Wealth Distribution under Idiosyncratic Investment Risk

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Abstract

I investigate the mechanisms through which idiosyncratic investment risk generates a fat tail of the wealth distribution. I set up a continuous time OLG model with a bequest motive and portfolio selection to show that idiosyncratic investment risk generates a Pareto tail in the wealth distribution. I calibrate my model to the U.S. economy and show that the wealth distribution in the simulated economy matches the Gini coefficient and has a Lorenz curve close to that of U.S. Extending the model to allow for the age-dependent death rates generates both realistic age cohort distribution and fat-tailed wealth distribution.

1. Introduction

The wealth distribution in U.S. has a fat tail. Wolff (2004), using 2001 Survey of Consumer Finances (SCF), documents that the top 1% of population holds 33.4% of the wealth in the economy. The wealth distribution is right-skewed. The median wealth is \$73.5 K (2001 USD), while the mean wealth is \$380.1 K (2001 USD). The mean of the wealth distribution is larger than the median. In this data, the Gini coefficient is 0.826. Among these three features of the wealth distribution: a fat tail, skewness to the right, and a high Gini coefficient, the fat tail is the most challenging one for the macroeconomists.

A standard tool of modern macroeconomics, the basic heterogeneous agent incomplete market model with idiosyncratic labor income shocks, such as Aiyagari (1994) and Huggett (1996), does not replicate the fat tail of the wealth distribution. The agent's precautionary saving to self-insure the labor income risk is not strong enough to generate the concentration of the wealth distribution.[1]

Researchers have tried different methods to tackle this challenge. Quadrini (2000) and Cagetti and De Nardi (2006, 2009) introduce entrepreneurship into the heterogeneous agent model and match the fat tail of wealth distribution. Similarly, Benhabib and Zhu (2008) and Benhabib, Bisin, and Zhu (2009) introduce the idiosyncratic investment risk into the heterogeneous agent model. Specifically these models produce a Pareto tail of the wealth distribution and then match the fat tail. Castaneda, Diaz-Gimenez, and Rios-Rull (2003) use a labor efficiency process with a large dispersion and match the fat tail. Krusell and Smith (1998) use a specific stochastic process of time discount factors and match the fat tail.

Introducing a portfolio selection problem into Benhabib, Bisin, and Zhu (2009), I set up a heterogeneous agent model with within lifetime idiosyncratic investment risk to replicate these three features of wealth distribution. I prove that the stationary wealth distribution in the model has a Pareto tail and theoretically derive the Pareto exponent. Thus it is not difficult for the model to produce the fat tail of wealth distribution in a simple calibration exercise. This way I show that the stochastic investment return is the main cause of these features.

Household wealth suffers from idiosyncratic investment risk. Households own housing, stocks, and private equity which are main channels of idiosyncratic investment risk. Wolff (2004), using 2001 SCF, documents that 67.7% of the households

[1] One common feature of these models is that the bounded risk aversion confines the wealth accumulation to be bounded. See Schechtman and Escudero (1977).

own principal residence. 16.8% own other real estate. 11.9% own unincorporated business equity. 21.3% of the household directly hold stock. In the composition of household wealth, the gross value of housing (including principal residence and other real estate) accounts for 38% of the gross assets and the net value of unincorporated business equity accounts for 17.2% of the gross assets.

The returns of these assets have a large dispersion. Case and Shiller (1989) document that individual housing prices have large standard deviation of annual percentage change, close to 15% a year. Flavin and Yamashita (2002), using the 1968-1992 waves of Panel Study of Income Dynamics (PSID), show that the housing prices have a large idiosyncratic component. The standard deviation of return of housing, at the level of the individual house, is about 0.14. Moskowitz and Vissing-Jorgensen (2002) find investment in private equity to be extremely concentrated, and therefore risky. About 75% of all private equity is owned by households for whom it constitutes at least half of their total net worth. Their estimates of the entrepreneurial returns show that substantial risk exists even conditional on survival.[2]

The model is a continuous time OLG model. The economy is populated by a continuum of agents with measure 1. The agent has finite life. When an agent dies, he gives birth to one child. The agent has a "joy of giving" bequest motive. Each agent has a portfolio selection problem. The agent can invest wealth in a risk free public account and a private investment project with idiosyncratic return risk. The value of private investment follows a Geometric Brownian motion with jump. The agent has constant labor income at any time. After analytically deriving the agent's policy functions, I do a simple calibration exercise to simulate the wealth distribution. The simulation results show that the model has a fat tail of the wealth distribution and produces a Gini coefficient and a Lorenz curve close to those of the U.S. wealth distribution.

Using the model I discuss the effects of capital income taxes and estate tax on wealth inequality and social welfare. I also illustrate that my model could be extended to include age-dependent death rates to generate a fat-tailed wealth distribution and a realistic demographic distribution.

[2]Using the 1989 SCF, they find that the median of the capital gain distribution is 6.9% per year, while the first quartile is 0 and the third quartile is 18.6% per year.

1.1. Literature review

Huggett (1996) sets up a life cycle model with idiosyncratic earnings shock to study the wealth distribution. De Nardi (2004) incorporates bequest motive and inheritance of earnings ability into a life cycle model to investigate the wealth distribution. Quadrini (2000) incorporates the death rate into a dynasty model and takes into account inheritance of earnings ability. Cagetti and De Nardi (2006, 2009) and Castaneda, Diaz-Gimenez, and Rios-Rull (2003) use probabilities of retiring and dying to mimic the life cycle pattern in dynasty models. Within these works Quadrini (2000) and Cagetti and De Nardi (2006, 2009) include the idiosyncratic investment risk into the model. Another strand of literature which emphasizes investment risk includes Benhabib and Zhu (2008) and Benhabib, Bisin, and Zhu (2009). Benhabib and Zhu (2008) is a dynasty model with probability of death, while Benhabib, Bisin, and Zhu (2009) is a life cycle model. In Benhabib, Bisin, and Zhu (2009) the agent only draws the uncertain rate of return at the beginning of life. Table 1.1 summarizes the model structure of these works in the literature.

Table 1.1: Model structure of literature

Model	Life cycle/Dynasty	Investment risk
Huggett (1996)	Life cycle	No
De Nardi (2004)	Life cycle	No
Krusell and Smith (1998)	Dynasty	No
Quadrini (2000)	Dynasty with probability of death	Yes
Cagetti and De Nardi (2006, 2009)	Dynasty with p. of retiring and death	Yes
Castaneda et al. (2003)	Dynasty with p. of retiring and death	No
Benhabib and Zhu (2008)	Dynasty with probability of death	Yes
Panousi (2008)	Dynasty with probability of death	Yes
Benhabib et al. (2009)	Life cycle	Yes Draw at age 0

Table 1.1 shows that no research has investigated the impacts of the within lifetime idiosyncratic investment risk on wealth inequality in a life cycle environment.

De Nardi (2004) uses the "joy of giving" bequest motive and death rate to increase the concentration of wealth. The mechanism of the combination of the "joy of giving" bequest motive, estate tax, and death rate to generate the Pareto tail of the wealth distribution is also illustrated by Benhabib and Bisin (2006).

Levy (2003) formulates a general stochastic process of wealth accumulation by capital investment and analyze the conditions required to ensure convergence to the empirically observed Pareto wealth distribution. Using the Forbes 400 lists during 1988-2003, Klass et al. (2006) find that the top end of the wealth distribution follows a Pareto distribution with an average exponent of 1.49. Champernowne (1953) employs the same mechanism to study the income distribution.

Panousi (2008) studies the stationary wealth distribution in a heterogeneous-agent economy with uninsurable entrepreneurial risk. However the right tail of the wealth distribution of Panousi (2008) is much thinner than that of the U.S. wealth distribution.[3]

Benhabib and Zhu (2008) incorporate two kinds of uncertainty–investment risk and death risk–to produce the Pareto tails of wealth distribution. Benhabib and Zhu (2008) analytically solve the density function of the wealth distribution: a double Pareto distribution. In this paper I emphasize the investment risk during lifetime itself is enough to produce the Pareto tail. In section 6 I introduce the age-dependent death rate into the model; however this is only to produce a realistic age distribution.

The framework of this paper follows Benhabib, Bisin, and Zhu (2009). Benhabib, Bisin and Zhu (2009) studies the impacts of the inheritance of the investment ability on wealth inequality. The agent draws an interest rate and labor income at the beginning of his life. And during his lifetime, there is no uncertainty of the investment return and labor income. The agent in this paper suffers from the investment return uncertainty during his life. However, there is no stochastic draw of labor income in this model. Benhabib, Bisin, and Zhu (2009) theoretically show that the wealth distribution has a fat tail and use the tail index to characterize the fatness of the tail. This paper, following Benhabib, Bisin, and Zhu (2009), theoretically shows that the wealth distribution has a Pareto tail and derives the equation that the Pareto exponent satisfies.

The rest of the paper is organized as follows. Section 2 presents the basic model. I derive the Pareto tail of the wealth distribution in section 3. Section

[3] Angeletos (2007) concentrates on the aggregate variables of the economy with uninsured idiosyncratic investment risk.

4 contains the simulated economy. Section 5 studies comparative statics and welfare. I introduce the age-dependent death rate into the basic model in section 6. Section 7 concludes the paper. Most of the proofs and the derivation are in Appendix.

2. Model

The agent lives from age 0 to age T. At the end of life, each agent gives birth to one child. There are a continuum of measure 1 of agents in the economy. In the benchmark model, the agent has the certain life span. This helps to highlight the mechanism for the idiosyncratic investment risk to produce the Pareto tail of wealth distribution.

Agents can invest in a risk-free account and an individual investment opportunity. The rate of return in the risk-free asset is r. The individual investment is a risky asset. Its price follows a Geometric Brownian motion with jump

$$\frac{dP(s)}{P(s)} = \alpha ds + \sigma dz(s) + \delta dq(s)$$

where $z(s)$ is a standard Brownian motion and $q(s)$ is a Poisson process with arriving intensity λ. α is the mean return without jump. σ is the standard deviation of the return without jump. δ is the jump size. The investment risk is idiosyncratic and the agents are not permitted to pool their investment risk.

At any time during his life the agent supplies one unit of labor inelastically. The wage rate is ω. Thus the agent receives labor income ω. At any time the alive agent also receives the lump-sum transfer Γ from the government. Agent chooses to invest a fraction $\phi(t)$ of his wealth $w(t)$ in the risky asset. Agent obtains utility from consumption, $c(t)$, and the bequest he leaves to his child. The agent has a 'joy-of-giving' bequest motive. His utility functions on consumption and on the after-estate-tax bequest have the same Constant Relative Risk Aversion (CRRA) form. Let χ denote the bequest motive intensity. The agent's utility maximization problem is

$$J(w,t) = \max_{c(s),\phi(s)} \{E_t \int_t^T \frac{c(s)^{1-\gamma}}{1-\gamma} e^{-\theta(s-t)} ds + \chi \frac{[(1-\zeta)w(T)]^{1-\gamma}}{1-\gamma} e^{-\theta(T-t)}\}$$

$$
\begin{aligned}
s.t. \quad dw(s) &= [(r-\tau)w(s) + ((\alpha-\tilde{\tau})-(r-\tau))\phi(s)w(s) - c(s) + \omega + \Gamma]ds \\
&\quad +\sigma\phi(s)w(s)dz(s) + \delta\phi(s)w(s)dq(s)
\end{aligned}
$$

where τ is the capital income tax on risk-free asset. $\tilde{\tau}$ is the capital income tax on risky asset. ζ is the estate tax. γ is the coefficient of relative risk aversion and is assumed to greater than or equal to 1. θ is the time discount rate.

Let $h(t)$ denote the agent's human wealth which is the discounted sum of future labor income and lump-sum transfer.

$$
\begin{aligned}
h(t) &= \int_t^T (\omega+\Gamma)e^{-(r-\tau)(s-t)}ds \\
&= \frac{1-e^{-(r-\tau)(T-t)}}{r-\tau}(\omega+\Gamma)
\end{aligned}
$$

Solving the agent's utility maximization problem, we have

Proposition 1. *The agent's policy functions are*

$$
c(t) = a(t)^{-\frac{1}{\gamma}}(w(t)+h(t))
$$

and

$$
\phi(t)w(t) = \Pi(w(t)+h(t))
$$

where Π *solves*

$$
(\alpha-\tilde{\tau}) - (r-\tau) - \gamma\sigma^2\Pi + \lambda\delta(1+\delta\Pi)^{-\gamma} = 0 \tag{2.1}
$$

and

$$
a(t) = \left(\frac{e^{\eta(T-t)}-1}{\eta} + \left(\chi(1-\zeta)^{1-\gamma}\right)^{\frac{1}{\gamma}} e^{\eta(T-t)}\right)^{\gamma}
$$

where η *is a function of parameters expressed in Appendix 8.1.*

The amount of wealth invested in the risky asset is a constant fraction of the agent's total wealth. The fraction is independent of age. The bequest motive intensity, χ, and the estate tax, ζ, do not influence the portfolio selection. We can divide the terms in equation (2.1) into three parts which are related to different factors influencing the portfolio decision. The first part is the risk premium between the risky asset and the risk free asset, $(\alpha-\tilde{\tau})-(r-\tau)$. The second part, $-\gamma\sigma^2\Pi$, reflects the volatility caused by the Brownian motion in the return of the risky asset. The third part, $\lambda\delta(1+\delta\Pi)^{-\gamma}$, reflects the influence of the jump in the return of the risky asset.

The optimal consumption is a linear function of the total wealth of the agent, which is the sum of the physical wealth and the human wealth. The function of $a(t)$, which determines the consumption propensity, is a deterministic function of age t. The three main factors influencing $a(t)$ are the time discount rate, the average rate of return of wealth, adjusted by the volatility of the risky asset, and the bequest motive intensity. The terms in the big bracket of the expression of $a(t)$ are a weighted average of two terms of $-\frac{1}{\eta}$ and $(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}$. Here $-\frac{1}{\eta} > 0$ since we assume that $\gamma \geq 1$. Note that η is independent of χ and ζ. The stronger the bequest motive intensity is, the lower the consumption propensity is. But the relationships of the time discount rate and the average rate of return of wealth on the consumption propensity are ambiguous, since the influences of η on the term $-\frac{1}{\eta}$ and the weight $(1-e^{\eta(T-t)})$ are on the two opposite ways.

From the stochastic differential equation of $(w(t)+h(t))$, we know that the total wealth, $(w(t)+h(t))$ is positive. And since Π is positive so is $\phi(t)w(t)$. Thus the agent always has a positive investment in the private project. There is no market for trading the private investment opportunity. But the agent can save and borrow money through the risk free account. The young and the poor borrow money to smooth the consumption.

2.1. Wealth accumulation within life

Let $x(t)$ be the agent's total wealth, i.e. the sum of physical wealth and human wealth, at age t.

$$x(t) = w(t) + h(t)$$

Using Proposition 1 we can derive

$$dx(t) = e(t)x(t)dt + \Psi x(t)dz(t) + \delta\Pi x(t)dq(t)$$

where $e(t) = (r-\tau) + ((\alpha-\tilde{\tau}) - (r-\tau))\Pi - a(t)^{-\frac{1}{\gamma}}$ and $\Psi = \sigma\Pi$. Thus $x(t)$ is a Geometric Brownian motion with jump. The mean of the total wealth accumulation rate, $e(t)$, is age-dependent. This gives the life cycle wealth accumulation pattern. We have the explicit form expression of $x(t)$,

$$x(t) = \left(\frac{a(t)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda t + \Psi z(t)](1+\delta\Pi)^{q(t)}x(0) \tag{2.2}$$

where $\Lambda = \frac{(r-\tau)-(\theta+\lambda)+((\alpha-\tilde{\tau})-(r-\tau))\Pi}{\gamma} - \frac{1}{2}(2-\gamma)\sigma^2\Pi^2 + \frac{\lambda}{\gamma}(1+\delta\Pi)^{1-\gamma}$. For an individual with starting wealth $x(0)$, the logarithm of his total wealth at age t is a mixture of normal distributions. The tail of the distribution of $x(t)$ conditional on $x(0)$ is thinner than that of any Pareto distribution, since $E[x^v(t)|x(0)]$ is finite for any $v \geq 0$.[4]

The wealth of generations in a lineage is connected through bequest. At the end of life the agent's human wealth is zero, i.e. $h(T) = 0$. Thus $w(T) = x(T)$. By formula (2.2) $w(T)$, the end-of-life wealth, which is also the before-estate-tax bequest, is a function of the starting wealth of the agent. The child of the dying agent receives the bequest subtracting the estate tax, i.e. a fraction of $(1-\zeta)$ of the before-estate-tax bequest. The newborn's endowment includes two parts: the endowment from his father, i.e. the bequest, and the natural endowment, i.e. the human wealth. The sum of after-estate-tax bequest and the human wealth at age 0, $h(0)$, is the starting total wealth of the newborn.

2.2. Intergenerational connection

Let $x_1, x_2, x_3, \cdots, x_n, \cdots$ be the starting wealth of the generation $1, 2, 3, \cdots, n, \cdots$ of a lineage. In the Appendix I derive the equation of wealth accumulation process across generations,

$$x_{n+1} = \rho_{n+1} x_n + h(0) \tag{2.3}$$

where the random variable ρ_{n+1} is given by

$$\rho_{n+1} = \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda T + \Psi z(T)](1+\delta\Pi)^{q(T)} \tag{2.4}$$

This equation reflects the wealth connection of the two consecutive generations in a lineage. From equation (2.4) we know that the bequest motive intensity, χ, and the estate tax, ζ, influence ρ_{n+1}, but do not influence the stochastic part of ρ_{n+1}. This is because the bequest motive intensity, χ, and the estate tax, ζ, influence the

[4] Note that

$$E[x^v(t)|x(0)] = \left(\frac{a(t)}{a(0)}\right)^{\frac{v}{\gamma}} \exp\left(t[\Lambda v + \frac{1}{2}\Psi^2 v^2 + ((1+\delta\Pi)^v - 1)\lambda]\right) x^v(0).$$

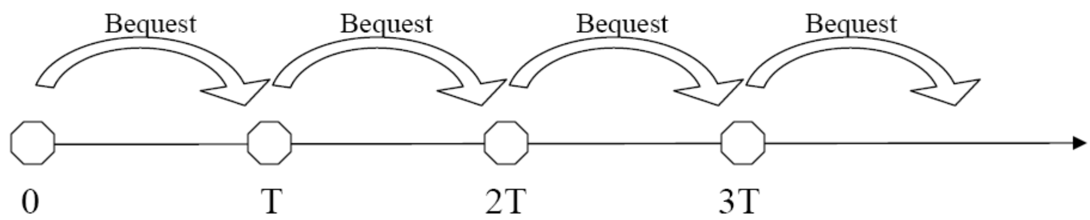

Figure 2.1: Illustration of Bequest Movement through Lineage

propensity to consume but the stochastic part of the wealth accumulation is only determined by the portfolio selection, while the portfolio selection is independent of the bequest motive intensity, χ, and the estate tax, ζ, by Proposition 1.

Suppose that the generation 0 of a lineage is born at time 0. Then the generation 1, 2, 3, $\cdots$, n, $\cdots$ are born at time T, $2T$, $3T$, $\cdots$, nT, $\cdots$ respectively. The movement of the bequest could be illustrated in Figure 2.1. The bequest movement is governed by equation (2.3) while the wealth accumulation within life time is governed by equation (2.2) in section 2.1. The wealth connection of other lineages are similar.

2.3. Government budget

The government collects revenue through capital income taxes and estate tax. The government then gives lump-sum transfer, Γ, to all of the alive agent and finance the government expenditure, G. The government expenditure, G, is exogenously given. The government budget constraint is

$$\Gamma + G = \tilde{\tau} K + \tau B + \frac{\zeta E^s w(T)}{T}$$

where K and B are the per capita risky asset and per capita risk free asset respectively. The first two terms are the capital income tax on the risky asset and on the risk free asset respectively. The third term is the estate tax.

Given the capital income tax rate, the estate tax rate, and government expenditures, the lump-sum transfer is determined in the stationary distribution.

For the aggregate variables, the government budget balance is a linear equation for the lump-sum transfer, Γ. Solving this equation we can find the government transfer, Γ. See Appendix 8.4.

3. Pareto tail

Following Benhabib, Bisin, and Zhu (2009), I prove that the wealth distribution of the whole economy has a Pareto tail in this section. This theoretical result gives us the confidence that the wealth distribution of the model economy might have a fat tail. I do the simulation exercise in section 4 to show that this model actually replicates the fat tail observed in the U.S. data.

3.1. The distribution of starting wealth

In section 2.2 I show that the starting wealth of the agents follows

$$x_{n+1} = \rho_{n+1} x_n + h(0) \tag{3.1}$$

By the theorem of Sornette (2006),[5] Goldie (1991), and Kesten (1973), this equation permits a stationary distribution of x_n, with a Pareto upper tail, if there exists a $\mu > 1$ such that[6]

$$E\rho_{n+1}^{\mu} = 1$$

The above theorem also points out that in the stationary distribution the Pareto exponent is μ, i.e.

$$P(x(0) > x) \sim x^{-\mu} \qquad as\ x \to +\infty$$

In equation (3.1) ρ_{n+1} is the fraction of the total initial wealth passed on as a bequest after lifetime accumulation and consumption are taken into account. From the expression of ρ_{n+1} in equation (2.4) we know that low bequest motive intensity and high estate tax can help to keep $E\rho_{n+1} < 1$. At the same time human wealth $h(0)$ acts as a reflecting barrier. The combination of these two forces generates a mean reverting mechanism with a non-trivial stationary wealth distribution.

But a mean reverting mechanism does not necessarily guarantee a fat tail of the stationary distribution. The mechanism to produce the Pareto tail is the

[5] See page 374 of Sornette (2006).

[6] Note that by Holder's inequality, this condition implies that $E\rho_{n+1} < \left(E\rho_{n+1}^{\mu}\right)^{\frac{1}{\mu}} = 1$.

multiplicative stochastic term. Some realizations of ρ_{n+1} are greater than 1, while some realizations of ρ_{n+1} are smaller than 1. In the group of agents who draw the good realizations of ρ_{n+1} (the realizations greater than 1), the rich benefit more from the good luck. Even though the logarithm of ρ_{n+1} is a mixture of normal distributions, the distribution of ρ_{n+1} itself is not enough to generate a Pareto tail in wealth.[7] The multiplicative mechanism amplifies the dispersion of ρ_{n+1} and generates the Pareto tail of the distribution of wealth.

From the equation (2.4) we know μ solves

$$\frac{\mu}{\gamma}\ln\left(\frac{\chi(1-\zeta)}{a(0)}\right)+T\left((\Lambda\mu+\frac{1}{2}\Psi^2\mu^2)+((1+\delta\Pi)^{\mu}-1)\lambda\right)=0 \tag{3.2}$$

As in Benhabib, Bisin, and Zhu (2009), we have

Proposition 2. *The higher the bequest motive χ, or the lower the estate tax ζ, the smaller is μ.*

A smaller μ implies a fatter tail. Thus the impacts of χ and ζ on μ are in line with our intuition about the role of bequest on wealth inequality: the more persistent the bequest process, the higher is the inequality in wealth distribution.

3.2. Wealth distribution conditional on age

The newborn draws a realization of his starting wealth from the distribution presented in section 3.1. Stochastic wealth accumulation is governed by the normal random variable $z(t)$ and Poisson random variable $q(t)$ of equation (2.2) in section 2.1. These two kinds of uncertainty determine the total wealth distribution of an individual at age t, which is also the stationary total wealth distribution of age cohort t. By the linear relationship of the total wealth and the physical wealth, we have Proposition 3.

Proposition 3. *The stationary wealth distribution in every age cohort has a Pareto tail with the same Pareto exponent as that of the starting wealth distribution,*

$$P(w(t)>w)\sim w^{-\mu} \qquad \textit{as } w\rightarrow+\infty$$

[7] Any order moment of ρ_{n+1} is finite. The tail of ρ_{n+1} is thinner than that of any Pareto distribution. Also see the remarks following equation (2.2) in section 2.1.

3.3. Wealth distribution of the whole economy

The wealth distribution of the whole economy is a weighted average of the wealth distribution of all age cohorts. Let $F_t(w)$ be the cumulative distribution function of wealth in age cohort t, then the wealth distribution function $F(w)$ of the whole economy is

$$F(w) = \frac{1}{T}\int_0^T F_t(w)dt$$

Proposition 4. *The stationary wealth distribution in the whole economy has a Pareto tail with the same Pareto exponent as that of the starting wealth distribution,*

$$P(W > w) \sim w^{-\mu} \qquad as\ w \to +\infty$$

Following Benhbib, Bisin, and Zhu (2009), I show that the wealth distribution of the whole economy has a Pareto tail. The main difference between Benhabib, Bisin, and Zhu (2009) and this paper is that this paper incorporates within life idiosyncratic investment risk into the model. And I show that the results of Pareto tail in Benhabib, Bisin, and Zhu (2009) still hold in my model.

Here each age cohort has equal size. In the data the unequal cohort distribution also contributes to the wealth inequality. In section 6 I introduce age-dependent death rate into the benchmark model and show that the model can produce not only a fat tail of wealth distribution but also a realistic age cohort distribution.

4. Simulated economy

In the previous section, I theoretically show that idiosyncratic investment risk would produce a Pareto tail of wealth distribution. However this theoretical result only predicts the tail of the distribution. In this section I simulate the wealth distribution to investigate inequality not only of the rich but also of the poor.

4.1. Parameters and calibration

All of the parameters in the model are divided into two groups: for the parameters in the first group, I choose the values in the reasonable region. For the parameters in the second group, I calibrate the values through the model by targeting some variables in U.S. data.

The parameters in the first group include the length of life, T, the time discount rate, θ, the coefficient of relative risk aversion, γ, the rate of return of the risk-free asset, r, the mean return of the risky asset without jump, α, the standard deviation of the return of the risky asset without jump, σ, the capital income tax rate on risk-free asset, τ, and the capital income tax rate on risky asset, $\tilde{\tau}$.

The rest of the parameters are in the second group. I calibrate the intensity of the bequest motive intensity, χ, to cause the ratio of bequest to wealth close to the U.S. data, around 0.3%. I set the estate tax rate close to the effective estate tax rate in U.S. I set the arrival intensity of the jump in the return of the risky asset $\lambda = 0.0001$ and the size of jump in the return of the risky asset $\delta = 180$ to replicate the large dispersion of wealth changes in PSID, and to match the ratio of aggregate capital income to aggregate labor income in U.S. The labor income ω is chosen close to the average household labor income in U.S. And government expenditure G is chosen to match the ratio of the government expenditure to GDP in U.S. The target is 4% which is much lower than that in U.S. data, since in the model government only collect capital income taxes and estate tax. Given the government expenditure, the government lump-sum transfer, Γ, is determined by the balance of government budget. The ratio of the government transfer to labor income, $\frac{\Gamma}{\omega} = 0.011$.

Table 4.1: Calibration of Parameters

Parameter	What's it?	Value	Target
T	length of life	60	chosen
θ	time discount rate	0.03	chosen
γ	risk aversion	3.5	chosen
r	risk-free interest rate	0.03	chosen
α	mean equity return, no jump	0.08	chosen
σ	s.d. of equity return, no jump	0.28	chosen
τ	capital income tax rate	0.006	chosen
$\tilde{\tau}$	capital income tax rate	0.006	chosen
χ	bequest motive intensity	2	bequest/wealth
ζ	estate tax rate	0.2	effective estate tax rate in U.S.
λ	arriving intensity of jump	0.0001	household wealth change in PSID
δ	size of jump	180	$\frac{\text{aggregate capital income}}{\text{aggregate labor income}}$
ω	labor income	50000	average household labor income in U.S.
G	government expenditure	2000	about 4% of GDP

The following table shows that the distribution of household wealth change in PSID has large dispersion. This justifies the assumption that there is a jump component in the return of the private project.

Table 4.2: The Times of Wealth Change[8]

	Percentiles		
Periods	$P90$	$P95$	$P99$
2001-2003	3.8	8.6	47.8
2003-2005	3.5	8.5	28.9
2005-2007	3.4	8.3	69.3

4.2. Aggregate economy

In the model economy the ratio of aggregate capital income to labor income is 0.502. And the aggregate saving in the risk free account B is positive. From the comments after the Proposition 1, we know that each agent invests in the private project. Thus the aggregate risky asset is positive in the economy. Each agent selects his own portfolio, depending on age and wealth. But we can see from Table 4.3 that the aggregate risky asset accounts for 69.4% of the total wealth in the economy.

Table 4.3: Aggregate Variables

	capital inc./labor inc.	B	K	$K/(B+K)$
Model	0.502	107738	244658	0.694

4.3. Wealth distribution

I report the Gini coefficient and the Lorenz curve of the simulated wealth distribution[9] and compare them with those of the U.S. data in Table 4.4. The Gini coefficient and the quintiles of U.S. data are from the 2007 SCF.

[8] To calculate the real change rate I use the yearly inflation rate from 2001-2006 in the 2009 Statistical Abstract of the United States.

[9] In the simulated economy $E\rho_{n+1} = 0.0621$. Note that $E\rho_{n+1} < 1$ is a necessary condition for the existence of a stationary wealth distribution with finite mean.

Table 4.4: Gini and Quintiles of Wealth

Economy	*Gini*	*Quintiles*				
		First	*Second*	*Third*	*Fourth*	*Fifth*
U.S.	0.816	-0.2	1.1	4.5	11.2	83.4
Model	0.803	-7.3	3.7	10.2	19.7	73.7

The simulated results show that my model replicates the Gini coefficient of the wealth distribution in U.S. But the Lorenz curve deviates from that of the U.S. data to some extent. The model economy produces more negative wealth than the data. In U.S. data the negative wealth accounts for only -0.2% of the total wealth, while in the model economy the negative wealth accounts for -7.3% of the total wealth. Without the borrowing constraint the poor agent could short his human wealth to smooth consumption. This produces large negative wealth in the model. The same phenomenon happens in Huggett (1996) when an agent can borrow to a limit.

To see more closely the fat tail of the wealth distribution in the model, I report the percentiles of the top tail. I also report the Pareto exponent of the wealth distribution in the model. Instead of estimating the Pareto exponent from the simulated data, I calculate the Pareto exponent from equation (3.2).

Table 4.5: Pareto Tail and Top Percentiles of Wealth

Economy	*Pareto* exp*onent*	*Percentiles*		
		$90th - 95th$	$95th - 99th$	$99th - 100th$
U.S.	1.49	11.1	26.7	33.6
Model	1.95	10.8	12.4	34.7

The simulated data match the top 1% of the wealth distribution in the data very well. For the $90\% - 95\%$ the model almost replicates the percentage of wealth in the data. For the $95\% - 99\%$ of the wealth distribution, the model underestimates the percentage of the wealth. This causes the wealth holding of the fifth quintile of the model to be lower than that of the data, shown in Table 4.2.

Quadrini (2000) introduces the entrepreneurship into the heterogeneous agent model and replicates the extreme concentration of wealth on the top tail. He incorporates the financial market friction and the entrepreneurial risk into his model. But he emphasizes the role of financial market friction on the higher wealth accumulation pattern of entrepreneurs: To benefit from the high return of entrepreneurial activities the agents have to accumulate much wealth to avoid the borrowing constraint. Cagetti and De Nardi (2006, 2009) also emphasize this mechanism in their works. The mechanism that I emphasize here is the second one that Quadrini (2000) uses. The entrepreneurial risk itself is enough to generate the extreme concentration of wealth.

4.4. Age cohorts

The life cycle structure of the model permits us to investigate the wealth accumulation and distribution along the age cohorts. Huggett (1996) investigates wealth accumulation and wealth concentration within age groups. But Hugget (1996) does not consider idiosyncratic investment risk. Benhabib, Bisin, and Zhu (2009) study a life cycle model with idiosyncratic investment risk. But Benhabib, Bisin, and Zhu (2009) do not take into account investment risk within lifetime.

I calibrate the lenth of life, $T = 60$. And I assume that age 0 in the model corresponds to age 20 in the data. Thus the life span in the model corresponds to $20 - 80$ in the data.

Figure 4.1 shows that the mean wealth of age cohorts has a hump shape as in U.S. data. The hump shape of wealth accumulation comes from the relative strength of high rates of return of assets and the low bequest motive intensity. The composite rate of return of the risk free asset and the risky asset is higher than the time discount rate. This effect causes the agent to accumulate wealth. But the low intensity of the bequest motive intensity causes the agent to decummulate wealth when the agent approaches the end of life.

In section 3.2 I theoretically show that wealth distribution of each cohort has a Pareto tail. Figure 4.2 shows that the simulated data mimic the whole pattern of the wealth dispersion by age in the U.S. data. The wealth dispersion first increases for the young age cohorts and then decreases for the old age cohorts. The reason for this pattern in the model is that the wealth accumulation by age has a hump shape and the stochastic term is multiplicative.

4.5. Consumption distribution

I also investigate the distribution of consumption. I report the Gini coefficient and the Lorenz curve of consumption and compare them with those of the U.S. data in Tables 4.6 and 4.7.[10] As in Castande et al. (2003) and Cagetti and De

[10] The U.S. data are from Castaneda et al. (2003) who use the 1991 CEX.

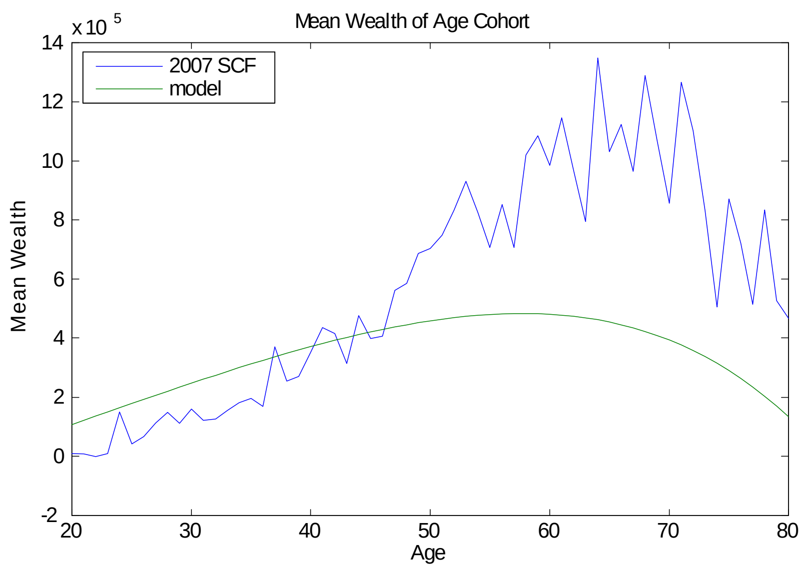

Figure 4.1: Mean Wealth of Age Cohort

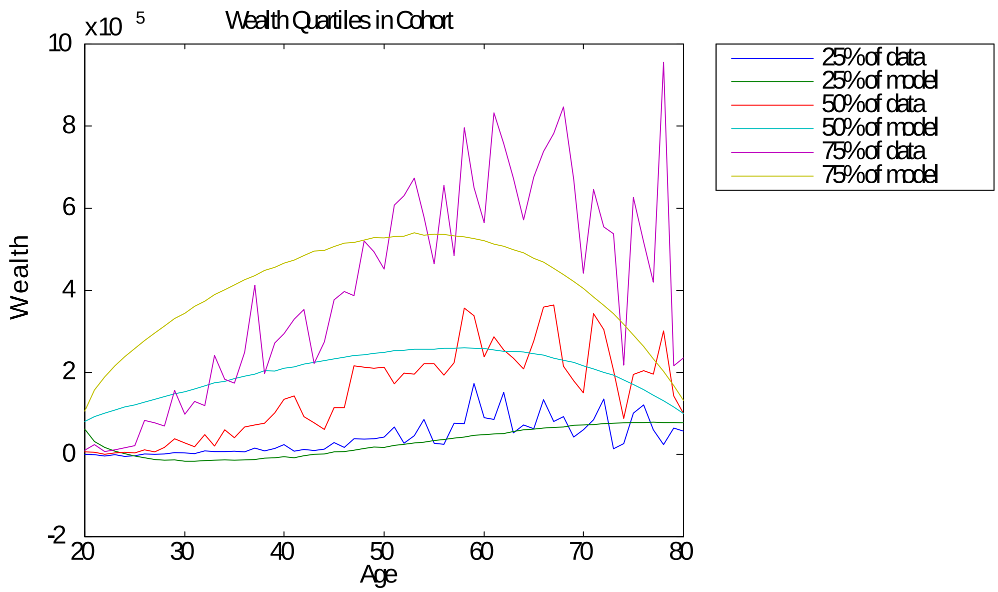

Figure 4.2: Wealth Quartiles in Cohort

Nardi (2007), the model predicts a fat tail of consumption distribution. Note that this pattern is not found in the data because CEX data has top-coding.

Table 4.6: Gini and Quintiles of Consumption

		Quintiles				
Economy	*Gini*	*First*	*Second*	*Third*	*Fourth*	*Fifth*
U.S. (Nondurables)	0.32	6.87	12.27	17.27	23.33	40.27
*U.S. (Nondurables+)**	0.30	7.19	12.96	17.80	23.77	38.28
Model	0.248	11.9	14.9	16.8	19.4	37.0

$*$ includes imputed services of consumer durables.

Table 4.7: Top Percentiles of Consumption

	Pareto	*Percentiles*		
Economy	*exponent*	$90th - 95th$	$95th - 99th$	$99th - 100th$
U.S. (Nondurables)		9.71	10.30	4.83
*U.S. (Nondurables+)**		9.43	9.69	3.77
Model	1.95	6.7	6.7	12.1

$*$ includes imputed services of consumer durables.

5. Comparative statics and welfare

After investigating the implications of the benchmark model, I discuss how the parameters of the economy influence wealth inequality and welfare.

5.1. Risk aversion

From Table 5.1 we see that the higher the risk aversion, the thinner the tail and the higher the Gini coefficient. The reason is that high risk aversion implies that the agent invests a smaller fraction of his wealth in the risky asset. This causes a lower volatility of the wealth accumulation process and a lower inequality of the stationary wealth distribution.

Table 5.1: The Effect of Risk Aversion

γ	1.5	2	2.5	3	3.5
μ	1.414	1.587	1.724	1.840	1.945
Gini	0.869	0.853	0.839	0.821	0.803

5.2. Taxes

In section 3.1 we know that the lower the estate tax ζ is, the smaller the Pareto exponent μ is. A smaller μ implies a fatter tail of wealth distribution. Castaneda et al. (2003) study the steady-state implications of abolishing estate taxation. They find that abolishing estate taxation brings about very little change in wealth inequality. Cagetti and De Nardi (2009) study the effect of abolishing estate taxation on the stationary wealth distribution in different policy change experiments. They also find that in each experiment abolishing estate taxation has little effect on the wealth inequality.

From the calculation results in Appendix 8.8 we see that the higher the capital income tax on risk free asset is, the fatter the tail is. The lower the capital income tax on risky asset is, the fatter the tail is. The reason is that in both cases agents shift away from the risk free asset and toward the risky asset. The wealth accumulation process becomes more volatile.

For the effects of taxes on Gini of the wealth distribution I do the experiments by simulation. In the experiment the government adjusts the lump-sum transfer to keep the expenditure while the government changes the taxes.

From the simulation results in Appendix 8.9 we see that a higher the capital income tax on a risk-free asset causese higher inequality. The reason is that the agent invests more in the risky asset when the net return on the risk free asset is low. Consistent with our intuition, the higher the estate tax, the lower the inequality.

5.3. Welfare

The capital income taxes and estate tax influence the wealth accumulation of individuals and the wealth distribution in the society. To study the influence of the capital income tax and the estate tax on welfare, I define the egalitarian welfare object function as

$$U = \int J(w(0), 0) dv(w(0))$$

where $J(w(0), 0)$ is the indirect utility function of agent at age 0, and $v(w(0)$ is the stationary distribution of the starting physical wealth.

Cagetti and De Nardi (2009) study the welfare change along the transition path after government abolishes estate tax. I only study the effect of the taxes on the welfare in the stationary distribution. In the experiment government adjusts the lump-sum transfer to keep the expenditure constant when government changes the taxes.

From the simulation results in Appendix 8.10 we see that the higher the capital income tax on risk free asset, the higher the welfare. The higher the capital income tax on risky asset, the lower the welfare. The higher the estate tax, the higher the welfare.

6. Age-dependent death rate

In order to replicate the demographic structure of the U.S. economy, I introduce age-dependent death rate to the basic model. Let $\pi(t)$, $t \in [0, T]$ be the agent's death rate. Suppose that there is a perfect life insurance market in the economy.

In Appendix 8.7, I show that the newborn's endowment process follows

$$x_{n+1} = \rho_{n+1} x_n + h(0)$$

and the agent's starting wealth has a stationary distribution with a Pareto upper tail. The Pareto exponent, μ, of the distribution solves

$$\left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{\mu}{\gamma}} \int_0^T \exp\left([\Lambda\mu + \frac{1}{2}\Psi^2\mu^2 + ((1+\delta\Pi)^\mu - 1)\lambda]t\right) \pi(t)dt = 1 \quad (6.1)$$

Equation 6.1 shows that the death rate can affect the tail of the wealth distribution. This finding is consistent with the simulation work of De Nardi (2004).

As in section 3.2 we have

Proposition 5. *The stationary wealth distribution in every age cohort has a Pareto tail with the same Pareto exponent as that of the starting wealth distribution.*

Let $F_t(w)$ be the cumulative distribution function of wealth in age cohort t. Now the wealth distribution function $F(w)$ of the whole economy is

$$F(w) = \int_0^T F_t(w)\pi(t)dt$$

Thus we have

Proposition 6. *The stationary wealth distribution in the whole economy has a Pareto tail with the same Pareto exponent as that of the starting wealth distribution.*

For the proofs of Proposition 5 and Proposition 6 see Appendix 8.7.

7. Conclusion

Idiosyncratic investment risk plays a very important role in causing wealth inequality. Multiplicative shock generates a fat tail of the wealth distribution.

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8. Appendix

8.1. Proof of Proposition 1.

Proof: Hamilton-Jacobi-Bellman equation of this problem is

$$\begin{aligned}(\theta+\lambda)J(w,t) &= \max_{c(t),\phi(t)}\{\frac{c(t)^{1-\gamma}}{1-\gamma} \\ &+J_w(w,t)[(r-\tau)w(t)+((\alpha-\tilde{\tau})-(r-\tau))\phi(t)w(t)-c(t)+\omega+\Gamma] \\ &+\frac{1}{2}J_{ww}(w,t)\sigma^2\phi(t)^2w(t)^2 \\ &+J_t(w,t) \\ &+\lambda J(w+\delta\phi(t)w(t),t)\}\end{aligned}$$

We have the F.O.C.

$$c(t)^{-\gamma}=J_w(w,t)$$

$$J_w(w,t)((\alpha-\tilde{\tau})-(r-\tau))+J_{ww}(w,t)\sigma^2\phi(t)w(t)+\lambda\delta J_w(w+\delta\phi(t)w(t),t)=0$$

Guess

$$J(w,t)=\frac{a(t)}{1-\gamma}(w(t)+h(t))^{1-\gamma}$$

and

$$\phi(t)w(t)=\Pi(w(t)+h(t))$$

We have

$$c(t) = a(t)^{-\frac{1}{\gamma}}(w(t) + h(t))$$

and

$$(\alpha - \tilde{\tau}) - (r - \tau) - \gamma\sigma^2\Pi + \lambda\delta(1 + \delta\Pi)^{-\gamma} = 0$$

Plugging these expressions into the HJB, we have

$$\frac{1}{\gamma}a(t)^{\frac{1}{\gamma}-1}\dot{a}(t) + \eta a(t)^{\frac{1}{\gamma}} + 1 = 0$$

where $\eta = \frac{(1-\gamma)(r-\tau)-(\theta+\lambda)}{\gamma} + \frac{1-\gamma}{\gamma}\left(((\alpha - \tilde{\tau}) - (r - \tau))\Pi - \frac{1}{2}\gamma\sigma^2\Pi^2 + \frac{\lambda}{1-\gamma}(1 + \delta\Pi)^{1-\gamma}\right)$.

Using the boundary condition

$$a(T) = \chi(1 - \zeta)^{1-\gamma}$$

we have

$$a(t) = \left(\frac{e^{\eta(T-t)} - 1}{\eta} + \left(\chi(1 - \zeta)^{1-\gamma}\right)^{\frac{1}{\gamma}} e^{\eta(T-t)}\right)^{\gamma}$$

And we have

$$\begin{aligned} d(w(t) + h(t)) &= [(r - \tau) + ((\alpha - \tilde{\tau}) - (r - \tau))\Pi - a(t)^{-\frac{1}{\gamma}}](w(t) + h(t))dt \\ &\quad + \sigma\Pi(w(t) + h(t))dz(t) + \delta\Pi(w(t) + h(t))dq(t). \end{aligned}$$

■

8.2. Derivation of equation (2.3)

We know $w(T) = x(T)$. From formula (2.2) we have

$$w(T) = \left(\frac{\chi(1 - \zeta)^{1-\gamma}}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda T + \Psi z(T)](1 + \delta\Pi)^{q(T)}x(0)$$

Let $w_n(T)$ be the wealth of the agent at age T of generation n. Thus

$$\begin{aligned} x_{n+1} &= (1 - \zeta)w_n(T) + h(0) \\ &= \left(\frac{\chi(1 - \zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda T + \Psi z(T)](1 + \delta\Pi)^{q(T)}x_n + h(0) \end{aligned}$$

■

8.3. Proof of Proposition 2

Proof: Following Benhabib, Bisin and Zhu (2009), I prove this proposition by combining two results of monotonicity. The first one is that $E\rho_{n+1}^{\mu}$ is increasing in ρ_{n+1} when $\mu > 1$. The second one is that $E\rho_{n+1}^{\mu}$ is increasing with μ in the neighborhood of $\hat{\mu}$ such that $E\rho_{n+1}^{\hat{\mu}} = 1$ since by Lemma 2.2 of Goldie (1991), $E(\rho_{n+1}^{\hat{\mu}} \log \rho_{n+1}) > 0$. Then the results of the proposition follows the fact that ρ_{n+1} is an increasing function of χ and is a decreasing function of ζ. ■

8.4. Procedure to solve the equilibrium Γ

We compute the aggregate starting wealth in the stationary distribution.

$$E^s x(0) = \frac{h(0)}{1 - E\rho_{n+1}}$$

where $E^s(\cdot)$ is the expectation with respect to the stationary distribution. We compute the aggregate total wealth X

$$\begin{aligned} X &= \frac{1}{T}\int_0^T E^s x(t)dt \\ &= \frac{1}{T}\frac{E^s x(0)}{a(0)^{\frac{1}{\gamma}}}\left(\frac{\frac{1}{\Phi+\delta\Pi\lambda} + (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}}{\Phi+\delta\Pi\lambda-\eta}e^{(\Phi+\delta\Pi\lambda)T} - \frac{\frac{1}{\eta} + (\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}}{\Phi+\delta\Pi\lambda-\eta}e^{\eta T} + \frac{1}{\eta(\Phi+\delta\Pi\lambda)}\right) \end{aligned}$$

The aggregate human capital, H, is

$$\begin{aligned} H &= \frac{1}{T}\int_0^T h(t)dt \\ &= \frac{\omega+\Gamma}{r-\tau}\left(1 - \frac{1-e^{-(r-\tau)T}}{T(r-\tau)}\right) \end{aligned}$$

Compute the aggregate risky asset, K

$$K = \Pi X$$

Compute the aggregate risk free asset, B

$$\begin{aligned} B &= W - K \\ &= X - H - K \\ &= (1 - \Pi) X - H \end{aligned}$$

From government budget balance, we have

$$\Gamma = \tilde{\tau} K + \tau B + \frac{\zeta E^s w(T)}{T} - G \tag{8.1}$$

Note that $h(0)$ is a linear function of Γ. The first three terms on the right hand side of equation (8.1) are linear functions of Γ. Thus we can solve Γ from linear equation (8.1). ■

8.5. Proof of Proposition 3.

Proof: By the linear relationship of the total wealth and the physical wealth, we only need to prove that the proposition holds for the total wealth.

Let $m_n(t) = \frac{1}{\gamma} \ln\left(\frac{a(t)}{a(0)}\right) + \Lambda t + n \ln(1 + \delta \Pi)$. Thus

$$P(x(t) > x | q(t) = n, x(0) = y) = 1 - \Phi\left(\frac{\ln x - (\ln y + m_n(t))}{\Psi \sqrt{t}}\right)$$

where Φ is the cumulative distribution function of a standard normal distribution.

$$P(x(t) > x | x(0) = y) = \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} P(x(t) > x | q(t) = n, x(0) = y)$$

Now I prove that if the distribution of $x(0)$, $F_0(x)$, has the property that $1 - F_0(x) = P(x(0) > x) \sim x^{-\mu}$ as $x \to +\infty$, then the distribution of $x(t)$ has the property that $P(x(t) > x) \sim x^{-\mu}$ as $x \to +\infty$.

Note that there exists $c > 0$ such that

$$\lim_{x \to +\infty} \frac{P(x(0) > x)}{x^{-\mu}} = c$$

and

$$\begin{aligned}
P(x(t) &> x) \\
&= \int_0^{+\infty} P(x(t) > x | x(0) = y) dF_0(y) \\
&= \int_0^{+\infty} \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} P(x(t) > x | q(t) = n, x(0) = y) dF_0(y) \\
&= \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} \int_0^{+\infty} \left(1 - \Phi\left(\frac{\ln x - (\ln y + m_n(t))}{\Psi\sqrt{t}}\right)\right) dF_0(y) \\
&= \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} \int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) dy
\end{aligned}$$

where $f(y; x, m_n(t), \Psi\sqrt{t}) = \frac{1}{y} \frac{1}{\sqrt{2\pi\Psi^2 t}} \exp\left(-\frac{(\ln y - (\ln x - m_n(t)))^2}{2\Psi^2 t}\right)$.

Thus we have

$$\frac{P(x(t) > x)}{x^{-\mu}} = \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} \int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy$$

and

$$\lim_{x \to +\infty} \frac{P(x(t) > x)}{x^{-\mu}} = \sum_{n=0}^{+\infty} e^{-\lambda} \frac{\lambda^n}{n!} \lim_{x \to +\infty} \int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy$$

Note that

$$
\begin{aligned}
&|\int_0^{+\infty} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy - c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)| \\
&= |\int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ \int_{\hat{x}}^{+\infty} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy - c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)| \\
&\le \int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ |\int_{\hat{x}}^{+\infty} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy - c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)| \\
&\le \int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ |\int_{\hat{x}}^{+\infty} \frac{P(x(0)>y)}{y^{-\mu}}y^{-\mu}f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy - c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)| \\
&\le \int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ |\int_{\hat{x}}^{+\infty} \left(\frac{P(x(0)>y)}{y^{-\mu}} - c\right)y^{-\mu}f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy| \\
&+ |\int_{\hat{x}}^{+\infty} cy^{-\mu}f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy - c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)| \\
&\le \int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ \int_{\hat{x}}^{+\infty} |\frac{P(x(0)>y)}{y^{-\mu}} - c|y^{-\mu}f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy \\
&+ c\Phi\left(\frac{\ln\hat{x} - (\ln x - m_n(t) - \Psi^2 t\mu)}{\Psi\sqrt{t}}\right)\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)
\end{aligned}
$$

Now pick a large $\hat{x}$ such that $|\frac{P(x(0)>y)}{y^{-\mu}} - c| < \frac{\varepsilon}{3}\exp\left(-\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)$. For this specific $\hat{x}$, when x is sufficiently large we have $\int_0^{\hat{x}} P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy <$

$\frac{\varepsilon}{3}$ and $c\Phi\left(\frac{\ln\hat{x}-(\ln x-m_n(t)-\Psi^2t\mu)}{\Psi\sqrt{t}}\right)\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2t\mu\right)\right)<\frac{\varepsilon}{2}$. Thus

$$|\int_0^{+\infty}P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy-c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2t\mu\right)\right)|<\frac{\varepsilon}{3}+\frac{\varepsilon}{3}+\frac{\varepsilon}{3}=\varepsilon$$

when x is sufficiently large. Thus

$$\lim_{x\to+\infty}\int_0^{+\infty}P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy=c\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2t\mu\right)\right)$$

Thus

$$\lim_{x\to+\infty}\frac{P(x(t)>x)}{x^{-\mu}}=c\sum_{n=0}^{+\infty}e^{-\lambda}\frac{\lambda^n}{n!}\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2t\mu\right)\right).$$

■

8.6. Proof of Proposition 4.

Proof: By the linear relationship of the total wealth and the physical wealth, we only need to prove that the proposition holds for the total wealth.

For the stationary distribution of the total wealth of the whole economy we have

$$P(X>x)=\frac{1}{T}\int_0^T P(x(t)>x)dt$$

Thus

$$\begin{aligned}
&\lim_{x\to+\infty}\frac{P(X>x)}{x^{-\mu}}\\
=&\frac{1}{T}\lim_{x\to+\infty}\int_0^T\frac{P(x(t)>x)}{x^{-\mu}}dt\\
=&\frac{1}{T}\lim_{x\to+\infty}\int_0^T\left(\sum_{n=0}^{+\infty}e^{-\lambda}\frac{\lambda^n}{n!}\int_0^{+\infty}P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy\right)dt\\
=&\frac{1}{T}\sum_{n=0}^{+\infty}e^{-\lambda}\frac{\lambda^n}{n!}\lim_{x\to+\infty}\int_0^T\left(\int_0^{+\infty}P(x(0)>y)f(y;x,m_n(t),\Psi\sqrt{t})\frac{1}{x^{-\mu}}dy\right)dt
\end{aligned}$$

by the results in section 8.5.

For any $\hat{x} \in (0, +\infty)$ and $t \in (0, T]$

$$\begin{aligned}
& \int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
= & \int_0^{\hat{x}} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy + \int_{\hat{x}}^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy
\end{aligned}$$

Now pick a sufficiently large $\hat{x}$ such that $\frac{P(x(0)>y)}{y^{-\mu}} - c < \varepsilon$.

$$\begin{aligned}
\int_{\hat{x}}^{+\infty} P(x(0) &> y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&= \int_{\hat{x}}^{+\infty} \frac{P(x(0) > y)}{y^{-\mu}} y^{-\mu} f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&= \int_{\hat{x}}^{+\infty} \left(\frac{P(x(0) > y)}{y^{-\mu}} - c \right) y^{-\mu} f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&\quad + \int_{\hat{x}}^{+\infty} c y^{-\mu} f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&< (c + \varepsilon) \int_0^{+\infty} y^{-\mu} f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&= (c + \varepsilon) \exp\left(\mu \left(m_n(t) + \frac{1}{2} \Psi^2 t \mu \right) \right)
\end{aligned}$$

And when x is sufficiently large

$$\begin{aligned}
\int_0^{\hat{x}} P(x(0) &> y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&\le \int_0^{\hat{x}} f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \\
&= \int_0^{\hat{x}} \frac{1}{y} \frac{1}{\sqrt{2\pi\Psi^2 t}} \exp\left(- \frac{(\ln y - (\ln x - m_n(t)))^2}{2\Psi^2 t} \right) \frac{1}{x^{-\mu}} dy \\
&= \int_0^{\hat{x}} y^{\mu} \left(\frac{y}{x}\right)^{-\mu} \frac{x}{y} \frac{1}{\sqrt{2\pi\Psi^2 t}} \exp\left(- \frac{\left(\ln \frac{y}{x} + m_n(t)\right)^2}{2\Psi^2 t} \right) d\left(\frac{y}{x}\right) \\
&\le \hat{x}^{\mu} \int_0^{+\infty} \left(\frac{y}{x}\right)^{-\mu} \frac{x}{y} \frac{1}{\sqrt{2\pi\Psi^2 t}} \exp\left(- \frac{\left(\ln \frac{y}{x} + m_n(t)\right)^2}{2\Psi^2 t} \right) d\left(\frac{y}{x}\right) \\
&= \hat{x}^{\mu} \exp\left(\mu \left(m_n(t) + \frac{1}{2}\Psi^2 t\mu \right) \right)
\end{aligned}$$

Thus

$$\int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy < (c+\varepsilon+\hat{x}^{\mu}) \exp\left(\mu \left(m_n(t) + \frac{1}{2}\Psi^2 t\mu \right) \right)$$

Since $\exp\left(\mu\left(m_n(t) + \frac{1}{2}\Psi^2 t\mu\right)\right)$ is bounded for $t \in (0, T]$, there exits $M > 0$ such that

$$\int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy < M$$

Thus by bounded convergence theorem we have

$$\begin{aligned}
&\lim_{x\to+\infty} \int_0^T \left(\int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \right) dt \\
= &\int_0^T \lim_{x\to+\infty} \left(\int_0^{+\infty} P(x(0) > y) f(y; x, m_n(t), \Psi\sqrt{t}) \frac{1}{x^{-\mu}} dy \right) dt
\end{aligned}$$

Thus

$$\begin{aligned}
\lim_{x\to+\infty}\frac{P(X>x)}{x^{-\mu}} &= \frac{1}{T}\lim_{x\to+\infty}\int_0^T\frac{P(x(t)>x)}{x^{-\mu}}dt \\
&= \frac{1}{T}\int_0^T\lim_{x\to+\infty}\frac{P(x(t)>x)}{x^{-\mu}}dt \\
&= \frac{1}{T}\int_0^T C(t)dt
\end{aligned}$$

where

$$C(t)=c\sum_{n=0}^{+\infty}e^{-\lambda}\frac{\lambda^n}{n!}\exp\left(\mu\left(m_n(t)+\frac{1}{2}\Psi^2 t\mu\right)\right)$$

by the results in section 8.5. ■

8.7. Age-dependent death rate

Agents have a portfolio selection problem among a risky asset, a risk-free asset, and life insurance. The agent purchases life insurance, $P(t)$, during the lifetime and leaves bequest, $Z(t)$, to his child.

Define

$$G(t)=\int_t^T \pi(s)ds$$

and

$$\pi(v,t)=\frac{\pi(v)}{G(t)}$$

The conditional death rate is

$$u(t)=\frac{\pi(t)}{G(t)}$$

There is a perfect life insurance market. Thus the price of the life insurance is $u(t)$. Thus

$$Z(t)=w(t)+\frac{P(t)}{u(t)}$$

Consumer's problem is

$$J(w,t)=\max_{c,P,\phi}\{E_t\int_t^T\pi(v,t)[\int_t^v\frac{c(s)^{1-\gamma}}{1-\gamma}e^{-\theta(s-t)}ds+\chi\frac{[(1-\zeta)Z(v)]^{1-\gamma}}{1-\gamma}e^{-\theta(v-t)}]dv\}$$

Using integration by parts, I rewrite the agent's problem,

$$J(w,t)=\max_{c,P,\phi}\frac{E_t\int_t^T\left(G(s)\frac{c(s)^{1-\gamma}}{1-\gamma}+\pi(s)\chi\frac{[(1-\zeta)Z(s)]^{1-\gamma}}{1-\gamma}\right)e^{-\theta(s-t)}ds}{G(t)}$$

$$\begin{aligned} s.t.\quad dw(s) \;=\;& [(r-\tau)w(s)+((\alpha-\tilde{\tau})-(r-\tau))\phi(s)w(s)-c(s)-P(s)+\omega+\Gamma]ds \\ & +\sigma\phi(s)w(s)dz(s)+\delta\phi(s)w(s)dq(s) \end{aligned}$$

Define human wealth $h(t)$ as

$$h(t)=\int_t^T(\omega+\Gamma)e^{-\int_t^s[(r-\tau)+u(v)]dv}ds$$

Proposition 7. [11] *The agent's policy functions are*

$$c(t)=a(t)^{-\frac{1}{\gamma}}(w(t)+h(t))$$

$$P(t)=u(t)\left(\left(\frac{a(t)}{\chi(1-\zeta)^{1-\gamma}}\right)^{-\frac{1}{\gamma}}(w(t)+h(t))-w(t)\right)$$

and

$$\phi(t)w(t)=\Pi(w(t)+h(t))$$

where $a(t)=\left(\left(1+(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}\eta\right)\int_t^T e^{\int_t^s(\eta-\lambda(v))dv}ds+(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}\right)^{\gamma}$ *in which* $\eta=\frac{(1-\gamma)(r-\tau)-(\theta+\lambda)}{\gamma}+\frac{1-\gamma}{\gamma}\left(((\alpha-\tilde{\tau})-(r-\tau))\Pi-\frac{1}{2}\gamma\sigma^2\Pi^2+\frac{\lambda}{1-\gamma}E(1+\delta\Pi)^{1-\gamma}\right)$.

As in section 2.1 let

$$x(t)=w(t)+h(t)$$

Thus

$$dx(t)=f(t)x(t)dt+\Psi x(t)dz(t)+\delta\Pi dq(t)$$

where $f(t)=(r-\tau)+u(t)+((\alpha-\tilde{\tau})-(r-\tau))\Pi-[1+(\chi(1-\zeta)^{1-\gamma})^{\frac{1}{\gamma}}u(t)]a(t)^{-\frac{1}{\gamma}}$

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[11] *I put this proof on my webpage to shorten the Appendix. For the proof of this Proposition please visit my webpage*
http://homepages.nyu.edu/~sz436/

and $\Psi = \sigma\Pi$. The explicit form solution of $x(t)$ is

$$x(t) = \left(\frac{a(t)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda t + \Psi z(t)](1+\delta\Pi)^{q(t)} x(0) \tag{8.2}$$

where $\Lambda = \frac{(r-\tau)-(\theta+\lambda)+((\alpha-\tilde{\tau})-(r-\tau))\Pi}{\gamma} - \frac{1}{2}(2-\gamma)\sigma^2\Pi^2 + \frac{\lambda}{\gamma}(1+\delta\Pi)^{1-\gamma}$.

Now let t_1, t_2, t_3,$\cdots$, t_n,$\cdots$ be the born time of generation 1, 2, 3,$\cdots$, n,$\cdots$. Let

$$x_1 = x(t_1), x_2 = x(t_2), x_3 = x(t_3), \cdots, x_n = x(t_n), \cdots$$

Thus

$$x_{n+1} = \rho_{n+1} x_n + h(0)$$

where

$$\rho_{n+1} = \left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{1}{\gamma}} \exp[\Lambda(t_{n+1}-t_n) + \Psi(z(t_{n+1}) - z(t_n))](1+\delta\Pi)^{q(t_{n+1})-q(t_n)} \tag{8.3}$$

For fixed t_{n+1}, ρ_{n+1} is a random variable. But $t_{n+1} - t_n$ follows the density function of $\pi(\cdot)$. Thus ρ_{n+1} is a compound random variable. By Sornette, the agent's starting wealth has a stationary distribution with a Pareto upper tail, if there exists a $\mu > 1$ such that

$$E\rho_{n+1}^{\mu} = 1$$

Thus by the formula 8.3 μ solves

$$\left(\frac{\chi(1-\zeta)}{a(0)}\right)^{\frac{\mu}{\gamma}} \int_0^T \exp\left([\Lambda\mu + \frac{1}{2}\Psi^2\mu^2 + ((1+\delta\Pi)^{\mu} - 1)\lambda]t\right) \pi(t)dt = 1$$

The proof of Proposition 5 is the same as the proof in section 8.5, since the equations (8.2) and (8.2) has the same form.

The proof of Proposition 6 is almost the same as the proof in section 8.6. The only difference is that here we apply the Lebesgue dominated convergence theorem in the place where we apply the bounded convergence theorem in section 8.6. ∎

8.8. The dependence of Pareto exponent on taxes

$\zeta = 0$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	1.938	1.953	1.968	1.983
0.007	1.927	1.941	1.956	1.972
0.008	1.916	1.931	1.945	1.960
0.009	1.906	1.920	1.934	1.949
$\zeta = 0.1$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	1.941	1.956	1.971	1.987
0.007	1.930	1.945	1.960	1.975
0.008	1.920	1.934	1.949	1.964
0.009	1.910	1.924	1.938	1.952
$\zeta = 0.2$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	1.945	1.960	1.975	1.991
0.007	1.934	1.949	1.964	1.979
0.008	1.924	1.938	1.953	1.967
0.009	1.914	1.927	1.942	1.956
$\zeta = 0.3$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	1.949	1.964	1.980	1.995
0.007	1.939	1.953	1.968	1.983
0.008	1.928	1.942	1.957	1.972
0.009	1.918	1.932	1.946	1.960

8.9. The dependence of Gini on taxes

$\zeta = 0$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	0.805	0.810	0.815	0.820
0.007	0.812	0.818	0.823	0.828
0.008	0.821	0.823	0.830	0.836
0.009	0.825	0.830	0.836	0.844
$\zeta = 0.1$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	0.804	0.810	0.815	0.819
0.007	0.812	0.818	0.822	0.828
0.008	0.817	0.823	0.829	0.835
0.009	0.825	0.830	0.836	0.844
$\zeta = 0.2$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	0.804	0.808	0.813	0.821
0.007	0.809	0.814	0.822	0.827
0.008	0.819	0.822	0.829	0.835
0.009	0.823	0.827	0.834	0.842
$\zeta = 0.3$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	0.801	0.809	0.812	0.816
0.007	0.809	0.814	0.820	0.825
0.008	0.816	0.821	0.825	0.833
0.009	0.821	0.826	0.834	0.838

8.10. The dependence of welfare on taxes (welfare$\times 10^{11}$)

$\zeta = 0$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	-1.360	-1.366	-1.372	-1.379
0.007	-1.336	-1.340	-1.347	-1.353
0.008	-1.309	-1.313	-1.321	-1.328
0.009	-1.282	-1.288	-1.294	-1.301
$\zeta = 0.1$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	-1.355	-1.360	-1.367	-1.373
0.007	-1.329	-1.335	-1.341	-1.347
0.008	-1.303	-1.308	-1.316	-1.322
0.009	-1.275	-1.283	-1.289	-1.296
$\zeta = 0.2$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	-1.348	-1.353	-1.360	-1.368
0.007	-1.321	-1.328	-1.333	-1.341
0.008	-1.296	-1.302	-1.308	-1.315
0.009	-1.268	-1.274	-1.282	-1.288
$\zeta = 0.3$				
$\tau \backslash \tilde{\tau}$	0.006	0.007	0.008	0.009
0.006	-1.338	-1.344	-1.351	-1.358
0.007	-1.312	-1.318	-1.325	-1.332
0.008	-1.286	-1.291	-1.299	-1.306
0.009	-1.258	-1.264	-1.272	-1.279