everything is driven by a single shock. Neither of these assumptions is restrictive. Even if the stochastic discount factor were endogenous and driven by multiple sources of uncertainty, most of the results of the paper would survive.

Specifically, the fact that (29) provides an upper bound to problem 1 remains valid for any continuous stochastic discount factor H_t . It is also straightforward to show that the portfolio insurance policy would attain the upper bound of proposition 2 for any stochastic discount factor. However, the result that seems to depend on the constant nature of the investment opportunity set is the optimality of the constant income policy. Nevertheless, since the upper bound (29) is attainable for any stochastic discount factor, one can use

t_b cohort coincides with the welfare of problem 1, as long as the value function of agents who experience idiosyncratic shocks in retirement is finite. Because of the welfare system, agents who experience idiosyncratic shocks can keep their consumption bounded from below, and hence their value function is finite

²⁹For economic applications of the concept of “multiple selves”, see e.g. Amador et al. (2006).

equation (29) to evaluate the magnitude of potential welfare losses, and balance these losses against the simplicity of a constant income policy.

In summary, the qualitative findings of the model would survive even in a closed, general equilibrium economy.³⁰ In that case, the prices of the guarantees and all the parametric formulas would be altered. However, the key results of the paper, namely the nature of the upper bound of equation (29) and the the existence of a portfolio insurance policy that attains the upper bound, would remain unchanged.

10 Conclusion

By exploiting borrowing restrictions of agents, this paper proposed a framework to discuss optimal transfer processes that can ensure a minimum standard of living in retirement.

Within the framework of the baseline life-cycle model, two policies were shown to be optimal: According to the first policy, retirees use part of their accumulated assets to purchase a fixed annuity that pays off a constant income stream. The second policy is an appropriate form of portfolio insurance that ensures retirees against further negative returns, once their assets approach zero. Somewhat surprisingly, even though the two alternative policies lead to the same welfare, the cost of these guarantees is different in general. This suggests caution in drawing welfare conclusions from the literature that prices guarantees as contingent claims.

Several issues are unexplored by the present paper. A first question concerns unobserved preference heterogeneity. If agents have different risk aversions, or discount factors, then the government needs to offer menus of contracts in the spirit of discriminatory pricing. It appears straightforward to extend the analysis to allow for this possibility. An open question is whether the need to enforce sorting into different types of contracts would affect the qualitative features of the guarantees.

³⁰Of course in general equilibrium care should be taken to make sure that aggregate consumption stays above the level ξ multiplied by the mass of retirees. In an endowment economy this could be done by boundedness assumptions on the aggregate endowment. Alternatively one could introduce production and relax the bounds on the fundamental shocks.

A second question concerns the implications of such guarantees for asset prices. Even though the results of the paper go through for arbitrary stochastic discount factors, it is likely that extensive coverage of retirees by these guarantees would affect the stochastic discount factor in general equilibrium. Studying these two questions is left for future research.

A Appendix

Proof of proposition 1. Subject to minor modifications, the proof of this proposition is identical to the first theorem of He and Pages (1993) and the reader is referred to that paper for a proof. ■

Proof of Lemma 1. The proof of this lemma is contained in the proof of proposition 2 (Particularly Lemma 4). ■

Proof of Proposition 2. The proof of this proposition is established in steps. The following Lemma contains a useful first result.

Lemma 3 Take any $\lambda \in (0, \xi^{-\gamma}]$ and any process G_t and define

$$\widehat{X}_t \equiv \arg \min_{X_s \in \mathcal{D}} E_0 \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s (X_s - 1) dG_s \right). \quad (43)$$

Then:

$$\lambda E_0 \left(\int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s \right) = E_0 \int_0^\infty e^{-(\rho+q)s} \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{1-\frac{1}{\gamma}} \left(1 - \frac{1}{\widehat{X}_s} \right) ds. \quad (44)$$

Proof of Lemma 3. Let $\Lambda_t \equiv 1 - \frac{1}{\widehat{X}_t}$. Applying Ito's Lemma to Λ_t , one obtains $d\Lambda_t \equiv \frac{d\widehat{X}_t}{(\widehat{X}_t)^2}$. Hence Λ_t changes when and only when \widehat{X}_t changes. By Theorem 1 of He and Pages (1993):

$$\int_0^\infty \left[E_t \left(\int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) - E_t \left(\int_t^\infty \widehat{X}_s e^{-qs} H_s c_s ds \right) \right] d\widehat{X}_t = 0, \quad (45)$$

where c_s is given explicitly by (23). Plugging (23) into (45), and observing that Λ_t changes when and only when \widehat{X}_t changes implies that

$$\int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s - E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{-\frac{1}{\gamma}} ds \right) d\Lambda_t = 0.$$

Then, for any admissible G_t and \widehat{X}_t given by (43)

$$\begin{aligned} \lambda E_0 \left(\int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s \right) = \\ \lambda E_0 \left[\int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s - \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \right] \\ + \lambda E_0 \left\{ \int_0^\infty E_t \left[\int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{-\frac{1}{\gamma}} ds \right] d\Lambda_t \right\}. \end{aligned} \quad (46)$$

Next consider the martingale

$$\mathcal{M}_t \equiv E_t \int_0^\infty \widehat{X}_s e^{-qs} H_s dG_s = \int_0^t \widehat{X}_s e^{-qs} H_s dG_s + E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s. \quad (47)$$

According to the martingale representation theorem, there exists a square integrable $\widetilde{\psi}_s$ such that

$$\mathcal{M}_t = \mathcal{M}_0 + \int_0^t \widetilde{\psi}_s dB_s. \quad (48)$$

Combining (47) and (48) gives

$$\begin{aligned} d \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) &= d\mathcal{M}_t - \widehat{X}_t e^{-qt} H_t dG_t \\ &= \widetilde{\psi}_t dB_t - \widehat{X}_t e^{-qt} H_t dG_t. \end{aligned}$$

Now, fixing an arbitrary $\varepsilon > 0$, letting τ^ε be the first time t such that $|\Lambda_t| \geq \frac{1}{\varepsilon}$, applying integration by parts and using the fact that $\Lambda_0 = 0$, gives

$$\begin{aligned} -E_0 \int_0^{T \wedge \tau^\varepsilon} \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t &= -E_0 \int_0^{T \wedge \tau^\varepsilon} \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s + E_0 \int_0^{T \wedge \tau^\varepsilon} \Lambda_s \widetilde{\psi}_s dB_s \\ &\quad - E_0 \left[\Lambda_{T \wedge \tau^\varepsilon} \left(E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) \right]. \end{aligned}$$

Since ψ_s is square integrable and $|\Lambda_s|$ is bounded in $[0, \frac{1}{\varepsilon}]$ the second term on the right hand side of the above expression is 0. Also note that

$$-E_0 \left[\Lambda_{T \wedge \tau^\varepsilon} \left(E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) \right] = -E_0 \left[\widehat{X}_{T \wedge \tau^\varepsilon} \Lambda_{T \wedge \tau^\varepsilon} J \right], \quad (49)$$

where

$$J \equiv \left(E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty \frac{\widehat{X}_s}{\widehat{X}_{T \wedge \tau^\varepsilon}} e^{-qs} H_s dG_s \right) \leq E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty e^{-qs} H_s dG_s, \quad (50)$$

since \widehat{X}_t is non-increasing. Combining (50) with (49) and noting that $0 < \widehat{X}_t \leq 1$,

$$-E_0 \left[\widehat{X}_{T \wedge \tau^\varepsilon} \Lambda_{T \wedge \tau^\varepsilon} J \right] = E_0 \left[\left(1 - \widehat{X}_{T \wedge \tau^\varepsilon} \right) J \right] \leq E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty e^{-qs} H_s dG_s. \quad (51)$$

Given that $E \int_0^\infty e^{-qs} H_s dG_s < \infty$ it follows that

$$E_{T \wedge \tau^\varepsilon} \int_{T \wedge \tau^\varepsilon}^\infty e^{-qs} H_s dG_s \rightarrow 0, \quad (52)$$

as $\varepsilon \rightarrow 0, T \rightarrow \infty$. This leads to the inequalities:

$$\begin{aligned} -E_0 \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t &\geq -E_0 \int_0^{T \wedge \tau^\varepsilon} \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \\ &\geq -E_0 \int_0^{T \wedge \tau^\varepsilon} \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s. \end{aligned}$$

Letting $\varepsilon \rightarrow 0, T \rightarrow \infty$, using the monotone convergence theorem, and using (51) and (52), gives

$$- \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t = -E_0 \int_0^\infty \Lambda_s \widehat{X}_s e^{-qs} H_s dG_s. \quad (53)$$

Using (53) and the definition of Λ_t gives

$$\begin{aligned} \lambda E_0 \left[\int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s - \int_0^\infty \left(E_t \int_t^\infty \widehat{X}_s e^{-qs} H_s dG_s \right) d\Lambda_t \right] &= \\ = E_0 \left[\lambda \int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s - \lambda \int_0^\infty e^{-qs} H_s \widehat{X}_s \Lambda_s dG_s \right] &= 0. \end{aligned}$$

Returning now to (46) and using the above equation yields

$$\lambda E_0 \left(\int_0^\infty e^{-qs} H_s (\widehat{X}_s - 1) dG_s \right) = \lambda E_0 \left\{ \int_0^\infty E_t \left[\int_t^\infty \widehat{X}_s e^{-qs} H_s \left(e^{\rho s} \lambda H_s \widehat{X}_s \right)^{-\frac{1}{\gamma}} ds \right] d\Lambda_t \right\} \quad (54)$$

$$= E_0 \left[\int_0^\infty e^{-(\rho+q)t} \left(e^{\rho t} \lambda H_t \widehat{X}_s \right)^{1-\frac{1}{\gamma}} \Lambda_t dt \right], \quad (55)$$

where (55) follows from a similar integration by parts argument as the one in equations (47)-(53).

■

The next Lemma uses Lemma 3 to prove (24).

Lemma 4 *For all admissible processes $G_t \in \mathcal{G}$:*

$$\max_{G_t \in \mathcal{G}} V(W_0) \leq \min_{\lambda \in (0, \xi^{-\gamma}] } \left[E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right] \quad (56)$$

Proof of Lemma 4. Proposition 1 along with Lemma 3 implies that for any admissible process

G_t there exists a $\lambda^G > 0$ and a decreasing process $X_t^G \in \mathcal{D}$ that minimizes (17) such that

$$\begin{aligned} V(W_0) &= E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda^G e^{\rho s} H_s X_s^G c_s \right) ds + \lambda^G \int_0^\infty e^{-qs} H_s (X_s^G - 1) dG_s \right) + \lambda^G W_0 \\ &= E \int_0^\infty e^{-(\rho+q)s} \left(\frac{(e^{\rho s} \lambda^G H_s X_s^G)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda^G e^{\rho s} H_s (e^{\rho s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} \right) ds + \lambda^G W_0. \end{aligned} \quad (57)$$

Moreover, since the process G_t enforces $c_t \geq \xi$, equation (16) implies that $\lambda^G \leq \xi^{-\gamma}$. Next take an arbitrary $\lambda > 0$. Since $c_t = (e^{\rho t} \lambda^G H_t X_t^G)^{-\frac{1}{\gamma}}$ is an optimal consumption process, it exhausts the “budget constraint” of the consumer so that

$$E \int_0^\infty e^{-(\rho+q)s} e^{\rho s} H_s (e^{\rho s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} ds = W_0 - D_0 + E \int_0^\infty e^{-qs} H_s dG_s.$$

Using (8), this implies that $E \int_0^\infty e^{-(\rho+q)s} e^{\rho s} H_s (e^{\rho s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} ds = W_0$. This furthermore implies that (57) can be rewritten as

$$V(W_0) = E \int_0^\infty e^{-(\rho+q)s} \left(\frac{(e^{\rho s} \lambda^G H_s X_s^G)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s (e^{\rho s} \lambda^G H_s X_s^G)^{-\frac{1}{\gamma}} \right) ds + \lambda W_0. \quad (58)$$

Next define X_t^* as in equation (28), and let the process N_t be given as $N_t \equiv \frac{\lambda^G X_t^G}{\lambda X_t^*}$. Using N_t , one can rewrite equation (58) as

$$V(W_0) = E \int_0^\infty e^{-(\rho+q)s} \left(\frac{(e^{\rho s} \lambda H_s X_s^* N_s)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s (e^{\rho s} \lambda H_s X_s^* N_s)^{-\frac{1}{\gamma}} \right) ds + \lambda W_0. \quad (59)$$

Since $\lambda^G X_t^G$ is a decreasing process that starts at λ^G and always stays below $\xi^{-\gamma}$, the Skorohod equation³¹ implies that there exists another decreasing process $\lambda^G X_t^{*G}$ that also starts at λ^G and stays below $\xi^{-\gamma}$, with the property

$$\lambda^G X_t^G \leq \lambda^G X_t^{*G}. \quad (60)$$

This process is given by $X_t^{*G} = \min \left[1, \frac{\xi^{-\gamma} / \lambda^G}{\max_{0 \leq s \leq t} (e^{\rho s} H_s)} \right]$. Note that X_t^{*G} is identical to X_t^* with the

³¹For the Skorohod equation see Karatzas and Shreve (1991) p. 210.

exception that λ replaces λ^G . Using (60) and the definition of N_t yields

$$N_t = \frac{\lambda^G X_t^G}{\lambda X_t^*} \leq \frac{\lambda^G X_t^{*G}}{\lambda X_t^{**}}. \quad (61)$$

Using (61) and (59) leads to

$$V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} A(s) ds + \lambda W_0, \quad (62)$$

where

$$A(s) \equiv \max_{N_s \leq Q_s} \left(\tilde{A}(s) \right), \quad (63)$$

and $\tilde{A}(s)$ is defined as $\tilde{A}(s) \equiv \frac{(e^{\rho s} \lambda H_s X_s^* N_s)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda e^{\rho s} H_s (e^{\rho s} \lambda H_s X_s^* N_s)^{-\frac{1}{\gamma}}$, while $Q_s \equiv \max \left[1, \frac{\lambda^G X_s^{*G}}{\lambda X_s^{**}} \right]$.

To study the maximization problem of equation (63) it is useful to compute the derivative of \tilde{A}_s with respect to N_s . Performing this computation and combining terms gives

$$\frac{\partial \tilde{A}_s}{\partial N_s} = -\frac{1}{\gamma} (e^{\rho s} \lambda H_s X_s^* N_s)^{1-\frac{1}{\gamma}} N_s^{-1} \left(1 - \frac{1}{N_s X_s^*} \right). \quad (64)$$

At this stage it is useful to consider two cases separately. The first case is $\lambda > \lambda^G$. In this case, it is straightforward to show that $Q_s = 1$. Hence in maximizing $\tilde{A}(s)$, one can constrain attention to values of $N_s \leq 1$. An examination of (64) reveals that $\frac{\partial \tilde{A}(s)}{\partial N_s} \geq 0$ for all $N_s \leq 1$ and all X_s^* , since $X_s^* \leq 1$. Hence the solution to (63) is $N_s = 1$ when $\lambda > \lambda^G$.

In the case where $\lambda < \lambda^G$ it is also true that the optimal N_s in (63) is equal to one. To see this, observe that

$$Q_s = \begin{cases} \frac{\lambda^G X_s^{*G}}{\lambda X_s^{**}} & \text{when } X_s^* = 1 \\ 1 & \text{when } X_s^* < 1 \end{cases}$$

Using this observation in (64) reveals that the optimal choice for N_s is always equal to 1.³²

The above reasoning shows that the optimal solution of (63) is given by $N_s = 1$. Returning to (62), this implies that

$$V(W_0) \leq E \int_0^\infty e^{-(\rho+q)s} \left(\frac{(e^{\rho s} \lambda H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda e^{\rho s} H_s (e^{\rho s} \lambda H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) + \lambda W_0.$$

³²To see this distinguish cases. When $X_s^* = 1$, then solving $\frac{\partial \tilde{A}(s)}{\partial N_s} = 0$ gives $N_s = 1 \leq Q_s$. Hence N_s is the unique interior solution. When $X_s^* < 1$, then $\frac{\partial \tilde{A}(s)}{\partial N_s} > 0$ for all $N_s \leq Q_s = 1$. Hence the solution is given by the corner $N_s = Q_s = 1$.

Since this bound holds for arbitrary $\lambda \in (0, \xi^{-\gamma}]$ and arbitrary $G_t \in \mathcal{G}$, it also holds for the $\lambda \in (0, \xi^{-\gamma}]$ that minimizes the right hand side of the above equation and the $G_t \in \mathcal{G}$ that maximizes the right hand side. Hence (56) follows. ■

The next part of the proof of Proposition 2 is to show that equation (29) holds. A first step is to show that (29) provides an upper bound to $J(W_0)$.

Lemma 5 *The value function of problem 2 is bounded above by*

$$J(W_0) \leq \min_{\lambda \in (0, \xi^{-\gamma}]} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds + \lambda W_0 \right) \right]. \quad (65)$$

Proof of Lemma 5. The proof of this Lemma follows identical steps to the proof of the previous Lemma. To see this, take an arbitrary triplet $\langle \hat{\lambda}, X_t, c_t \rangle$ that satisfies equations (21)-(23) of Problem 2. Then for any $\lambda > 0$, one obtains

$$J(W_0) \leq E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\hat{\lambda} e^{\rho s} H_s X_s)^{1-\frac{1}{\gamma}}}{1-\gamma} - \lambda \int_0^\infty e^{-qs} H_s (\hat{\lambda} e^{\rho s} H_s X_s)^{-\frac{1}{\gamma}} + \lambda W_0 \right)$$

Notice that this equation is identical to equation (58), with the exception that λ^G is replaced by $\hat{\lambda}$ and X_t^G is replaced by X_t . Since the equations following (58) hold for any λ^G, X_t^G they also hold for $\hat{\lambda}, X_t$. Accordingly, by repeating the same steps, one can arrive at (65). ■

The next step in the proof of the proposition is to show that the inequality in (65) holds with equality for the optimal policy. The following Lemma presents a step in this direction.

Lemma 6 *Let $F(\lambda)$ be given by*

$$F(\lambda) = E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(\lambda e^{\rho s} H_s X_s^*)^{1-\frac{1}{\gamma}}}{1-\gamma} ds - \lambda \int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) \quad (66)$$

Then

$$F(\lambda) = -\frac{K \xi^{1-\gamma}}{\gamma \phi (\phi - 1)} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^\phi + K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}} \quad (67)$$

Assume moreover that (27) is met. Then

$$\min_{\lambda \in (0, \xi^{-\gamma}]} [F(\lambda) + \lambda W_0] = \min_{\lambda > 0} [F(\lambda) + \lambda W_0] \quad (68)$$

and (65) can be rewritten as $J(W_0) \leq \min_{\lambda>0} [F(\lambda) + \lambda W_0]$. Moreover, letting λ^* be given as $\lambda^* \equiv \arg \min_{\lambda>0} [F(\lambda) + \lambda W_0]$ implies that $E_0 \left[\int_0^\infty e^{-qs} H_s (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} \right] = W_0$, and accordingly $c_s^* = (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem 2.

Proof of Lemma 6. To save notation, let

$$Z_t \equiv \lambda e^{\rho t} H_t X_t^*, \quad (69)$$

and note that $Z_0 = \lambda$, and that $Z_t \in (0, \xi^{-\gamma}]$ by the definition of X_t^* in equation (28). Equation (66) can now be rewritten as

$$F(\lambda) = E \left[\int_0^\infty e^{-(\rho+q)s} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds - \int_0^\infty e^{-(\rho+q)s} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds \right]. \quad (70)$$

It will be convenient to compute the two terms inside equation (70) separately. Define first

$$G(Z_t) \equiv E \left[\int_t^\infty e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t \right]. \quad (71)$$

To compute $G(Z_t)$, it is easiest to let τ^ε be the first hitting time of Z_t to the level $\varepsilon > 0$, namely $\tau^\varepsilon \equiv \inf_{s \geq t} \{Z_s = \varepsilon\}$, and then compute the expression:

$$G^\varepsilon(Z_t) = E \left[\int_t^{\tau^\varepsilon} e^{-(\rho+q)s} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds | Z_t \right]. \quad (72)$$

To compute (72), apply first Ito's Lemma to (69) to obtain $\frac{dZ_t}{Z_t} = (\rho - r) dt - \kappa dB_t + \frac{dX_t^*}{X_t^*}$. Next, construct a function $G^\varepsilon(Z)$ that satisfies the ODE

$$\frac{\kappa^2}{2} G_{ZZ}^\varepsilon Z^2 + G_Z^\varepsilon Z (\rho - r) - (\rho + q) G^\varepsilon + \frac{1}{1-\gamma} (Z)^{1-\frac{1}{\gamma}} = 0, \quad (73)$$

subject to the boundary conditions $G_Z^\varepsilon(\xi^{-\gamma}) = 0, G^\varepsilon(\varepsilon) = 0$.

Equation (73) is a linear ordinary differential equation with general solution

$$G^\varepsilon(Z) = C_1 Z^{\phi^-} + C_2 Z^\phi + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}},$$

where C_1, C_2 are arbitrary constants, K is given in equation (26), $\phi > 0$ in (25), and ϕ^- is given by

$$\phi^- \equiv \frac{-\left(\rho - r - \frac{\kappa^2}{2}\right) - \sqrt{\left(\rho - r - \frac{\kappa^2}{2}\right)^2 + 2(\rho + q)\kappa^2}}{\kappa^2} < 0 \quad (74)$$

To satisfy the two boundary conditions $G_Z^\varepsilon(\xi^{-\gamma}) = 0$, $G^\varepsilon(\varepsilon) = 0$, the constants C_1 and C_2 must be chosen so that

$$\phi^- C_1 (\xi^{-\gamma})^{\phi^-} + \phi C_2 (\xi^{-\gamma})^\phi - \frac{1}{\gamma} K (\xi^{-\gamma})^{1-\frac{1}{\gamma}} = 0, \quad C_1 \varepsilon^{\phi^-} + C_2 \varepsilon^\phi + K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}} = 0.$$

Solving this system yields:

$$C_2 = \frac{K \left[\frac{1}{\gamma \phi^-} (\xi^{-\gamma})^{1-\frac{1}{\gamma}-\phi^-} \varepsilon^{\phi^-} + \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}} \right]}{\frac{\phi}{\phi^-} (\xi^{-\gamma})^{\phi-\phi^-} \varepsilon^{\phi^-} - \varepsilon^\phi}, \quad C_1 = -C_2 \varepsilon^{\phi-\phi^-} - K \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\phi^-}.$$

It remains now to verify that $G^\varepsilon(Z_t)$ satisfies (72). To this end, apply Ito's Lemma to $e^{-(\rho+q)t} G^\varepsilon(Z_t)$ to obtain for any time $T \wedge \tau^\varepsilon$

$$\begin{aligned} e^{-(\rho+q)T} G^\varepsilon(Z_{T \wedge \tau^\varepsilon}) - e^{-(\rho+q)t} G^\varepsilon(Z_t) &= \int_t^{T \wedge \tau^\varepsilon} \left(\frac{\kappa^2}{2} G_{ZZ}^\varepsilon Z_s^2 + G_Z^\varepsilon Z_s (\rho - r) - (\rho + q) G^\varepsilon \right) e^{-(\rho+q)s} ds \\ &\quad - \int_t^{T \wedge \tau^\varepsilon} e^{-(\rho+q)s} \kappa G_Z^\varepsilon Z_s dB_s + \int_t^{T \wedge \tau^\varepsilon} e^{-(\rho+q)s} G_Z^\varepsilon (\xi^{-\gamma}) \xi^{-\gamma} \frac{dX_s^*}{X_s^*}. \end{aligned}$$

Using (73) inside the first term on the right hand side of the above equation along with $G_Z^\varepsilon(\xi^{-\gamma}) = 0$ inside the third term, letting $T \rightarrow \infty$ along with $G^\varepsilon(\varepsilon) = 0$, and using the monotone convergence theorem gives

$$G^\varepsilon(Z_t) = E_t \left[\int_t^{\tau^\varepsilon} e^{-(\rho+q)(s-t)} \frac{1}{1-\gamma} (Z_s)^{1-\frac{1}{\gamma}} ds + \int_t^{\tau^\varepsilon} e^{-(\rho+q)(s-t)} \kappa G_Z^\varepsilon Z_s dB_s \right]. \quad (75)$$

Since $G_Z^\varepsilon Z$ is bounded between t and τ^ε , the second term in the above expression is a martingale and hence (94) follows. Next, letting $\varepsilon \rightarrow 0$, it is straightforward to show that

$$C_2 = \frac{K \left[\frac{1}{\gamma \phi^-} (\xi^{-\gamma})^{1-\frac{1}{\gamma}-\phi^-} - \frac{1}{1-\gamma} \varepsilon^{1-\frac{1}{\gamma}-\phi^-} \right]}{\frac{\phi}{\phi^-} (\xi^{-\gamma})^{\phi-\phi^-} - \varepsilon^{\phi-\phi^-}} \rightarrow K \frac{1}{\gamma \phi} (\xi^{-\gamma})^{1-\frac{1}{\gamma}-\phi},$$

since $\varepsilon^{\phi-\phi^-} \rightarrow 0$ and $\varepsilon^{1-\frac{1}{\gamma}-\phi^-} \rightarrow 0$. By a similar argument it is easy to show that $C_1 \rightarrow 0$ and hence:

$$\lim_{\varepsilon \rightarrow 0} G^\varepsilon(Z) = G(Z) = \frac{1}{\phi} \frac{1}{\gamma} K \xi^{1-\gamma} \left(\frac{Z}{\xi^{-\gamma}} \right)^\phi + K \frac{1}{1-\gamma} Z^{1-\frac{1}{\gamma}}. \quad (76)$$

Equation (71) follows as a consequence of the monotone convergence theorem.

It remains to compute the expression

$$N(Z_t, X_t^*) = E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds \right). \quad (77)$$

Following similar steps as for $G(Z_t)$, $N(Z, X^*)$ is given by

$$N(Z, X^*) = \frac{1}{(\phi-1)\gamma} \frac{1}{K} \frac{K(\xi^{-\gamma})^{1-\frac{1}{\gamma}}}{X^*} \left(\frac{Z}{\xi^{-\gamma}} \right)^\phi + K \frac{Z^{1-\frac{1}{\gamma}}}{X^*}. \quad (78)$$

It is now possible to compute $F(\lambda)$ which is given by

$$F(\lambda) = G(\lambda) - N(\lambda, 1) = -\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^\phi + K \frac{\gamma}{1-\gamma} \lambda^{1-\frac{1}{\gamma}}. \quad (79)$$

To show the second part of the proposition, observe that (77), (69) and (78) imply that

$$\begin{aligned} \frac{N(\lambda, 1)}{\lambda} &= \frac{1}{\lambda} E_0 \left(\int_0^\infty e^{-(\rho+q)s} \frac{Z_s^{1-\frac{1}{\gamma}}}{X_s^*} ds \right) = E_0 \left(\int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) = \\ &= \frac{K\xi^{1-\gamma}}{(\phi-1)\gamma} \frac{1}{\lambda} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^\phi \frac{1}{\lambda} + K\lambda^{-\frac{1}{\gamma}}. \end{aligned} \quad (80)$$

Moreover, computing $F'(\lambda)$ in (79) yields

$$F'(\lambda) = -\frac{K\xi^{1-\gamma}}{(\phi-1)\gamma} \frac{1}{\lambda} \left(\frac{\lambda}{\xi^{-\gamma}} \right)^\phi \frac{1}{\lambda} - K\lambda^{-\frac{1}{\gamma}}. \quad (81)$$

Combining (80) and (81) leads to

$$F'(\lambda) = -\frac{N(\lambda, 1)}{\lambda} = -E_0 \left(\int_0^\infty e^{-qs} H_s (\lambda e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right). \quad (82)$$

Using the formula for $F(\lambda)$, equation (65) can be expressed as $\min_{\lambda \in (0, \xi^{-\gamma}]} \{F(\lambda) + \lambda W_0\}$, which leads to the first order condition $F'(\lambda^*) = -W_0$. Using (82) leads to

$$W_0 = E_0 \left(\int_0^\infty e^{-qs} H_s (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) = E_0 \left(\int_0^\infty e^{-qs} H_s c_s^* ds \right).$$

This last equation implies that λ^* , X_t^* and the associated consumption process $c_t^* = (\lambda^* e^{\rho t} H_t X_t^*)^{-\frac{1}{\gamma}}$ satisfy (21) and (23). To show that the choice $\langle \lambda^*, X_t^*, c_t^* \rangle$ constitutes a feasible triplet, it remains to show that it also satisfies (22). By construction of X_t^* this will be the case as long as $\lambda^* < \xi^{-\gamma}$. This will indeed be the case as long as W_0 satisfies (27). To see this, note that $\xi^{-\gamma}$ is the unique

solution of $F'(\lambda^*) = -W_0$, when W_0 is given by $W_0 = \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} K\xi$. Moreover, equation (81) implies that:

$$\begin{aligned} F''(\lambda) &= -K (\xi^{-\gamma})^{1-\frac{1}{\gamma}} \frac{1}{\gamma} \left(\frac{1}{\xi^{-\gamma}} \right)^\phi \lambda^{\phi-2} + \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1} \\ &= \frac{1}{\gamma} K \lambda^{-\frac{1}{\gamma}-1} \left[1 - \left(\frac{\lambda}{\xi^{-\gamma}} \right)^{\phi+\frac{1}{\gamma}-1} \right] > 0. \end{aligned} \quad (83)$$

The above equation shows that $F'(\lambda)$ is an increasing function of λ for $0 < \lambda < \xi^{-\gamma}$ and hence the solution λ^* of equation $F'(\lambda^*) = -W_0$ is a decreasing function of W_0 . Hence, as long as W_0 satisfies (27), then $\lambda^* < \xi^{-\gamma}$. Since the interior solution λ^* is smaller than $\xi^{-\gamma}$, equation (68) follows. ■

Combining the above Lemma with (65) implies that

$$\begin{aligned} J(W_0) &\leq \min_{\lambda > 0} [F(\lambda) + \lambda W_0] = F(\lambda^*) + \lambda^* W_0 = \\ &= E \left(\int_0^\infty e^{-(\rho+q)s} \frac{\left((\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} \right)^{1-\gamma}}{1-\gamma} ds \right) \\ &= E \left(\int_0^\infty e^{-(\rho+q)s} \frac{(c_s^*)^{1-\gamma}}{1-\gamma} ds \right) \leq J(W_0). \end{aligned}$$

The last inequality follows because $c_s^* = (\lambda^* e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}}$ is a feasible consumption process for problem for problem 2 and $J(W_0)$ is the value function of the problem. The above three lines imply that equation (65) holds with equality as long as one chooses the optimal solution in the statement of the proposition. This concludes the proof of Proposition 2. ■

Proof of Proposition 3. The proof of this Proposition is just a special case of Section 6 in He and Pages (1993) and hence I give only a sketch and refer the reader to He and Pages (1993) for details. To start, define

$$\tilde{V}(\lambda) = \min_{X_s \in \mathcal{D}} E \left[\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s y_0 ds \right]. \quad (84)$$

By equation (8) and equation (17) of Proposition 1

$$V(W_0) = \min_{\lambda > 0} \left[\tilde{V}(\lambda) + \lambda \left(W_0 - \frac{y_0}{r+q} \right) \right], \quad (85)$$

since $y_0 E \int_0^\infty H_s ds = \frac{y_0}{r}$. Next, for an arbitrary decreasing process X_t let Z_t be defined as $Z_t \equiv$

$\lambda e^{\rho s} H_s X_s$, and note that $Z_0 = \lambda$. Applying Ito's Lemma to Z_t gives:

$$\frac{dZ_t}{Z_t} = (\rho - r) dt - \kappa dB_t + \frac{dX_t}{X_t}. \quad (86)$$

With this definition of Z_t one can solve the maximization problem inside (84) and rewrite $\tilde{V}(\lambda)$ as

$$\tilde{V}(Z_0) = \min_{X_s \in \mathcal{D}} E \left[\int_0^\infty e^{-(\rho+q)s} \left(\frac{\gamma}{1-\gamma} Z_s^{1-\frac{1}{\gamma}} + y_0 Z_s \right) ds \right] \quad (87)$$

From this point on, one can use similar arguments to He and Pages (1993), and treat (87) as a singular stochastic control problem over the set of decreasing processes X_t . As He and Pages (1993) show, the optimal solution is to always decrease X_t appropriately, so as to keep Z_t in the interval $(0, \bar{Z}]$. \bar{Z} is a free boundary that is determined next.

Using this conjecture for the optimal policy one can now proceed as He and Pages (1993) to establish that $\tilde{V}(Z)$ satisfies the ordinary differential equation:

$$\frac{\kappa^2}{2} \tilde{V}_{ZZ} Z^2 + (\rho - r) \tilde{V}_Z Z - (\rho + q) \tilde{V} + \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + y_0 Z = 0 \text{ for all } Z \in (0, \bar{Z}].$$

The general solution to this equation is

$$\tilde{V}(Z) = C_1 Z^\phi + C_2 Z^{\phi^-} + K \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q} Z, \quad (88)$$

where K is given in (26), ϕ in (25) and ϕ^- in (74) and C_1, C_2 are arbitrary constants. By arguments similar to He and Pages (1993), one can set $C_2 = 0$ (since $\phi^- < 0$). Hence it remains to determine C_1 and the free boundary \bar{Z} . As most singular stochastic control problems, one can employ a ‘‘smooth pasting’’ and ‘‘high contact’’ principle, namely by determining C_1 and \bar{Z} so that $\tilde{V}_Z(\bar{Z}) = 0$, $\tilde{V}_{ZZ}(\bar{Z}) = 0$. Using the ‘‘smooth pasting’’ and ‘‘high contact’’ conditions, along with the general solution in (88) and $C_2 = 0$, one can solve for C_1 and \bar{Z} to obtain

$$\bar{Z}^{-\frac{1}{\gamma}} = \frac{1}{K} \frac{y_0}{r+q} \left(\frac{\phi - 1}{\frac{1}{\gamma} + \phi - 1} \right) \quad (89)$$

$$C_1 = - \frac{\frac{1}{\gamma} \frac{y_0}{r+q}}{\phi \bar{Z}^{\phi-1} \left[\frac{1}{\gamma} + \phi - 1 \right]} \quad (90)$$

The next steps to verify that the conjectured policy is indeed optimal are identical to He and Pages (1993) and are left out.

To conclude the proof, note that sofar the calculations were true for an arbitrary y_0 . To determine the y_0 that will safeguard that $c_t \geq \xi$ observe that $c_t = Z^{-\frac{1}{\gamma}}$ by equation (16). Since the optimal policy is to control X_t so as to “keep” Z_t in the interval $(0, \bar{Z}]$ it follows that the minimum level of consumption is given by $\bar{Z}^{-\frac{1}{\gamma}}$. Hence, in order to guarantee condition $c_t \geq \xi$ it suffices to determine y_0 so that

$$\xi = \bar{Z}^{-\frac{1}{\gamma}} = \frac{1}{K} \frac{y_0}{r+q} \left(\frac{\phi-1}{\frac{1}{\gamma} + \phi - 1} \right).$$

Solving for y_0 gives

$$y_0 = \xi(r+q)K \frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1}.$$

One can now substitute that level of y_0 into (90), (89) and use the resulting expressions to obtain from (88) the following expression for $\tilde{V}(Z)$:

$$\tilde{V}(Z) = -\frac{K\xi^{1-\gamma}}{\gamma\phi(\phi-1)} \left(\frac{Z}{\xi^{-\gamma}} \right)^\phi + K \frac{\gamma}{1-\gamma} Z^{1-\frac{1}{\gamma}} + \frac{y_0}{r+q} Z.$$

Evaluating this expression at $Z_0 = \lambda$ and using equation (85) gives equation (30), which shows that the “constant income” policy of the current proposition attains the upper bound of Proposition 2.

■

Proof of Lemma 2. First note that $\lim_{\gamma \rightarrow \infty} \left(\frac{y_0}{\xi} \right) = 1$. To show the result, it suffices to show that $\frac{d\left(\frac{y_0}{\xi}\right)}{d\gamma} < 0$. Differentiating $\frac{y_0}{\xi}$ with respect to γ gives

$$\frac{d\left(\frac{y_0}{\xi}\right)}{d\gamma} = \frac{(r+q)}{\phi-1} \frac{B}{\left(\frac{\gamma-1}{\gamma} \frac{\kappa^2}{2} + \gamma(r+q) + \rho - r\right)^2},$$

where

$$\begin{aligned} B &\equiv (\phi-1)(\rho-r) - (r+q) + (\phi-1) \frac{\kappa^2}{2} \\ &- (\phi-1) \frac{1}{\gamma} \frac{\kappa^2}{2} - [\gamma(\phi-1) + 1] \frac{1}{\gamma^2} \frac{\kappa^2}{2}. \end{aligned}$$

Since $\phi > 1$ and $r+q > 0$, it follows that $\frac{d\left(\frac{y_0}{\xi}\right)}{d\gamma} < 0$, as long as $(\phi-1)(\rho-r) - (r+q) + (\phi-1) \frac{\kappa^2}{2} < 0$. Since ϕ solves the quadratic equation $\frac{\kappa^2}{2} \phi^2 + \left(\rho - r - \frac{\kappa^2}{2}\right) \phi - (\rho+q) = 0$, it follows

that $(\phi - 1)(\rho - r) - (r + q) + (\phi - 1)\frac{\kappa^2}{2} = -(\phi - 1)^2\frac{\kappa^2}{2} < 0$. ■

Proof of Proposition 4. The proof of this proposition proceeds in steps. The first two Lemmas establish that the proposed transfer policy will make it possible for an agent who follows the optimal consumption process of proposition 4 to satisfy the intertemporal budget constraint. The proof then continues to show that the wealth process associated with the optimal consumption process of proposition 4, along with the portfolio process (34), will lead to non-negative levels of wealth at all times. Finally, it is shown that the consumption policy of proposition 4, along with the portfolio choice (34), are optimal for an agent who is faced with transfers given by (32) and attain the upper bound of proposition 2.

Lemma 7 *Let K and ϕ be given by (26) and (25) and for any $0 < \lambda < \xi^{-\gamma}$ let $Z_t = \lambda e^{\rho s} H_s X_s^*$. Then*

$$\int_0^\infty E_t \left(\int_t^\infty e^{-q(s-t)} H_s X_s^* dG_s - \int_t^\infty e^{-q(s-t)} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right) dX_t^* = 0. \quad (91)$$

Proof of Lemma 7. It will simplify notation to let

$$\eta \equiv -K\xi \left(\phi - 1 + \frac{1}{\gamma} \right). \quad (92)$$

The first step is to compute

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \frac{E_t \int_t^\infty e^{-qs} H_s dX_s^*}{e^{-qt} H_t X_t^*}. \quad (93)$$

Applying integration by parts and using the definition of Z_t gives

$$E_t \left(\int_t^\infty e^{-qs} H_s dX_s^* \right) = \frac{1}{\lambda} \left[-e^{-(\rho+q)t} Z_t + E_t \left(\int_t^\infty (r+q) e^{-(\rho+q)s} Z_s ds \right) \right]. \quad (94)$$

Using (94) in equation (93) gives

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = \eta \left[(r+q) \frac{E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right)}{Z_t} - 1 \right]. \quad (95)$$

By using a logic similar to equations (73)-(75),

$$E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s ds \right) = -\frac{1}{\phi} \frac{\xi^{-\gamma}}{r+q} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^\phi + \frac{1}{r+q} Z_t, \quad (96)$$

where ϕ is defined in equation (25). Plugging back (96) into (95) gives

$$\frac{E_t \int_t^\infty e^{-qs} H_s X_s^* dG_s}{e^{-qt} H_t X_t^*} = -\frac{\eta}{\phi} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1}. \quad (97)$$

To conclude the proof, note that equations (71) and (76) imply that

$$\frac{E_t \left(\int_t^\infty e^{-qs} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right)}{e^{-qt} H_t X_t^*} = \frac{E_t \left(\int_t^\infty e^{-(\rho+q)(s-t)} Z_s^{1-\frac{1}{\gamma}} ds \right)}{Z_t} = \frac{\frac{1}{\phi} \frac{1-\gamma}{\gamma} K \xi^{1-\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^\phi + K Z_t^{1-\frac{1}{\gamma}}}{Z_t}. \quad (98)$$

Combining (98) with (97) gives:

$$\begin{aligned} & \frac{E_t \left(\int_t^\infty e^{-qs} H_s X_s^* dG_s - \int_t^\infty e^{-qs} H_s X_s^* Z_s^{-\frac{1}{\gamma}} ds \right)}{e^{-qt} H_t X_t^*} = \\ & = -\frac{\eta}{\phi} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} - \frac{\frac{1}{\phi} \frac{1-\gamma}{\gamma} K \xi^{1-\gamma} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^\phi + K Z_t^{1-\frac{1}{\gamma}}}{Z_t}. \end{aligned}$$

Since $dX_t^* \neq 0$ when and only when $Z_t = \xi^{-\gamma}$, equation (91) amounts to checking that:

$$-\frac{\eta}{\phi} - \left(\frac{1}{\phi} \frac{1-\gamma}{\gamma} + 1 \right) K \xi = 0$$

which follows easily from the definition of η . ■

Lemma 8 *Let Z_s be as in the statement of the proposition 4 and let G_t be as in (32). Then the consumption policy:*

$$c_s^* = (Z_s)^{-\frac{1}{\gamma}} \quad (99)$$

satisfies:

$$E \int_0^\infty e^{-qs} H_s X_s^* c_s^* ds = W_0 + \int_0^\infty e^{-qs} H_s (X_s^* - 1) dG_s \quad (100)$$

Proof of Lemma 8. Taking any $\lambda \in (0, \xi^{-\gamma}]$, using the definition of X_t^* , and equation (91), the same reasoning behind (46) leads to

$$E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s^* c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s (X_s^* - 1) dG_s \right) + \lambda W_0 = (101)$$

$$= E \left[\int_0^\infty e^{-(\rho+q)s} \frac{\gamma}{1-\gamma} (e^{\rho s} \lambda H_s X_s^*)^{\frac{\gamma-1}{\gamma}} ds + \int_0^\infty e^{-(\rho+q)s} (e^{\rho s} \lambda H_s X_s^*)^{1-\frac{1}{\gamma}} \left(1 - \frac{1}{X_s^*}\right) ds \right] + \lambda W_0 \quad (102)$$

Hence the λ^* that minimizes (30) (and hence minimizes [102]) also minimizes (101). But since λ minimizes (101), the same argument as in He and Pages (1993) (Proof of Theorem 1) leads to (100). ■

Lemma 8 has asserted that the consumption policy (99) satisfies the intertemporal budget constraint (100). It remains to show that this consumption policy along with the portfolio policy (34) will lead to a process for financial wealth that satisfies $W_t \geq 0$. To that end let η be given as in (92) and define:

$$W^*(Z_t) = -K (\xi^{-\gamma})^{-\frac{1}{\gamma}} \left(\frac{Z_t}{\xi^{-\gamma}} \right)^{\phi-1} + K Z_t^{-\frac{1}{\gamma}} \quad (103)$$

It is straightforward to verify the following facts about $W^*(Z_t)$:

$$\frac{\kappa^2}{2} Z^2 W_{ZZ}^* + (\rho - r + \kappa^2) Z W_Z^* - (r + q) W + (Z)^{-\frac{1}{\gamma}} = 0 \quad (104)$$

$$W^*(\xi^{-\gamma}) = 0, W^*(Z) \geq 0 \text{ for all } Z \in (0, \xi^{-\gamma}] \quad (105)$$

$$W_Z^*(\xi^{-\gamma}) = -K \xi \left(\phi - 1 + \frac{1}{\gamma} \right) (\xi^{-\gamma})^{-1} = \frac{\eta}{\xi^{-\gamma}} \quad (106)$$

The next step is to verify that $W^*(Z_t)$ is the stochastic process for the financial wealth of the agent. To see this, use the definition of c_s^* (equation [99]) along with the definitions of dG_t, W_t^* (equations [32] and [103] respectively) and apply Ito's Lemma to obtain:

$$\begin{aligned} d \left(\int_0^t c_s^* ds - \int_0^t dG_s + W_t^* \right) &= \\ &= c_t^* dt - \eta \frac{dX_t^*}{X_t^*} + W_Z^* dZ_t + \frac{\kappa^2}{2} W_{ZZ}^* Z_t^2 dt \\ &= \left(c_t^* - Z_t^{-\frac{1}{\gamma}} \right) dt + [W_Z^*(\xi^{-\gamma}) \xi^{-\gamma} - \eta] \frac{dX_t^*}{X_t^*} + (r + q) W_t^* dt - \kappa^2 Z_t W_Z^* dt - \kappa W_Z^* Z_t dB_t = \\ &= (r + q) W_t^* dt - \kappa^2 Z_t W_Z^* dt - \frac{\kappa}{\sigma} W_Z^* Z_t \left(\frac{dP_t}{P_t} - \mu dt \right) \end{aligned}$$

$$\begin{aligned}
&= (r + q)W_t^* dt - \kappa^2 Z_t W_Z^* dt - \frac{\kappa}{\sigma} W_Z^* Z_t \left(\frac{dP_t}{P_t} - (\mu - r) dt - r dt \right) = \\
&= qW_t^* dt + r \left(W_t^* + \frac{\kappa}{\sigma} W_Z^* Z_t \right) dt - \frac{\kappa}{\sigma} W_Z^* Z_t \frac{dP_t}{P_t} = \\
&= qW_t^* dt + r (W_t^* - \pi_t^*) dt + \pi_t^* \frac{dP_t}{P_t}.
\end{aligned}$$

Integrating gives

$$\int_0^t c_s^* ds + W_t^* = W_0 - D_0 + \int_0^t dG_s + \int_0^t qW_s^* dt + \int_0^t r (W_t^* - \pi_t^*) dt + \int_0^t \pi_t^* \frac{dP_t}{P_t}.$$

Hence the process W_t^* satisfies the equation (10) for an agent who chooses a consumption policy given by (99) and a portfolio policy given by (34). Accordingly, it is the financial wealth process that is associated with that policy pair. Moreover, by equation (105) the financial wealth process is non-negative. Accordingly, the policies given by (99) and (34) are feasible for an agent who is faced with the transfer process (32).

Verifying the optimality of the stated policy pair is simple. According to proposition 1

$$V(W_0) = \min_{\lambda > 0, X_s \in \mathcal{D}} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s dG_s \right) \right] \leq Q(W_0),$$

where

$$Q(W_0) \equiv \min_{\lambda > 0} \left[E \left(\int_0^\infty e^{-(\rho+q)s} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho s} H_s X_s^* c_s \right) ds + \lambda \int_0^\infty e^{-qs} H_s X_s^* dG_s \right) \right].$$

One can use now Lemma 8 to illustrate that the consumption policy (99) leads to a payoff for the agent equal to $Q(W_0)$ which is an upper bound to the value function of the agent $V(W_0)$. Since the consumption policy (99) is also feasible, the payoff associated with that policy also provides a lower bound to the value function $V(W_0)$. Hence this policy must be optimal. Finally, the easiest way to show that

$$D_0 = K \xi^{\frac{1}{\gamma} + \phi - 1} \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi - 1},$$

is to observe that the intertemporal budget constraint implies that

$$E_{\tau_0} \left(\int_{\tau_0}^\infty e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} c_s^* ds \right) = E_{\tau_0} \left(\int_{\tau_0}^\infty e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} dG_s \right),$$

where τ_0 is the first time that $X_{\tau_0} \geq 1$ (or equivalently the first time that $W_{\tau_0} = 0$ and $\lambda^* e^{\rho\tau_0} H_{\tau_0} = \xi^{-\gamma}$). A few manipulations can be used to show that

$$E_{\tau_0} \left(\int_{\tau_0}^{\infty} e^{-q(s-\tau_0)} \frac{H_s}{H_{\tau_0}} c_s^* ds \right) = \frac{N(\xi^{-\gamma}, 1)}{\xi^{-\gamma}} = K \xi^{\frac{1}{\gamma} + \phi - 1}$$

where N is defined and computed in (78) and (77). Finally, since there are no transfers between 0 and τ_0 :

$$\begin{aligned} D_0 &= E(e^{-q\tau_0} H_{\tau_0}) K \xi^{\frac{1}{\gamma} + \phi - 1} = \frac{1}{\lambda^*} E(e^{-(\rho+q)\tau_0} \lambda^* e^{\rho\tau_0} H_{\tau_0}) K \xi^{\frac{1}{\gamma} + \phi - 1} = \\ &= \frac{\xi^{-\gamma}}{\lambda^*} E(e^{-(\rho+q)\tau_0}) K \xi^{\frac{1}{\gamma} + \phi - 1} = \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi-1} K \xi^{\frac{1}{\gamma} + \phi - 1} \end{aligned}$$

where the proof of $E(e^{-(\rho+q)\tau_0}) = \left(\frac{\lambda^*}{\xi^{-\gamma}} \right)^{\phi}$ is identical to the one given in Oksendal (1998), Chapter 10. ■

Proof of Proposition 5. Take any transfer process G_t such that the resulting consumption process of the agent satisfies $c_t \geq \xi$. Proposition 1 implies then that there exists a cumulative multiplier process X_t^G and a constant λ^G such that $c_t = (\lambda^G e^{\rho t} H_t X_t^G)^{-\frac{1}{\gamma}} \geq \xi$. Letting $X_t^* \equiv \min \left[1, \frac{\xi^{-\gamma} \lambda^G}{\max_{0 \leq s \leq t} (e^{\rho s} H_s)} \right]$, and $P \equiv E \left(\int_0^{\infty} e^{-qs} H_s c_s ds \right)$ gives

$$P = E \left(\int_0^{\infty} e^{-qs} H_s (\lambda^G e^{\rho s} H_s X_s^G)^{-\frac{1}{\gamma}} ds \right) \geq E \left(\int_0^{\infty} e^{-qs} H_s (\lambda^G e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) \quad (107)$$

since³³ $X_s^* (\lambda^G) \geq X_s^G$. Equation (80) implies that

$$E \left(\int_0^{\infty} e^{-qs} H_s (\lambda^G e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) = \frac{K \xi^{1-\gamma}}{(\phi-1) \gamma} \left(\frac{\lambda^G}{\xi^{-\gamma}} \right)^{\phi} \frac{1}{\lambda^G} + K (\lambda^G)^{-\frac{1}{\gamma}}.$$

Combining (82) and (83) implies that the right hand side of the above equation is decreasing in λ^G whenever $\lambda^G \leq \xi^{-\gamma}$. Since $c_0 = (\lambda^G)^{-\frac{1}{\gamma}} \geq \xi$ this implies furthermore

$$\begin{aligned} E \left(\int_0^{\infty} e^{-qs} H_s (\lambda^G e^{\rho s} H_s X_s^*)^{-\frac{1}{\gamma}} ds \right) &\geq \frac{K \xi^{1-\gamma}}{(\phi-1) \gamma} \frac{1}{\xi^{-\gamma}} + K \xi = K \xi \left(1 + \frac{1}{\phi-1} \frac{1}{\gamma} \right) \\ &= K \xi \left(\frac{\frac{1}{\gamma} + \phi - 1}{\phi - 1} \right). \end{aligned} \quad (108)$$

Combining (107) and (108) concludes the proof. ■

³³This is an implication of the Skorohod equation. See Karatzas and Shreve (1991).

Proof of Proposition 6. First note that a marginal increase in the minimum savings rate χ in each period prior to retirement can raise the agents' minimum assets by

$$\omega = Y \int_{-\tau}^0 e^{-(r+q)s} ds = Y \frac{e^{(r+q)\tau} - 1}{r + q}.$$

By an argument similar to Proposition 1, the agent's value function at birth (time $-\tau$) can be rewritten as

$$F = \min_{\tilde{X}_s, \lambda > 0} E_{(-\tau)} \left[\begin{array}{l} \int_{-\tau}^0 e^{-(\rho+q)(s+\tau)} \max_{c_s} \left(\frac{c_s^{1-\gamma}}{1-\gamma} - \lambda e^{\rho(s+\tau)} \tilde{X}_s \frac{H_s}{H_{(-\tau)}} c_s \right) ds \\ + \lambda(1-\chi)Y \int_{-\tau}^0 e^{-q(s+\tau)} \frac{H_s}{H_{(-\tau)}} \tilde{X}_s ds + \\ \max_{W_{0+} \geq 0} \left(e^{-(\rho+q)\tau} J(W_{0+} + \chi\omega) - \lambda \tilde{X}_0 e^{-q\tau} \frac{H_0}{H_{(-\tau)}} W_{0+} \right) \end{array} \right], \quad (109)$$

where $J(W_{0+} + \chi\omega)$ is given in proposition 2 and \tilde{X}_s is a decreasing process starting at $\tilde{X}_{(-\tau)} = 1$. Let the expected value of the expression inside the square brackets be denoted as $U(\tilde{X}_s, \lambda)$, so that

$$F(\chi) = \min_{\tilde{X}_s, \lambda} U(\tilde{X}_s, \lambda; \chi).$$

Differentiating $U(\tilde{X}_s, \lambda; \chi)$ with respect to χ gives

$$U_\chi = E_{(-\tau)} \left[1_{\left\{ \lambda \tilde{X}_0 \frac{H_0}{H_{(-\tau)}} < \xi^{-\gamma} \right\}} e^{-(\rho+q)\tau} J'(W_{0+} + \chi\omega) \omega - \lambda Y \int_{-\tau}^0 e^{-q(s+\tau)} \frac{H_s}{H_{(-\tau)}} \tilde{X}_s ds \right]. \quad (110)$$

Whenever $\lambda \tilde{X}_0 \frac{H_0}{H_{(-\tau)}} \geq \xi^{-\gamma}$, so that the constraint $W_{0+} \geq 0$ does not bind, one can use the first order condition from the second maximization problem inside the square brackets of (109) to obtain

$$J'(W_{0+} + \chi\omega) = \lambda \tilde{X}_0 e^{\rho\tau} \frac{H_0}{H_{(-\tau)}}.$$

This allows one to rewrite expression (110) as

$$\begin{aligned} U_\chi &= E_{(-\tau)} \left[\left(1_{\left\{ \lambda \tilde{X}_0 \frac{H_0}{H_{(-\tau)}} < \xi^{-\gamma} \right\}} \lambda \tilde{X}_0 e^{-q\tau} \frac{H_0}{H_{(-\tau)}} \omega - \lambda Y \int_{-\tau}^0 e^{-q(s+\tau)} \frac{H_s}{H_{(-\tau)}} \tilde{X}_s ds \right) \right] \\ &\leq E_{(-\tau)} \left[\left(\lambda \tilde{X}_0 e^{-q\tau} \frac{H_0}{H_{(-\tau)}} \omega - \lambda Y \int_{-\tau}^0 e^{-q(s+\tau)} \frac{H_s}{H_{(-\tau)}} \tilde{X}_s ds \right) \right] \\ &= \lambda \delta e^{-q\tau} \omega - \lambda \delta E_{(-\tau)} \left(\int_{-\tau}^0 e^{-q(s+\tau)} Y \frac{H_s}{H_{(-\tau)}} \frac{\tilde{X}_s}{\delta} ds \right), \end{aligned} \quad (111)$$

where $\delta \equiv E_{(-\tau)} \left(\tilde{X}_0 \frac{H_0}{H_{(-\tau)}} \right)$. Furthermore,

$$\begin{aligned}
E_{(-\tau)} \left(\int_{-\tau}^0 e^{-q(s+\tau)} Y \frac{H_s}{H_{(-\tau)}} \frac{\tilde{X}_s}{\delta} ds \right) &= \int_{-\tau}^0 e^{-q(s+\tau)} Y \frac{E_{(-\tau)} \left(H_s \tilde{X}_s \right)}{E_{(-\tau)} \left(H_0 \tilde{X}_0 \right)} ds = \\
&= e^{r\tau} \int_{-\tau}^0 e^{-(r+q)(s+\tau)} Y \frac{E_{(-\tau)} \left(e^{r(s+\tau)} \frac{H_s}{H_{(-\tau)}} \tilde{X}_s \right)}{E_{(-\tau)} \left(e^{r\tau} \frac{H_0}{H_{(-\tau)}} \tilde{X}_0 \right)} ds \\
&\geq Y e^{r\tau} \int_{-\tau}^0 e^{-(r+q)(s+\tau)} ds = \omega e^{-q\tau}, \tag{112}
\end{aligned}$$

where the inequality follows from the fact that $e^{rs}H_s$ is a martingale while \tilde{X}_s is a decreasing process, so that $\tilde{X}_s \geq \tilde{X}_0$ for all $s \in [-\tau, 0]$. Combining (111) and (112) leads to $U_\chi \leq 0$.

Hence, letting χ^{\min} denote the minimum savings rate that will satisfy (27) as given by (41), it follows that $U(\tilde{X}_s, \lambda; \chi^{\min}) > U(\tilde{X}_s, \lambda; \chi)$ for all $\chi \in (\chi^{\min}, 1)$. This furthermore implies that

$$F^{(\chi^{\min})} = U(\tilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi^{\min}) \geq U(\tilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}; \chi) \geq U(\tilde{X}_s^\chi, \lambda^\chi; \chi) = F(\chi),$$

where $\tilde{X}_s^\chi, \lambda^\chi$ denote the minimizers of U given χ and similar for $\tilde{X}_s^{\chi^{\min}}, \lambda^{\chi^{\min}}$. Hence it is never optimal to set the minimum savings rate above χ^{\min} . ■

References

- Abel, A. B. (1987). *The Ricardian Equivalence Theorem*, Volume 4 of *In: John Eatwell, Murray Milgate and Peter Newman (eds.), The New Palgrave: A Dictionary of Economic Theory and Doctrine*, pp. 174–179. The Macmillan Press Ltd.
- Amador, M., I. Werning, and G.-M. Angeletos (2006). Commitment vs. flexibility. *Econometrica* 74(2), 365–396.
- Ball, L. and N. G. Mankiw (2007). Intergenerational risk sharing in the spirit of arrow, debreu, and rawls, with applications to social security design. *Journal of Political Economy* 115(4), 523 – 547.
- Barro, R. J. (1974). Are government bonds net wealth? *Journal of Political Economy* 82(6), 1095 – 1117.
- Barro, R. J. (1979). On the determination of the public debt. *Journal of Political Economy* 87(5), p. 940 – 971.
- Basak, S. and D. Cuoco (1998). An equilibrium model with restricted stock market participation. *Review of Financial Studies* 11(2), 309 – 341.
- Blanchard, O. J. (1985). Debt, deficits, and finite horizons. *Journal of Political Economy* 93(2), 223–247.
- Bodie, Z., R. C. Merton, and W. F. Samuelson (1992). Labor supply flexibility and portfolio choice in a life cycle model. *Journal of Economic Dynamics and Control* 16(3-4), 427–49.
- Bovbjerg, B. D. (2009). Private pensions: Alternative approaches could address retirement risks faced by workers but pose tradeoffs. Technical Report GAO-09-642, United States Government Accountability Office.
- Campbell, J. Y. and M. Feldstein (2001). *Risk Aspects of Investment-Based Social Security Reform*. Cambridge, MA: NBER.

- Cole, H. L. and N. R. Kocherlakota (2001). Efficient allocations with hidden income and hidden storage. *Review of Economic Studies* 68(3), 523 – 542.
- Constantinides, G. M., J. B. Donaldson, and R. Mehra (2002). Junior must pay: Pricing the implicit put in privatizing social security. National Bureau of Economic Research, Working Paper: 8906.
- Cuoco, D. (1997). Optimal consumption and equilibrium prices with portfolio constraints and stochastic income. *Journal of Economic Theory* 72(1), 33 – 73.
- Duffie, D. (2001). *Dynamic asset pricing theory*. Princeton and Oxford: Princeton University Press.
- Farhi, E. and S. Panageas (2007). Saving and investing for early retirement: A theoretical analysis. *Journal of Financial Economics* 83(1), 87 – 121.
- Feldstein, M. (2005a). Reducing the risk of investment-based social security reform. National Bureau of Economic Research, Working Paper: 11084.
- Feldstein, M. (2005b). Rethinking social insurance. *American Economic Review* 95(1), 1–24.
- Feldstein, M. and E. Rangelova (2001). Individual risk in an investment-based social security system. *American Economic Review* 91(4), 1116–1125.
- Fuster, L., A. Imrohorglu, and S. Imrohorglu (2008). *Personal Security Accounts and Mandatory Annuitization in a Dynastic Framework*. In: Fenge, R., de Menil, G. and Pestieau, P. , eds., *Strategies for Pension Reform*. Cambridge: MIT Press.
- Gallmeyer, M. and B. Hollifield (2008). An examination of heterogeneous beliefs with a short-sale constraint in a dynamic economy. *Review of Finance* 12(2), 323 – 364.
- Golosov, M. and A. Tsyvinski (2006). Designing optimal disability insurance: A case for asset testing. *Journal of Political Economy* 114(2), 257 – 279.
- Harris, C. and D. Laibson (2001). Dynamic choices of hyperbolic consumers. *Econometrica* 69(4), 935–957.
- He, H. and H. F. Pages (1993). Labor income, borrowing constraints, and equilibrium asset prices. *Economic Theory* 3(4), 663–696.

- Karatzas, I. and S. E. Shreve (1991). *Brownian motion and stochastic calculus* (Second ed.), Volume 113 of *Graduate Texts in Mathematics*. New York: Springer-Verlag.
- Karatzas, I. and S. E. Shreve (1998). *Methods of mathematical finance*. Applications of Mathematics. New York: Springer-Verlag.
- Krueger, D. and F. Kubler (2006). Pareto-improving social security reform when financial markets are incomplete!? *American Economic Review* 96(3), 737–755.
- Marcet, A. and R. Marimon (1998). Recursive contracts. UPF Working Paper.
- Mitchell, O. S. and M.-E. Lachance (2003). Guaranteeing individual accounts. *American Economic Review* 93(2), 257–260.
- Oksendal, B. (1998). *Stochastic differential equations: An introduction with applications*. Universitext. Berlin: Springer-Verlag.
- Ramsey, F. P. (1927). A contribution of the theory of taxation. *Economic Journal*, 47 – 61.
- Shoffner, D., A. G. Biggs, and P. Jacobs (2005). Poverty-level annuitization requirements in social security proposals incorporating personal retirement accounts. *U.S. Social Security Administration Office of Policy*.