Eductive Stability in Real Business Cycle Models*

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Abstract

We reexamine issues of coordination in the standard RBC model. Is the unique rational expectations equilibrium attainable by rational agents who contemplate the possibility of small deviations from equilibrium? Surprisingly, we find that coordination cannot be expected. Even with strong common knowledge assumptions, rational agents anticipating small but persistent deviations are led to take actions that eventually contradict the common knowledge assumption. This "impossibility" theorem for eductive learning is not fully overcome when adaptive learning is incorporated into the framework.

1 Introduction

This paper examines the question of expectational coordination in a simple Real Business Cycle model. The long run focal point for expectational coordination, as usual in economic modelling, is the rational expectations (in this simple model, perfect foresight) equilibrium. Our analysis puts emphasis on the expectational robustness of the equilibrium, using what may be called the "eductive" viewpoint, (see Evans and Guesnerie (1993, 2005) and Guesnerie (2002) for an introductory conceptual assessment

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and Guesnerie (2005) for a collection of studies along these lines). "Eductive learning" can be contrasted with the more standard "evolutive" or "adaptive" learning viewpoint,[1] which is also introduced at a later stage in the paper. The logical framework and the central results for eductive learning, as well as the connections with evolutive learning, are well understood in many contexts, and in particular within simple models of the overlapping generations type in which agents are short lived (see Gauthier and Guesnerie (2004) for an assessment that puts emphasis on the consistency of different viewpoints on expectational coordination).

In contrast, in an RBC (real business cycle) model, agents are long lived and in fact have infinite life. The long-life assumption plays an important role in the working of the world under examination. In particular, long-lived agents take into account their permanent income, rather than income over a short horizon, a fact that has a key impact on the understanding and design of macroeconomic policies. The question under scrutiny here is the effect of the introduction of long-lived agents on expectational coordination: does it make expectational coordination more or less robust?

The answer, based on our "eductive" assessment of coordination, is that coordination is necessarily weak. There is no collective image of the future, close but not identical to the "true," self-fulfilling, image, which is able to trigger (a common knowledge of) the self-fulfilling image. Thus every such image is subject, at some stage, to be invalidated by facts: in this simple world, a "crisis," here an expectational crisis, is in some sense unavoidable. However, the extent of weakness of expectational coordination, and metaphorically the plausibility of the crisis, depends upon certain system characteristics that we identify. Furthermore, the "real-time" amendment of the collective image of the future must necessarily rely on adaptive learning, the success of which in maintaining the collective image of the future, i.e. in some sense in avoiding the crisis, also depends on the system features that we stress.

The paper proceeds as follows. In Section 2, we present the model and its equilibria. We then provide a number of preliminary results, together with their intuition, on the connections between long-run and short-run individual expectations and aggregate, long-run or short-run, effects. In Section 3, we present from two different viewpoints the eductive criteria that serve to assess expectational robustness of the equilibrium. Section 4 provides conditions for weak eductive stability, gathered in three propositions. Section 5 shows that strong eductive stability, whatever its exact definition, necessarily fails. Section 6 focuses on the possibility of maintaining a plausible image of the future in the presence of real-time adaptive learning. Section 7, which precedes the Conclusion, contrasts our results with those for a model of capital accumulation in which agents have short lives.

[1]For the adaptive learning approach see, for example, Marcet and Sargent (1989), Woodford (1990) and Evans and Honkapohja (2001).

2 The model, equilibrium and the influence of beliefs on states

2.1 The model and equilibrium

We consider a standard RBC model, except that for simplicity we assume a fixed labor supply and omit exogenous productivity shocks.[2] These simplifying assumptions, which amount to a focus on a nonstochastic discrete-time Ramsey model, are not critical to our results and are made in order to clarify the central features of our analysis. Elimination of both random shocks and labor-supply response to disequilibrium expectations can be expected to facilitate coordination on the REE.[3] Despite eliminating these influences we establish that a strong form of eductive stability fails.

2.1.1 The household problem

There is a continuum of identical infinitely-lived households, indexed by $\omega \in [0,1]$. Each household ω owns capital, $k_t(\omega)$, and one unit of labor, supplied inelastically. At time $t = 0$, household ω solves

$$\max \quad E_0(\omega)\sum_{t=0}^{\infty}\beta^t U(c_t(\omega)), \;\text{ where } 0<\beta<1, \tag{1}$$

$$\text{subject to} \quad k_{t+1}(\omega) = (1+r_t)k_t(\omega) + q_t - c_t(\omega), \tag{2}$$

with initial wealth $k_0(\omega)$ given. We will focus on the case in which $k_0(\omega)$ is the same for all agents, but it is convenient not to impose this initially. Here q_t is the wage rate, and r_t is the rental rate for capital, at time t, and the utility function $U(c)$ is increasing, strictly concave and smooth. We further impose a No Ponzi Game (NPG) condition that the present value of their limiting lifetime wealth be nonnegative. If, at this stage, one does not assume that the future is deterministic, $E_0(\omega)$ captures the expectations of agent ω formed using his subjective distribution.

Iterating forward the household flow budget constraint, and imposing the NPG and transversality conditions, gives the lifetime budget constraint of the household; thus, we rewrite the consumer program as:

$$\max E_0(\omega)\sum_{t=0}^{\infty}\beta^t U(c_t(\omega))$$

[2]The seminal papers developing the RBC model include Kydland and Prescott (1982), Long and Plosser (1983) and Prescott (1986).

[3]For example, the weak eductive stability conditions, given below, can be shown to be stricter when labor supply is elastic.

subject to

$$\begin{aligned}\sum_{t=0}^{\infty} R_t c_t(\omega) &= \sum_{t=0}^{\infty} R_t q_t + (1+r_0)k_0(\omega), \text{ where} \\ R_t &= \prod_{i=1}^{t}(1+r_i)^{-1}\end{aligned}$$

and $R_0 = 1$. The first-order condition for the household optimization problem is the Euler equation

$$U'(c_t(\omega)) = \beta E_t(\omega)\left((1+r_{t+1})U'(c_{t+1}(\omega))\right). \tag{3}$$

2.1.2 Equilibrium

Goods are produced by firms from capital and labor using a constant returns to scale production function $f(K, L)$, satisfying the usual assumptions, under conditions of perfect competition. Thus r_t, q_t are given by

$$\begin{aligned} r_t &= f_K(K_t, 1) - \delta \\ q_t &= f_L(K_t, 1), \end{aligned}$$

where $K_t = \int_0^1 k_t(\omega)d\omega$ and where $f_K = \partial f/\partial K$ and $f_L = \partial f/\partial L$. For convenience, below, we also write $f(K)$ in place of $f(K, 1)$ and use the notation $f' = f_K$ and $f'' = f_{KK}$. In addition we have the aggregate capital accumulation equation

$$K_{t+1} = (1-\delta)K_t + f(K_t, 1) - C_t,$$

where $C_t = \int_0^1 c_t(\omega)d\omega$.

We can now define the (unique) perfect foresight steady state.

Definition 1 *The perfect foresight steady state $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, $r_t = \bar{r}$ and $q_t = \bar{q}$ is given by*

$$\begin{aligned} 1 &= \beta(1+\bar{r}) \\ \bar{r} &= f_K(\bar{K}, 1) - \delta, \\ \bar{q} &= f_L(\bar{K}, 1) \\ \bar{C} &= f(\bar{K}, 1) - \delta\bar{K}, \end{aligned}$$

Since

$$f(\bar{K}, 1) = f_K(\bar{K}, 1)\bar{K} + \bar{q}$$

we also know that $\bar{C} = \bar{r}\bar{K} + \bar{q}$: in steady state, agents consume what is left after depreciated capital is replaced.

If $K_0 = \bar{K}$ then under perfect foresight the economy stays in the steady state for all t, and if $K_0 \neq \bar{K}$, then there is a unique perfect foresight path that converges to the steady state as $t \to \infty$. We now assume that we are initially in steady state, with $k_0(\omega) = \bar{K}$ for all ω, and we examine the robustness of expectational coordination on this equilibrium.

2.2 Beliefs, actions, plans and realizations

2.2.1 Preliminaries

Consider an individual agent facing the consumption/savings problem (1)-(2). The behavior of the agent is in part determined by his beliefs about the future values of wages and interest rates. At the most general level, an agent's beliefs may be stochastic, and so summarized by a sequence of joint density functions $\{F_t(q^t, r^t)\}$ where q^t and r^t are the time t wage and interest rate histories, respectively. Here, we restrict attention to deterministic beliefs, i.e. to point expectations, which is satisfactory in our nonstochastic setting for beliefs that are small deviations from perfect foresight.

The beliefs of agent ω may therefore be summarized by real sequences of expected wages and interest rates. We choose to assume that the agent understands the relationship between aggregate capital and input prices, that is, the agent knows

$$r_t = f'(K_t) - \delta, \quad \text{and} \quad q_t = f(K_t) - f'(K_t)K_t,$$

so that, in fact, his beliefs are completely captured by a sequence of real numbers identifying his point expectations of future capital stock: we denote these beliefs by $\tilde{K}^e(\omega) = \{K_t^e(\omega)\}_{t\geq 0}$, where $K_0^e(\omega) = K_0 = \bar{K}$ is known to all agents. A beliefs profile is the collection of all agents' beliefs: $\tilde{K}^e = \{\tilde{K}^e(\omega) : \omega \in [0,1]\}$.

The key ingredient of the analysis is the understanding of the effect of changes in individual expectations on changes in individual actions or plans (particularly when these changes occur around the equilibrium). Taking as reference point the perfect foresight steady-state path $K_0 = \bar{K}, K_t = \bar{K}$, for all t, we analyze the effects on an agent's plans of small changes, around the steady-state values, in the agent's individual initial capital $k_0(\omega)$, in the aggregate initial capital, and in the agent's point expectations $K_t^e(\omega)$. The analysis relies on the following lemma.

Lemma 1 *Let $V(k_0(\omega), \tilde{K}^e(\omega))$ be the value function associated with the problem (1)-(2) and the NPG condition, given initial stock $k_0(\omega)$, aggregate capital K_0 and beliefs $\tilde{K}^e(\omega)$. The derivatives of the value function, at the steady-state path $k_0(\omega) = \bar{K}$, $K_0 = \bar{K}$, $K_t^e(\omega) = \bar{K}$ for all $t \geq 1$, are as follows:*

1. $\frac{\partial V}{\partial k_0(\omega)} = \beta^{-1} U'(\bar{C})$

2. $\frac{\partial V}{\partial K_0} = 0$

3. $\frac{\partial V}{\partial K_t^e(\omega)} = 0.$

The proof of the Lemma is provided in the Appendix. The first result of the Lemma is fairly intuitive from examination of the intertemporal budget constraint. The third result relies on the fact that, given the individual budget constraint (1), the equilibrium plan $K_t = k_t(\omega) = \bar{K}$, $C_t = c_t(\omega) = \bar{C}$, is still feasible, to first-order approximation, when r_t changes from $\bar{r}$ to $\bar{r} + f_{KK}(\bar{K}, 1)dK_t^e(\omega)$ and q_t changes from $\bar{q}$ to $\bar{q} + f_{KL}(\bar{K}, 1)dK_t^e(\omega)$, since $\bar{K}f_{KK}(\bar{K}, 1) + f_{KL}(\bar{K}, 1) = 0$. (The second result uses the same argument at period 0). In other words, the expected price changes induced by the change in expectations of the capital stock have no income effect, to first-order approximation. For that reason, expected price changes have no welfare effect, as stressed in the second and third parts of the Lemma.

2.2.2 Further insights

Lemma 1 provides a useful computational conclusion, which is most easily exploited by allowing for a modified notation that measures quantities as deviations from steady state: set $dk_t(\omega) = k_t(\omega) - \bar{K}$ and $dK_t^e(\omega) = K_t^e(\omega) - \bar{K}$, and similarly for any individual or aggregate quantity or point expectation thereof. Using the Lemma we can now explicitly compute, to first-order, the consumption function.

Because we are restricting attention to beliefs time paths within a small neighborhood of the steady state, we will often identify a particular variable's time path with its first-order approximation; and we will use our deviation notation to capture this identification. Suppose, for example, that agent ω has beliefs $\tilde{K}^e(\omega)$, initial wealth $k_0(\omega)$, and that $c_t(\omega)$ is his optimal consumption decision at time t. Then we will also say that he has beliefs $d\tilde{K}^e(\omega)$, initial wealth $dk_0(\omega)$ and that $dc_t(\omega)$ is his optimal consumption decision at time t.[4] With this notation, an intuitively straightforward consequence of the previous lemma is:

Corollary 1 *Consider any time path of beliefs $dK_t^e(\omega)$, and any initial capital stock $dk_0(\omega)$ and dK_0. Let $dc_t(\omega)$ be the associated sequence of consumption decisions. Then*

$$\beta^{-1}dk_0(\omega) = \sum_{t\geq 0} \beta^t dc_t(\omega). \tag{4}$$

[4] We may compute the agent's optimal consumption as

$$dc_t(\omega) = \frac{\partial c_t(\omega)}{\partial k_0(\omega)}dk_0(\omega) + \sum_{s\geq 1} \frac{\partial c_t(\omega)}{\partial K_s^e(\omega)}dK_s^e(\omega),$$

where the partial derivatives are evaluated at steady-state beliefs.

Corollary 1 follows from Lemma 1 and implies in particular that given beliefs $dK_t^e(\omega)$, if $k_0(\omega) = \bar{K}$ then the optimal consumption path satisfies $\sum \beta^t dc_t(\omega) = 0$. The natural welfare interpretation of the second part of the Lemma can again be stressed: a change in the expected path of aggregate capital, and the corresponding expected price changes that it triggers, have no first-order impact on welfare. For this reason, we call this the *welfare corollary.*

The next Lemma exploits the welfare corollary as well as the above individual Euler equation.

Lemma 2 *Consider any time path of beliefs $dK_t^e(\omega)$ and assume $dk_0(\omega) = 0$. Let $dc_t(\omega)$ be the associated sequence of consumption decisions. Then*

$$\frac{dc_t(\omega)}{\bar{C}} = \frac{dc_0(\omega)}{\bar{C}} + \frac{\beta}{\sigma}\left(\sum_{s=1}^{t} dr_s^e(\omega)\right),$$

where $\sigma = -\bar{C}U''(\bar{C})/U'(\bar{C})$, and

$$dc_0(\omega) = -\left(\frac{\beta\bar{C}}{\sigma}\right)\left(\sum_{s=1}^{\infty} \beta^s dr_s^e(\omega)\right),$$

where $dr_t^e(\omega) = f''(\bar{K})dK_t^e(\omega)$.

The above Lemma (which is proved in the Appendix) brings two facts into sharp relief. The first, expressed in the second formula, is the sensitivity of current consumption, $dc_0(\omega)$, to expectations of the distant future. The second, captured in the first formula, is still more striking since it shows that $dc_t(\omega)$, the plans for even distant future levels of consumption, can be extremely sensitive to a change in expectations. As will be seen later, this is a key issue for the assessment of expectational stability as we approach it here.

3 The robustness of expectational coordination: "eductive stability" criteria

We shall first provide a definition of "eductive" stability based on rather abstract game-theoretical considerations. This will be the "high-tech" view on expectational coordination. We shall however see later how the sophisticated "high-tech" viewpoint meets simpler considerations, that may be termed "low-tech". This first analysis will voluntarily leave in the shadow the time dimension of the problem. We shall reintroduce time in order to see how to adapt the general ideas to our infinite horizon setting.

3.1 Local eductive stability: the high-tech view

To introduce the concepts we consider an abstract economy populated with rational economic agents (in all the following, we shall assume that these agents are infinitesimal and modelled as a continuum). The agents know the logic of the collective economic interactions (the underlying model). Both the rationality of the agents and the model are Common Knowledge (CK). The state of the system is denoted E and belongs to some subset $\mathcal{E}$ of some vector space.

Note that E can be a number (the value of an equilibrium price or a growth rate), a vector (of equilibrium prices,...), a function (an equilibrium demand function), an infinite trajectory of states, as will be the case in this paper, or a probability distribution.

Emphasizing the expectational aspects of the problem, we view an equilibrium of the system as a state E^* such that if everybody believes that it prevails, it does prevail.[5]

Under eductive learning, as described below, each agent contemplates the possible states of the economy implied by the beliefs and associated actions of the economy's agents. Coordination on a particular equilibrium outcome obtains when this contemplation, together with the knowledge that all agents are engaged in the same contemplation, rules out all potential economic outcomes except the equilibrium. If coordination on an equilibrium is implied by the eductive learning process, then we say that the equilibrium is "eductively stable"or "strongly rational."[6] The argument can be either global or local. We now introduce the local version.

Formally, we say that E^* is locally eductively stable (or locally strongly rational) if and only if one can find some *non-trivial* "small" neighbourhood of E^*, $V(E^*)$, such that Assertion A implies Assertion B:

Assertion A : It is "hypothetically" CK that $E \in V(E^*)$.

Assertion B : It is CK that $E = E^*$.

Assertion A is at this stage hypothetical.[7] In the stable case the mental process that leads from Assertion A to Assertion B is the following:

1. Because everybody knows that $E \in V(E^*)$, everybody knows that everybody

[5] Note that E^* is such that the assertion "it is CK that $E = E^*$" is meaningful.

[6] We remain rather vague on the full game theoretical background of our investigation (for a deeper discussion of some of the issues, the reader may refer to Guesnerie and Jara-Moroni (2011)). A study within a "normal form" framework echoing the preoccupations of the present paper can be found in Matsui and Oyama (2006). We also remark that we here view eductive stability as a zero-one criterion. Less stringent indices of stability could also be developed, e.g. see Desgranges and Ghosal (2010) for one such approach.

[7] Although it might be sustained by some policy commitment.

limits their responses to actions that are best responses to some probability distributions over $V(E^*)$. It follows that everybody knows that the state of the system will be in a set $\mathcal{E}(1) \subset \mathcal{E}$.

2. If $\mathcal{E}(1)$ is a proper subset of $V(E^*)$, the mental process goes on as in step 1, but based now on $\mathcal{E}(1)$ instead of $V(E^*)$. In this case it follows that everybody knows that the state of the system will be in a set $\mathcal{E}(2) \subset \mathcal{E}$.

3. The process continues inductively provided that at each stage, $\mathcal{E}(n)$ is a proper subset of $\mathcal{E}(n-1)$.

In the stable case, we then have a decreasing sequence $V(E^*) \supset \mathcal{E}(1) \supset \cdots \supset \mathcal{E}(n-1) \supset \mathcal{E}(n)$.[8] When the sequence converges to E^*, the equilibrium is "locally strongly rational" or "locally eductively" stable. Here "locally" refers to the fact that the initial neighbourhood is small.[9] Note also that intuitively, in the small neighbourhood case, whenever the first step conclusion obtains then the next step is likely to follow.[10]

3.2 Local eductive stability: the low-tech view

The above definition, based on the successful deletion of non-best responses and starting under the assumption that the state of the system is close to the equilibrium state, reflects the "local" version of a "hyper-rationality" viewpoint. Another plausible intuitive definition of local expectational stability would be the following: there exists a non-trivial neighbourhood of the equilibrium such that, if everybody believes that the state of the system is in this neighbourhood, it is necessarily the case, whatever the specific form taken by everybody's belief, that the state is in the given neighbourhood.[11] With the above formal apparatus, the definition would be: one can find some non-trivial "small" neighbourhood of E^*, $V(E^*)$, such that if everybody believes that $E \in V(E^*)$, then the state of the system will be in $\mathcal{E}(1)$, a proper subset of $V(E^*)$. Again, $V(E^*)$ is an initial belief assumption, a universally shared conjecture on the set of possible states. Actual facts which result from individual actions which are

[8] A given model or economic environment may be naturally allied with several distinct common knowledge assumptions: these common knowledge assumptions will all impose the recursive reasoning process described above, but will differ in the initial restrictions on agents' beliefs; and because of the central role the initial restriction plays in the eductive learning process, different common knowledge assumptions may produce different stability results.

[9] If the initial neighbourhood were equal to the whole space $\mathcal{E}$, then the word global would replace the word local.

[10] At step 1, $\mathcal{E}(1) \subset \mathcal{E}$ is CK and the mental process goes on in step 2, so that the first step contraction still acts with a decreased support.

[11] The conjectural equilibrium bounds discussed by Benhabib and Bull (1988), in the context of the overlapping generations model of money, has a similar motivation.

best responses to some probability distributions over $V(E^*)$ cannot falsify the initial beliefs. Equivalently, in the absence of such a neighbourhood $V(E^*)$, facts may falsify any "collective" conjecture, whatever the proximity of the conjectured set to the equilibrium (unless the conjecture is reduced to the equilibrium E^* itself). The argument is here low-tech, in the sense that it refers to the rationality of agents, but not to CK of rationality or of the model:[12] the criterion refers to agents' actions which depend only on their beliefs about the state of the system, and not on their beliefs about other agents' beliefs. To put it in another way, as stated, the criterion appeals only to the results of the first step of the high-tech criterion. However, we have argued above that it is intuitively plausible that the high-tech and low-tech criteria turn out to be equivalent in this abstract setting, as previously stressed in the literature.[13]

Finally, some words are in order concerning the connections between the "eductive" viewpoint and the "evolutive learning" viewpoint. At this point let us only say that the failure to find a set $V(E^*)$, for which the equilibrium is locally strongly rational, signals a tendency for any near-equilibrium states of beliefs, a priori reachable through some "reasonable" evolutive updating process, to be driven away in some cases, a fact that threatens the convergence of the corresponding learning rule.[14]

3.3 Eductive stability: the time dimension

The time dimension of our problem, and in particular the infinite horizon, as well as the fact that agents are infinitely-lived, brings some additional issues to our general framework.

The equilibrium E^* under consideration is given by $K_t = \bar{K}$ for all t, and the first issue is concerned with the notion of a neighbourhood $V(E^*)$, which is less straightforward here than in timeless or short-horizon contexts. We begin with a simple, natural restriction on the initial time paths of beliefs, that they lie within an ε-neighborhood of the steady state, which might be called the "cylinder" assumption. In later sections we consider alternative initial assumptions. The cylinder assumption is:

B1: For some $\varepsilon > 0$ sufficiently small, $K_t \in D(\varepsilon) \equiv [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$ for all t.[15]

Under the high-tech eductive approach we tentatively assume that B1 is common knowledge, which we refer to as

[12] It does not even require full knowledge of the model.

[13] A formal statement of the equivalence requires additional technical assumptions such as the (weak) assumptions stressed of Guesnerie and Jara-Moroni (2009). Their results also allows one to show that the local analysis may concentrate on heterogenous point-expectations.

[14] And certainly forbids a strong form of "monotonic" convergence. More on this subject can be found in Guesnerie (2002), Guesnerie and Woodford (1991) and Gauthier and Guesnerie (2004).

[15] We will later introduce alternative assumptions B1' and B2.

CK1: B1 is common knowledge.

On occasions when it is necessary to be explicit we index B1 by B1(ε) and CK1 by CK1(ε)

Given our emphasis on point expectations, which is justified for a small neighbourhood of beliefs, we say that a beliefs profile $\tilde{K}^e$ satisfies B1 provided that $K_t^e(\omega) \in D(\varepsilon)$ for all ω and t. Furthermore, with CK1, we want to assume that agent ω' knows that $K_t^e(\omega) \in D(\varepsilon)$, and that agent ω'' knows that agent ω' knows that $K_t^e(\omega) \in D(\varepsilon)$, etc. In the logic of the high-tech approach set out above, the equilibrium is eductively stable, if rational agents relying on CK1 and common knowledge of rationality, are able to deduce at time zero that $K_t = \bar{K}$ for all t. We will henceforth say in this case that the steady state is *strongly eductively stable*.[16]

A proof of strong eductive stability should proceed as follows. The first stage of the mental process places attention on the infinite-horizon aggregate plans of the agents, associated with any combination of initial beliefs in B1. If such first-order beliefs yields plans that restrict initial beliefs, then the process continues and generates some kind of second-order beliefs, and so on. At the end of such a mental process, agents correctly predict the next period's state, and all future states resulting from the guessed actions and plans. Hence, in our intertemporal context, the high-tech story is more sophisticated than it is in the abstract timeless or one-period context introduced above, but it follows the same logic.

What about the low-tech version? Here the connection is less straightforward. As just argued, strong eductive stability requires that the plans of the agents, as deduced at the first step of the iteration process by everybody from the B1 assumption, be compatible with B1. In a sense, as noted above, this condition should be almost sufficient for (high-tech) strong eductive stability. This line of argument is developed in Sections 5.1 - 5.2 and used to show the impossibility of strong eductive stability.

However, there are further considerations. There is no longer a one-to-one connection between the plans of the agents, after the first step of the collective mental process and the set of actual paths of the economy, assuming either that the initial beliefs are maintained through time, or a fortiori assuming that they may be reconsidered when time passes. In other words, the required contraction property of the first stage of the mental process, associated with strong eductive stability and the absence of falsification of initial beliefs when time passes, are no longer equivalent.

Hence, the low-tech version and the high-tech view of stability a priori differ, and there may several low-tech approaches. The low-tech option considered in Section 5.3

[16] We slightly depart from previous terminology, by leaving the local aspects of the analysis implicit (and not referring to local eductive stability) and by using the word "strong" to stress that the simultaneous coordination at time zero concerns the entire time path of aggregate capital. Later we will distinguish this notion of stability from less demanding concepts discussed below.

consists of looking at the real-time consequences of the beliefs: we ask whether beliefs satisfying B1 trigger actions that generate an *actual* path of the system compatible with B1. Here, we allow individual agents to hold fixed or varying beliefs over time, even though these beliefs may have been (perhaps repeatedly) contradicted, provided they stay close to the equilibrium in the sense of B1, and we require that this implies that aggregate capital continues to satisfy B1.

Finally, another real-time option is to mix adaptive learning considerations with the idea of non-falsifiability of a collective image of the future, associated with the low-tech version of the eductive approach just stressed. Hence, as is standard under adaptive learning, agents are assumed to modify their beliefs over time in response to the realizations of the capital stock according to a natural class of adaptive rules; and, in line with the eductive approach, for stability we require that both beliefs and realizations satisfy B1. This approach is studied in Section 6.

Whatever the approach to coordination is, a necessary condition for eductive stability is that the initial beliefs necessarily trigger a *first period realization* of aggregate capital that is compatible with the belief restriction B1. We call this "weak eductive stability," and now turn to this concept.

4 Weak Eductive Stability

4.1 Further on the eductive view: weak eductive stability as a necessary condition

As just argued, the "strong" stability question is whether CK1 will allow agents to coordinate on the unique perfect foresight equilibrium $K_t = \bar{K}$ for all $t \geq 1$. Answering this question will be the focus of Section 5. In the current Section we determine when a necessary condition for this is met: under B1 will the optimal plans of agents necessarily lead to consumption and saving decisions $c_0(\omega), k_1(\omega)$ that satisfy $K_1 \in [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$? If an equilibrium meets this necessary condition, we say that it satisfies *weak eductive stability.*

Definition 2 *The steady state $\bar{K}$ is weakly eductively stable if, given the initial condition $k_0(\omega) = \bar{K}$ for all agents ω, and that for all agents beliefs satisfy B1, the aggregate capital stock in period $t = 1$ implied by the agents' optimal plans satisfies $K_1 \in D(\varepsilon)$.*

4.2 Weak eductive stability: a first result

We establish Lemma A1 in the Appendix which identifies the worst-case expectations of the agents in the sense of inducing a maximum threat to the validity of the initial conjecture. In words the lemma says that the worst-case expectations are on the boundary, i.e. they arise when agents believe that the capital stock will remain at one of the boundaries of the cylinder.

Using this lemma, we may now establish our first stability result.[17]

Theorem 1 *Under B1, the steady state is weakly eductively stable if and only if*

$$\left|\frac{\beta^2\bar{C}f''(\bar{K})}{\sigma(1-\beta)}\right| < 1 \tag{5}$$

Proof: Let $\varepsilon > 0$ be small, so that linear approximations to optimal behavior are valid. Weak eductive stability is equivalent to showing that for any beliefs profile satisfying B1 then $dK_1 \in I(\varepsilon) \equiv (-\varepsilon, \varepsilon)$. By Lemma A1 it suffices to find necessary and sufficient conditions guaranteeing that $dK_1 \in I(\varepsilon)$ provided that $dK_t^e(\omega) = e$ for all ω and t, and for $e = \pm\varepsilon$. Using $dk_1(\omega) = -dc_0(\omega)$, recalling from Lemma 2 that

$$dc_0(\omega) = -\left(\frac{\beta\bar{C}}{\sigma}\right)\left(\sum_{s=1}^{\infty}\beta^s dr_s^e(\omega)\right),$$

and noting $dr_s^e(\omega) = |f''|\,\varepsilon$, where for convenience we now write f'' for $f''(\bar{K})$, we get that

$$dk_1(\omega) = \left|\frac{\beta^2\bar{C}f''}{(1-\beta)\sigma}\right|\varepsilon.$$

From $K_t = \int_0^1 k_t(\omega)d\omega$ we thus have

$$dK_1 = \left|\frac{\beta^2\bar{C}f''}{(1-\beta)\sigma}\right|\varepsilon.$$

Since weak eductive stability requires $|dK_1| < \varepsilon$, the result follows. ■

Note that the formula (5) makes intuitive sense: high σ and low f'' promote eductive stability. As an example, suppose that production is Cobb-Douglas, so that $f(K) = K^\theta$, for $0 < \theta < 1$, and that utility takes the constant elasticity form $U(c) = (c^{(1-\sigma)} - 1)/(1-\sigma)$, for $\sigma > 0$. Then

$$\bar{K} = \left(\frac{\theta}{\bar{r}+\delta}\right)^{1/(1-\alpha)} \text{ and } f'' = \theta(\theta-1)\bar{K}^{\theta-2}.$$

[17] In the following theorem we by assumption exclude the case in which $\left|\beta^2\bar{C}f''/\left(\sigma(1-\beta)\right)\right| = 1$. Throughout the paper we exclude analogous nongeneric cases.

Using these and the steady-state equations for β and $\bar{C}$ it can be computed that if, say, $\theta = 1/3$, $\bar{r} = 0.05$ and $\delta = 0.10$, then we have weak eductive stability if and only if $\sigma > 2/3$. This condition is perhaps plausibly satisfied, but we have obtained the somewhat surprising result that even weak eductive stability of the RBC model cannot be taken for granted.

4.3 Weak eductive stability: a further result

A natural question is whether weak eductive stability can be obtained if assumption B1 is strengthened. We consider the following alternative assumption in which the possible deviations of K_t from $\bar{K}$ shrink to 0 over time at a geometric rate.

B1': For some $0 < a < 1$ and for $\varepsilon > 0$ sufficiently small, $K_t \in [\bar{K} - a^t\varepsilon, \bar{K} + a^t\varepsilon]$ for all t.

Theorem 1' *For $a > 0$ sufficiently small, the steady state is weakly eductively stable under B1' if and only if*

$$-\frac{f''\bar{C}}{\sigma}\beta^2 < 1.$$

Proof: Suppose again that all agents have homogeneous expectations given by one of the extremes $K_t^e(\omega) = K_t^e = \bar{K} + a^{t-1}e$ for all $t = 1, 2, 3, \ldots$, where $e = \pm\varepsilon$. Then

$$\begin{aligned} dC_t &= dC_0 + \frac{\beta\bar{C}}{\sigma}f'' \textstyle\sum_{s=1}^{t} dK_s^e \\ &= dC_0 + \frac{\beta\bar{C}}{\sigma}f''e\frac{1-a^t}{1-a}. \end{aligned}$$

Since also $\sum_{t=0}^{\infty} \beta^t dC_t = 0$, we obtain :

$$\sum_{t=0}^{\infty} \beta^t dC_0 + \sum_{t=0}^{\infty} \beta^t \left(\frac{\beta\bar{C}}{\sigma} f''e\frac{1-a^t}{1-a} \right) = 0,$$

which yields

$$dC_0 = -\frac{\beta^2\bar{C}f''e}{\sigma(1-\beta a)}.$$

Considering $a \to 0$ establishes the result. ■

Unsurprisingly, by imposing stronger belief assumptions, the condition we obtain is weaker than the previous one. However, it puts emphasis on similar features of the system, i.e. $\beta^2, \bar{C}, f'', \sigma$, with similar intuitive effects. We reserve consideration of a more general class of assumptions to the case of strong eductive stability, to which we now turn.

5 Strong eductive stability: impossibility theorems

5.1 The impossibility of CK1 strong eductive stability

The question of local "strong eductive stability" was discussed in Section 3.3 as a suitable formulation of the high-tech approach that takes into account the time dimension. For our economic model we can state this explicitly as follows:

Definition 3 *The steady state is strongly eductively stable if CK1 implies that it is common knowledge that the equilibrium path, $K_t = \bar{K}$ for all t, will take place.*

In Theorem 1 we gave a condition for a minimal consistency requirement: given beliefs B1, the initial plans of agents will necessarily be consistent with B1 *in the first period* $t = 1$ if and only if the stated condition (5) is satisfied. However, this is only a weak necessary condition for consistency in the stronger sense just defined. In $t = 0$, given their expectations of the aggregate capital stock $\{K_t^e(\omega)\}_{t=1}^{\infty}$ each agent formulates an optimal dynamic consumption plan $\{c_t(\omega)\}_{t=0}^{\infty}$. This implies a trajectory for the aggregate consumption $\left\{C_t = \int_0^1 c_t(\omega)d\omega\right\}_{t=0}^{\infty}$ and hence, using

$$K_{t+1} = (1-\delta)K_t + f(K_t, 1) - C_t, \tag{6}$$

a trajectory for the aggregate capital stock $\{K_t\}_{t=1}^{\infty}$. Recall that $D(\varepsilon) = [\bar{K}-\varepsilon, \bar{K}+\varepsilon]$. For local strong eductive stability of the steady state, it is necessary – indeed, for the reasons sketched in Section 3.1, almost sufficient – that the following condition be met: for every $\varepsilon > 0$ sufficiently small, under CK1(ε) the implied trajectory of aggregate capital $\{K_t\}_{t=1}^{\infty}$ lies in a strictly smaller cylinder $D^{\infty}(\varepsilon') = D(\varepsilon') \times D(\varepsilon') \times \cdots$, i.e. $K_t \in D(\varepsilon')$ for all $t = 1, 2, 3, \ldots$, for some $0 < \varepsilon' < \varepsilon$. This of course implies that for strong eductive stability, for all expectations satisfying CK1(ε), we must have $K_t \in D(\varepsilon)$ for all $t = 1, 2, 3, \ldots$. We now show that, in fact, for all parameter values, the plans made by agents at time $t = 0$, based on CK1, can fail to be compatible with beliefs. That is,

Theorem 2 *The steady state is never CK1 strongly eductively stable.*

Proof: Suppose that all agents have homogeneous expectations $K_t^e(\omega) = K_t^e = \bar{K} + e$ for all t, where $e = \pm\varepsilon$. Their consumption plans thus satisfy

$$\frac{dC_t}{\bar{C}} = \frac{dC_0}{\bar{C}} + \frac{\beta}{\sigma}tdr = \frac{dC_0}{\bar{C}} + \frac{\beta}{\sigma}tf''e.$$

We have already shown that $dC_0 = -\left(\sigma(1-\beta)\right)^{-1}\beta^2 f'' e\bar{C}$, which implies

$$dC_t = -\frac{\beta f'' e}{\sigma}\bar{C}\left(\frac{\beta}{1-\beta} - t\right), \text{ for } t = 1, 2, 3, \ldots$$

Linearizing (6) around the steady state we have $dK_{t+1} = (1 + f' - \delta)dK_t - dC_t$, or

$$dK_{t+1} = \beta^{-1}dK_t - dC_t.$$

For $t \geqslant \beta/(1-\beta)$ it can be seen that dC_t is bounded away from 0, with the opposite sign to e, and grows linearly with t. Hence for t sufficiently large $|dC_t| > 2\beta^{-1}\varepsilon$, which implies $|dK_{t+1}| > \varepsilon$ if $|dK_t| < \varepsilon$. It follows that there must be a time t at which $|dK_t| > \varepsilon$. This establishes the result. ■

The Theorem 2 result should be contrasted with the fact that, with short-lived agents, eductive stability holds under assumptions that are reminiscent of (though often more stringent than) standard conditions (such as determinacy, saddle-path configuration). The long-run horizon dramatically affects expectational coordination, as evaluated from the strong eductive viewpoint. In the next Section we show that the instability result does not depend on our specific choice of the neighbourhood.

5.2 The general impossibility of strong eductive stability

We now consider a more general class of beliefs, which nests B1 and B1':

B2: There exists a specified deterministic sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ with $0 < \varepsilon_t < \bar{\varepsilon}$ and $\bar{\varepsilon}$ sufficiently small, such that $K_t \in [\bar{K} - \varepsilon_t, \bar{K} + \varepsilon_t]$ for all t.

CK2: B2 is (tentatively) common knowledge.

Under CK2, Definition 3 of strong eductive stability is modified in the obvious way, and we refer to CK2 strong eductive stability.

We will sometimes refer to the set of trajectories that satisfy B2 as the "B2 tube."

Theorem 3 *The steady state is never CK2 strongly eductively stable.*

Proof: Fix a CK assumption with $\{\varepsilon_t\}_{t=1}^{\infty}$ that satisfies CK2. Suppose that all agents have homogeneous expectations $K_t^e(\omega) = K_t^e = \bar{K} + \varepsilon_t$ for all t. Their consumption plans satisfy

$$\frac{dC_t}{\bar{C}} = \frac{dC_0}{\bar{C}} + \frac{\beta f''}{\sigma}\left(\sum_{s=1}^{t} \varepsilon_s\right). \tag{7}$$

From Lemma 2 we have

$$dC_0 = -\frac{\beta f''}{\sigma}\bar{C}\left(\sum_{s=1}^{\infty} \beta^s \varepsilon_s\right), \tag{8}$$

which implies

$$dC_t = \frac{\beta f''}{\sigma}\bar{C}\left(\sum_{s=1}^{t}\varepsilon_s - \sum_{s=1}^{\infty}\beta^s\varepsilon_s\right), \text{ for } t = 1, 2, 3, \ldots. \tag{9}$$

From the linearization of (6) around the steady, as in the proof of Theorem 2, we have

$$dK_{t+1} = \beta^{-1}dK_t - dC_t. \tag{10}$$

If $(\sum_{s=1}^{\infty}\varepsilon_s) = +\infty$ then (7)-(8) imply that $|dC_t| \to \infty$ as $t \to \infty$. It follows from (10) that for t sufficiently large, $|dK_t| < \varepsilon_t < \bar{\varepsilon}$ implies $|dK_{t+1}| > \bar{\varepsilon}$, so that strong eductive stability fails.

Suppose instead that $(\sum_{s=1}^{\infty}\varepsilon_s)$ is finite. Then $\varepsilon_t \to 0$ as $t \to \infty$. Furthermore, since $0 < \beta < 1$, we have $\sum_{s=1}^{\infty}\varepsilon_s = (1+2a)\sum_{s=1}^{\infty}\beta^s\varepsilon_s$ for some $a > 0$, and hence there exists $T > 0$ such that

$$\sum_{s=1}^{t}\varepsilon_s \geq (1+a)\sum_{s=1}^{\infty}\beta^s\varepsilon_s$$

for all $t \geq T$. Then $t \geq T$ and (9) imply

$$dC_t = \frac{\beta\,|f''|}{\sigma}\bar{C}\left(\sum_{s=1}^{\infty}\beta^s\varepsilon_s - \sum_{s=1}^{t}\varepsilon_s\right) \leq -Q,$$

where $Q = \frac{\beta|f''|}{\sigma}\bar{C}a\sum_{s=1}^{\infty}\beta^s\varepsilon_s > 0$. Choose $T_1 > T$ sufficiently large so that $\varepsilon_t < \beta Q/2$ for all $t > T_1$. By (10) it follows that $|dK_t| < \varepsilon_t$ and $t \geq T_1$ imply

$$|dK_{t+1}| \geq Q/2 > \varepsilon_{t+1}$$

and again strong eductive stability fails. ■

We remark that if the common knowledge assumption for a sequence $\{\varepsilon_t\}_{t=1}^{\infty}$ satisfies CK2 except that $\varepsilon_t = 0$ for some proper subset of times t, the contradiction obtains more directly by focusing attention on such times.

The negative result of Theorem 3 means that the hyper-rationalistic viewpoint of strong eductive stability is never conclusive.[18] Our hyper-sophisticated agents cannot convince themselves that the equilibrium will prevail. In a sense, here in the RBC model, expectational coordination must appeal to bounded rationality considerations.

[18] Instability results also appear in the adaptive learning literature. For example, Howitt (1992) and Evans and Honkapohja (2003) show instability for a class of interest-rate rules in monetary models. However, these models can also suffer from indeterminacy, and stability under adaptive learning can be restored with a suitable choice of interest-rate rule. The generic instability results of the current paper are particularly striking since the RBC model is in general well-behaved.

5.3 A "real-time" alternative view of stability: another impossibility theorem

Another way to read the above statement is that there are always initial plans in line with initial beliefs of the type B2, which turn out to be inconsistent with such beliefs. It follows that CK2 cannot trigger common knowledge of the equilibrium.

In this Section we consider an alternative, real-time version of strong eductive stability, reflecting the low-tech viewpoint described in Section 3.3. B2 continues to describe a (common) set of possible beliefs: the beliefs of individual agents are taken from this set, and now may or may not change through time. At time t the earlier specific beliefs of agents will generally have been falsified by events. However we suppose that their beliefs always obey the initial restrictions B2 as long as the actual path up to t also satisfies $K_s \in [\bar{K} - \varepsilon_s, \bar{K} + \varepsilon_s]$ for $s \leq t$. Will some of the possible paths of the system generated from such beliefs falsify the assumed initial restrictions B2? The connection of this question with the notion of strong stability indicated in Definition 3 is not immediate.[19]

Let us go to the formal analysis. Our alternative low-tech definition of strong eductive stability, for collective beliefs of type B2, is the following.

Definition 4 *We say that $\bar{K}$ is strongly stable in the alternative sense if, for all t, $K_t \in [\bar{K} - \varepsilon_t, \bar{K} + \varepsilon_t]$ implies $K_{t+1} \in [\bar{K} - \varepsilon_{t+1}, \bar{K} + \varepsilon_{t+1}]$ when each agent ω chooses $c_t(\omega)$ optimally given expectations $\{K^e_s(\omega)\}_{s=t+1}^{\infty}$ consistent with B2.*

This definition says that if up to period t, the actual path of capital has remained compatible with the initially assumed restriction on beliefs, (making them close to the equilibrium beliefs, in the sense of assumption B2), then it will remain compatible at period $t+1, \ldots$, and hence for ever. In other words, when Definition 4 is satisfied then collective beliefs B2 are strongly "real-time" stable, in the sense that the realized trajectory of K_t will necessarily satisfy B2.

For this definition we still have:

Theorem 4 *Under the beliefs restriction B2, the steady state is never strongly stable in the sense of the alternative definition.*

Equivalently, Theorem 4 states that there do not exist collective beliefs B2 that are real-time stable in the sense just discussed.

[19] We know from the previous analysis that initial plans turn out to be incompatible with B2; however, the actual path of aggregate capital will clearly differ from the path determined by these initial plans, and so violation of B2 by the actual path of aggregate capital is not obvious: for more on the connection between planned and actual paths, see below.

Our argument will be somewhat informal.[20] To demonstrate Theorem 4, we compare the planned and actual aggregate trajectories of capital associated with particular beliefs consistent with B2. Consider the time $t = 0$ homogeneous beliefs $\{K_t^e\}_{t=1}^{\infty}$ examined in the Proof of Theorem 3. These beliefs, which satisfy B2, generate a path of aggregate capital $\{K_t\}_{t=1}^{\infty}$ as shown in the proof of Theorem 3, which exits the tube B2 in finite time. Let us call this path the "planned" or "virtual-time" trajectory of aggregate capital.

Turning to real-time trajectories, note that in general in the real-time approach, beliefs at time 0 over the whole trajectory, to infinity, determine aggregate K_1, given $K_0 = \bar{K}$; then the beliefs from time 1, up to infinity, determine K_2, and so on. If all real time trajectories remain in B2, as described in Definition 4, then it is also the case for the subset of trajectories with time invariant beliefs, i.e. beliefs maintained from the start forever.

Thus consider the actual path of capital generated by the beliefs $\{K_t^e\}_{t=1}^{\infty}$ just used to obtain the virtual-time path. That is, consider time-invariant beliefs, i.e. beliefs maintained from the start forever, that are identical to those just examined, and consider the corresponding actual path of capital. This real-time trajectory will differ from the virtual-time trajectory just considered, because along the real-time trajectory agents condition their time t decisions on the *realized* aggregate capital at time t as well as the expected future path of aggregate capital in $t + s$ for $s > 0$.

We claim that, for these beliefs, the real-time trajectory is close (in fact at a second-order distance) to the planned trajectory. This comes from the fact that although planned aggregate capital, and the realized aggregate capital along the real-time path, differ, this has only a second order effect on the change of consumption of individual agents. This follows from Corollary 1, which stresses that changes in *aggregate* K_0 do not matter for time $t = 0$ decisions. For the same reason, to first order, *aggregate* K_t has no impact on time t decisions. It follows that, to first-order approximation, the planned and real aggregate trajectories of capital, when beliefs are maintained as explained, are the same. Hence, since the virtual-time trajectory exits the B2 tube in finite time, the real-time trajectory associated with these same beliefs, maintained in real time, must exit the B2 tube in finite time. Hence Theorem 4 follows from Theorem 3.[21]

[20] A formal proof requires only notational care, and is omitted.

[21] In fact the above argument suggests that real-time beliefs stability, in the sense of Definition 4, implies strong eductive stability. The reason can be sketched as follows. If all real-time trajectories remain in B2, as described in Definition 4, then it is also the case for the subset of trajectories with time invariant beliefs, i.e beliefs maintained from the start for ever. But such trajectories, from the above "proximity" argument, are close to the virtual time trajectories of the first step of the "eductive" process, and so must also remain in the tube, so that the second step of the "eductive process" can start. (A precise argument would require some additional regularity assumptions of the kind introduced in the last section of Guesnerie-Jara-Moroni (2011)). The above suggested implication would then follow; again, from Theorem 3, this would imply Theorem 4.

This analysis confirms the pessimism of the first analysis. Trembling beliefs, of the type described here, are subject to real-time falsification, either in the long run or in the short run (leading then to what may be called a crisis?). This brings us to the next viewpoint which must necessarily mix bounded rationality with considerations associated with the eductive viewpoint. Bounded rationality will lead us to introduce learning of the evolutive type.

6 Combining eductive and evolutive learning

6.1 The framework

We have found that the steady state $\bar{K}$ is not strongly eductively stable according to the various definitions given above. At the same time it is known that it is locally asymptotically stable under certain statistical learning rules. At first sight, this suggests a significant contrast between stable adaptive and unstable eductive approaches. A better way to consider the connection is to combine these approaches.

We will again endow agents with expectations about the future path of the aggregate capital stock. These expectations are restricted to belong a set, which for convenience we will take to be described by B1. The set B1 can here be viewed as describing a collective belief that provides bounds to individual beliefs. Starting from $k_0(\omega) = \bar{K}$ these assumptions generate, in accordance with the analysis of Section 4, a range of possible values for K_1.

In line with the eductive approach of Sections 2 and 3, agents' decisions today are based on an assessment of the whole future.[22] Now, however, the "expected" trajectory at time t is supposed not only to reflect initial beliefs but to respond to observed actual capital, and in the spirit of evolutive approach, we specify a set of adaptive learning rules that determine the way initial expectations change along the real-time trajectory of aggregate capital. Then, coming back to the collective belief interpretation of B1, reminiscent of the "eductive" approach, we then ask if the implied paths $\{K_t\}_{t=1}^{\infty}$ will necessarily satisfy B1, i.e. if the collective belief which serves as a frame for the individual beliefs is subject to falsification. If for some nonempty subset of adaptive learning rules, falsification is impossible, then we say that the steady state is B1-stable under evolutive learning, and if this occurs for all adaptive learning rules within the set of adaptive learning rules under consideration, we say it is robustly B1-stable under evolutive learning.

[22] In the adaptive learning literature, within infinite-horizon models, this approach has been followed, for example, in Sargent (1993, pp. 122-125), Preston (2006), Eusepi and Preston (2008) and Evans, Honkapoha and Mitra (2009). An alternative approach in the adaptive learning literature is based on one-step-ahead "Euler equation" learning. See, e.g. Evans and Honkapohja (2001), Ch. 10, and Evans and McGough (2009).

Let us first make more precise what the evolutive learning process is about, and introduce a simple evolutive learning scheme in the nonstochastic RBC model. We assume, in line with standard adaptive learning studies, that all the agents use the same adaptive learning rule.[23] We also assume that what is learned is not the whole future trajectory, but some individually relevant summary statistics of the future. Let us be more precise.

6.2 The real-time system

In the real-time system we assume that at each time t each agent ω re-solves their dynamic optimization problem. That is, at each t agent ω chooses $c_t(\omega)$, given $k_t(\omega)$, to solve their consumer program, given their time t expectations about future wages and interest rates, where these are a function of their time t expectations of future aggregate capital, $\left\{K^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty}$. From Corollary 1, for any specified expectations, the optimal path of consumption satisfies $\beta^{-1}dk_t(\omega) = \sum_{s=0}^{\infty}\beta^s dc_{t+s}(\omega)$, and from Lemma 2 we have $dc_{t+s}(\omega) = dc_t(\omega) + \frac{\beta\bar{C}}{\sigma}\sum_{j=1}^{s} dr^e_{t,t+j}(\omega)$. Combining these equations and using $dr^e_{t,t+j}(\omega) = f'' dK^e_{t,t+j}(\omega)$ we obtain

$$dc_t(\omega) = \frac{(1-\beta)}{\beta}dk_t(\omega) - \frac{\beta\bar{C}}{\sigma}f''\sum_{j=1}^{\infty}\beta^s dK^e_{t,t+j}(\omega), \tag{11}$$

which gives, in deviation from steady-state form, the consumption at t of agent ω as a function of its current wealth $k_t(\omega)$ and the time path of expected future aggregate capital $\left\{dK^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty}$. It can be seen that the agent's decision $c_t(\omega)$ depends on a sufficient statistic for $\left\{dK^e_{t,t+j}(\omega)\right\}_{j=1}^{\infty}$, given by

$$d\hat{K}^e_t(\omega) = \beta^{-1}(1-\beta)\sum_{j=1}^{\infty}\beta^s dK^e_{t,t+j}(\omega).$$

Thus $d\hat{K}^e_t(\omega)$ is the time t discounted sum of the expected future aggregate capital stock. Together with $dk_t(\omega)$ the quantity $d\hat{K}^e_t(\omega)$ is a sufficient statistic for $dc_t(\omega)$. The proportionality factor $\beta^{-1}(1-\beta)$ ensures that if $dK^e_{t,t+j}(\omega) = e$ for all $j = 1, 2, 3, \ldots$, then $d\hat{K}^e_t(\omega) = e$.[24] Furthermore, it is easily seen that under B1, $d\hat{K}^e_t(\omega)$ must lie in the interval $[-\varepsilon, \varepsilon]$.

[23] However, our finding of a failure of robust stability under adaptive learning extends to the case in which heterogeneous adaptive learning rules are permitted, since small perturbations of each agent's learning rule leads to a small perturbation of the aggregate path for K_t.

[24] Adaptive learning in nonstochastic models with infinite horizons often assumes "steady state learning" in which forecasts are the same at all horizons. See, for example, Evans, Honkapohja and Mitra (2009). In the current context this would mean $dK^e_{t,t+j} = e$ at t for all j, with the value of e updated over time. Our formulation in terms of $d\hat{K}^e_t$ allows for greater generality, while retaining

From (11) we have

$$dc_t(\omega) = \frac{(1-\beta)}{\beta} dk_t(\omega) - \frac{\beta^2 \bar{C}}{(1-\beta)\sigma} f'' d\hat{K}_t^e(\omega),$$

and from the linearized household accumulation equation (as in the proof of Theorem 3) we have

$$dk_{t+1}(\omega) = \beta^{-1} dk_t(\omega) - dc_t(\omega).$$

Finally we specify a simple adaptive scheme for the revisions of $d\hat{K}_t^e(\omega)$ over time:

$$d\hat{K}_t^e(\omega) = (1-\alpha) d\hat{K}_{t-1}^e(\omega) + \alpha dK_t,$$

where $0 < \alpha \leq 1$ parameterizes how expectations adapt to current information about the actual capital stock.

We are now in a position to describe the real-time evolution of the system. For the sake of simplicity, we start from an initial situation, in which the time zero belief is the same for everybody $d\hat{K}_0^e(\omega) = d\hat{K}_0$, so that the system has homogeneous expectations for all t, together with initial actual K_0 at or near $\bar{K}$ (i.e., dK_0 is near 0). The homogeneity assumption allows us to calculate the resulting time-path, but is also illuminating in the sense that we would hope the system to be stable under learning if we start near the steady state and with a small commonly-held expected deviation of expected future capital from the steady state.

By combining the above expressions for $dc_t(\omega)$ and $dk_{t+1}(\omega)$ we obtain $dk_{t+1}(\omega) = dk_t(\omega) - \xi d\hat{K}_t^e(\omega)$, where

$$\xi = -\frac{\beta^2 \bar{C} f''}{\sigma(1-\beta)}$$

denotes parameter that determines weak eductive stability in the sense of Section 4. The system can thus be written as

$$dK_{t+1} = dK_t - \xi d\hat{K}_t^e \quad (12)$$

$$d\hat{K}_{t+1}^e = (1-\alpha) d\hat{K}_t^e + \alpha dK_{t+1}. \quad (13)$$

We can now return to the previously suggested definitions of B1-stability under evolutive (adaptive) learning and give formal definitions.

Definition 5 *The equilibrium is B1-stable under adaptive learning for a given* $0 < \alpha \leq 1$ *if, for all* $\varepsilon > 0$ *sufficiently small, the trajectory* $\{K_t\}_{t=1}^{\infty}$*, triggered by (12)-(13), remains in the cylinder* $D(\varepsilon)$*, defined in B1, for all* K_0 *near* $\bar{K}$ *(*dK_0 *near* 0*) and all* $|dK_{0,j}^e| \leq \varepsilon$*,* $j = 1, 2, 3, \ldots$*.*

---

a single sufficient statistic that is updated over time. In stochastic models, the time pattern of interest rates can be estimated and updated using recursive least squares. For technical reasons this procedure cannot be used in nonstochastic systems. Intuitively, in a nonstochastic equilibrium the asymptotic lack of temporal variation makes impossible consistent estimation of the time-series parameters. See Evans and Honkapohja (2001, pp. 152-154).

Definition 6 *The equilibrium is robustly B1-stable under adaptive learning if it is B1-stable under adaptive learning for all* $0 < \alpha \leq 1$.

If an equilibrium is B1-stable for some nonempty subset of $0 < \alpha \leq 1$, but is not robustly B1-stable, then we will say that it is partially B1-stable under adaptive learning.

Stability obtains under these definitions if, starting near the steady state $K_0 = \bar{K}$, the real-time evolution of K_t stays for all t in the cylinder $D(\varepsilon)$, provided the initial (homogeneous) expectations satisfy B1, so that $\left| d\hat{K}_0 \right| \leq \varepsilon$.

6.3 The results

The first result is again an impossibility result.

Theorem 5 *The equilibrium cannot be robustly B1-stable under adaptive learning.*

Proof: The result can be obtained as a consequence of the argument in Section 5.3. The trajectory under adaptive learning is continuous in $\alpha = 0$ (for small $\alpha > 0$, the $\alpha-$trajectory is close to the $\alpha = 0$ trajectory under an appropriate metric). Impossibility then follows by continuity as a result of the impossibility of strong eductive stability in the alternative sense. A second proof, obtained by explicit computation of the path under adaptive learning, is given in the Appendix. ■

A striking feature of this result is that the instability arises in the "small gain" limit of small α, which in the adaptive and least-squares learning literature is usually viewed as a stabilizing case.[25] In our approach, the problem is that in this case the initial collective belief will be falsified, which we view as a fragility of expectational coordination.

Our finding of a generic failure of robust stability under adaptive learning is even stronger than it may appear. Theorem 5 implies that at some time t_0 we must have $|dK_{t_0}| > \varepsilon$, but is it nonetheless possible that the discounted sum of future aggregate capital remains in $D(\varepsilon)$ for all t, i.e. that $|d\hat{K}_t| \leq \varepsilon$ for all t where $d\hat{K}_t = \beta^{-1}(1-\beta) \sum_{j=1}^{\infty} \beta^s dK_{t+j}$? The answer is again no: from the proof of Theorem 5 it can

[25] The connection between eductive stability and stability under evolutive (or adaptive) learning rules, has been discussed, for example, in Evans and Guesnerie (1993), Guesnerie (2002) and Hommes and Wagener (2010). In short-horizon set-ups, eductive instability is usually reflected in adaptive instability for large gains (here $\alpha < 1$ large). This is seen for the overlapping generations model with money, under adaptive learning, in Guesnerie and Woodford (1991) and Evans and Honkapohja (1995), and for the cobweb model under dynamic predictor-selection learning, this connection is apparent in Brock and Hommes (1997). Theorem 5 is thus particularly unexpected in that it establishes strong instability under adapative learning for small $\alpha > 0$.

be seen that for sufficiently small α there exists a time t_1 such that $|dK_{t_1}| > \varepsilon\beta(1-\beta)^{-1}$ and hence $|d\hat{K}_{t_1}| > \varepsilon$. Hence the collective belief associated with B1 can be destroyed in two senses: actual capital goes away from the cylinder, and the summary statistics of future capital also goes away from the cylinder.

We next turn to partial B1-stability under adaptive learning. We show first that it is necessary for our definition that the system (12)-(13) be asymptotically stable.[26]

Lemma 3 *The equilibrium is B1-stable under adaptive learning for a given* $0 < \alpha \leq 1$ *only if the system (12)-(13) is asymptotically stable for that* α*.*

Proof: This follows from standard results for discrete-time linear systems. If the equilibrium is weakly stable under adaptive learning then the system (12)-(13) must be stable for all initial conditions. Because the system is linear this implies that its eigenvalues lie inside the unit circle, which in turn implies asymptotic stability. ■

Lemma 4 *The evolutive system (12)-(13) is asymptotically stable if and only if*

$$\xi < 4\alpha^{-1} - 2.$$

Proof: The system can be written as

$$\begin{pmatrix} dK_{t+1} \\ d\hat{K}^e_{t+1} \end{pmatrix} = \begin{pmatrix} 1 & -\xi \\ \alpha & 1-\alpha(1+\xi) \end{pmatrix} \begin{pmatrix} dK_t \\ d\hat{K}^e_t \end{pmatrix}. \tag{14}$$

Let A denote the 2×2 matrix that governs the dynamics. For asymptotic stability we need both eigenvalues within the unit circle. Equivalently we require $|\det(A)| < 1$ and $|\mathrm{tr}(A)| < |1 + \det(A)|$. Since $\det(A) = 1 - \alpha$ the first condition is satisfied for all $0 < \alpha < 1$. Using $\mathrm{tr}(A) = 2 - \alpha(1 + \xi)$ leads to the stated condition. ■

Lemma 4 implies asymptotic stability for all $0 < \alpha < 1$ if $\xi < 2$ and for some $0 < \alpha < 1$ if $\xi \geq 2$. Asymptotic stability implies that $|dK_t|, |d\hat{K}^e_t| \leq \varepsilon$ for t sufficiently large. However this is only a necessary condition for B1-stability under adaptive learning. Based on Section 4 we already know that $\xi < 1$ is a requirement for B1-stability under adaptive learning, since this gives the condition for K_1 to satisfy B1, for $K_0 = \bar{K}$ and all possible beliefs that satisfy B1, as $\alpha \to 1$; and in fact for $\xi < 1$ B1-stability under adaptive learning holds for $\alpha < 1$ sufficiently close to one.[27]

[26] Recall that througout the paper we rule out nongeneric cases. Thus we rule out eigenvalues of (12)-(13) that lie on the unit circle.

[27] Based on the proof of Theorem 5 given in the Appendix, it can be shown that $\xi < 1$ implies partial eductive stability under adaptive learning. For $\alpha < 1$ near one, the eigenvalues λ_1, λ_2 of A are real and negative and tend to $\{0, \xi\}$ as $\alpha \to 1$. For $dK_0 = 0$ and $d\hat{K}^e_0 = e$ we have $dK_1 = -\xi e$ and the path of K_t is given by

$$dK_t = -(\lambda_1 - \lambda_2)^{-1}\xi e(\lambda_1^t - \lambda_2^t)$$

from which it can be seen that $|dK_t|$ reaches a maximum at $t = 1$, which implies the stated result.

We can summarize the results in the following Theorem.

Theorem 6 *Under adaptive learning the partial and asymptotic stability properties of (12)-(13) are as follows:*

1. *For* $0 < \xi < 1$, *the steady state is B1-stable under adaptive learning for all* $\alpha \in (\gamma(\xi), 1]$, *where* $\gamma(\xi)$ *is an increasing function from* $(0,1)$ *onto* $(0,1)$.

2. *For* $1 < \xi$, *the steady state does not have partial B1-stability under adaptive learning, although:*

3. *For* $1 \leq \xi < 2$, *the learning process is asymptotically stable whatever* α.

4. *For* $2 \leq \xi$, *the learning process is asymptotically stable for* $\alpha < \frac{4}{2+\xi}$.

This theorem emphasizes the relevance of what we earlier called weak eductive stability for the understanding of real-time learning. Indeed, the coefficient ξ, stressed in Theorem 1, plays a key role either in the understanding of B1-stability under adaptive learning or in the analysis of the plausibility of the asymptotic stability of the adaptive learning process. Whatever the viewpoint taken, a higher ξ signals higher expectational fragility. If $\xi < 1$ and if the learning rule reacts quickly enough to information on the path, i.e. $\alpha > \gamma(\xi)$, where the required reaction speed α is higher the greater is ξ, then the trajectory necessarily remains in the cylinder defining the collective belief. If $\xi > 1$ then the path may eventually move away and falsify the initial belief, *whatever* the specific (asymptotically stable or not) adaptive learning rule used by the agents

Finally, we remark that while for given ξ the asymptotic stability condition $\xi < 4\alpha^{-1} - 2$ is easier to satisfy when $\alpha > 0$ is small, it is small values of α that generate the failure of robust B1-stability under adaptive learning. Small $\alpha > 0$ under adaptive learning leads to a cumulative movement of aggregate capital away from the steady state value, which over finite time periods, as $\alpha \to 0$, track the possible K_t paths deduced by agents in our eductive setting.[28]

7 A finite-horizon model with capital

In the introduction we stated that the strong instability results of this paper result from the long planning horizon of agents in the RBC model. This is reflected in our proofs, which show that the violation of the CK assumption arises not because dK_1

[28] An interesting feature of the eductive instability under adaptive learning, which can be seen in the proof of Theorem 5 given in the Appendix, is that the instability is associated with long cyclical movements in K_t.

is too large, but because, eventually, dK_t is too large. However, one might still ask whether strong instability would arise in a model with capital in which agents are short-lived but the economy is infinitely-lived. To answer this question, we contrast our eductive stability results with those for an overlapping generations model with capital. This model can also be interpreted as a variant of our model in which agents are myopic (i.e. they only consider next period as the relevant horizon) instead of being far-sighted and envisaging the whole future. The previous analysis stresses that farsighted agents find it very difficult to coordinate expectations. The next analysis shows that indeed myopia makes expectational coordination easier.

7.1 The OLG Model

We stick here to the standard overlapping generations (OLG) terminology, although we keep in mind the just noted dimension of myopia versus farsightedness in the agent's problem.

Consider a two-period OLG model with capital, along the lines of Diamond (1965). Population is constant and normalized to one, and all markets are competitive. Let ω_t be an agent born at time t. He is endowed with one unit of labor, which he supplies inelastically for real wage q_t. He then allocates his income between savings $s(\omega_t) = k(\omega_t)$ and consumption $c_1(\omega_t)$. In period $t+1$, this agent is now old: he rents his savings for net real return r_{t+1}, consumes the gross return plus profits and dies. Thus agent ω_t solves the following problem

$$\max E(\omega_t)\left\{u\left(c_1(\omega_t), c_2(\omega_t)\right)\right\}$$

$$\text{s.t.} \quad c_1(\omega_t) + s(\omega_t) = q_t \tag{15}$$

$$c_2(\omega_t) = (1 + r^e_{t+1}(\omega_t))s(\omega_t) + \pi(\omega_t) \tag{16}$$

Notice that when agent ω_t makes his savings decision, he does not know the value of r_{t+1}. Below we assume constant returns to scale production so that $\pi = 0$.

The agent ω_t's first-order condition is given by

$$u_{c_1}\left(c_1(\omega_t), c_2(\omega_t)\right) = \beta(1 + r^e_{t+1}(\omega_t))u_{c_2}\left(c_1(\omega_t), c_2(\omega_t)\right). \tag{17}$$

Equations (15)–(17) may be used to compute the savings decision of agent ω_t based on current and expected future factor prices:

$$s(\omega_t) = s(q_t, r^e_{t+1}(\omega_t)).$$

Firms hire workers and rent capital in competitive factor markets, and employ constant returns to scale technology to manufacture goods: $Y = f(K, L)$; thus profits are zero and factors prices are given by the respective marginal products. Capital is inelastically supplied "in the morning" by the old and depreciation is zero: the capital accumulation equation is given accordingly by

$$K_{t+1} = \int s(\omega_t) d\omega_t = \int s(q_t, r_{t+1}^e(\omega_t)) d\omega_t.$$

Assuming agents know the relationship between real interest rates and marginal products, and so form expectations of aggregate capital instead of real interest rates, we have

$$K_{t+1} = \int s\left(f_L(K_t, 1), f_K(K_{t+1}^e(\omega_t), 1)\right) d\omega_t, \tag{18}$$

where $K_{t+1}^e(\omega_t)$ is agent ω_t's forecast of aggregate capital tomorrow. Equation (18) captures the dynamics of the economy: given aggregate capital today and forecasts of aggregate capital tomorrow, the actual value of aggregate capital tomorrow can be determined. It also highlights a key difference between the OLG model and the RBC model: in the OLG model aggregate capital depends only on one period ahead forecasts; in the RBC model, aggregate capital depends on forecasts at all horizons.

7.2 Common Knowledge and Eductive Stability

To motivate the definition of eductive stability, we consider the following thought experiment at time $t = 0$: Let $\bar{K}$ be a steady state of (18): $\bar{K} = s\left(f_L(\bar{K}, 1), f_K(\bar{K}, 1)\right)$. Assume that at time $t = 0$ every old household has capital $k_0(\omega) = \bar{K}$. This determines the wage, and therefore the income, of the young, as given by $\bar{q}$. Each young agent ω_t forecasts capital stock tomorrow, $K^e(\omega_t)$, and determines his savings decision $s(\bar{q}, f_K(K^e(\omega_t), 1))$. He then contemplates the savings decisions of other agents. We again make the common knowledge CK1:

CK1: It is common knowledge that for some $\varepsilon > 0$ sufficiently small, $K_t \in D(\varepsilon) \equiv [\bar{K} - \varepsilon, \bar{K} + \varepsilon]$ for all t.

These CK beliefs are assumed held by all agents at all times, i.e. for all ω_t for all t.

The definitions of weak and strong eductive stability under CK1 are identical to the definitions given in Sections 4 and 5. We have the following results:

Theorem 7 *The steady state $\bar{K}$ is strongly eductively stable if and only if*

$$\left|\partial \mathbf{s}/\partial \mathbf{r}\left(f_L(\bar{K}, 1), f_K(\bar{K}, 1)\right) \cdot f_{KK}\left(\bar{K}, 1\right)\right| < 1. \tag{19}$$

Proof: To see this, first note that (19) holds if and only if there is $\zeta \in (0, 1)$ such that for small $\varepsilon > 0$, whenever $|K^e(\omega_t) - \bar{K}| \leq \varepsilon$ it follows that

$$\left|s\left(f_L(\bar{K}, 1), f_K(K^e(\omega_t), 1)\right) - \bar{K}\right| < \zeta\varepsilon.$$

CK1 implies in particular that $K_1 \in D(\varepsilon)$. If (19) holds this implies that $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$. Because each agent ω_0 knows that $K^e(\hat{\omega}_0) \in D(\varepsilon)$, he concludes that $s(\hat{\omega}_0) \in D(\zeta\varepsilon)$ for all $\hat{\omega}_0$ and hence that $K_1 \in D(\zeta\varepsilon)$. Thus it is common knowledge that $K_1 \in D(\zeta\varepsilon)$. Iterating the argument it follows that $K_1 \in \bigcap_{n=1}^{\infty} D(\zeta^n\varepsilon) = \{\bar{K}\}$. In contrast if (19) fails then for some beliefs compatible with CK1, the aggregate capital stock at $t = 1$ implied by the agents' optimal plans fails to satisfy $K_1 \in D(\varepsilon)$.

It follows that it is common knowledge that $K_1 = \bar{K}$ if and only if (19) holds. For agents at time $t = 1$ the situation is identical to the situation at time $t = 0$. Thus at $t = 1$ agents will conclude that $K_2 = \bar{K}$ and this implies that it will be the case that $K_2 = \bar{K}$ and this will also be common knowledge for agents at $t = 0$. By induction it follows that the fact that the equilibrium path will be $K_t = \bar{K}$, all t, is common knowledge. ■

Note that this stability result is local and can provide a refinement criterion in the case of multiple steady states.

As an exercise for illustrating our results, we specify particular functional forms and conduct numerical analysis. Assume utility is time separable and takes the constant relative risk-aversion form

$$u(c_1, c_2) = \frac{1}{1-\sigma}\left(c_1^{1-\sigma} + c_2^{1-\sigma} - 2\right),$$

for $0 < \sigma < 1$, and assume that production is Cobb-Douglas, $f(K, L) = K^{\theta}L^{1-\theta}$. In this case there is a unique positive steady-state level of capital, and parameter values for θ and σ completely characterize the model. For all parameters examined – $\theta \in (0, 1)$ and $\sigma \in (0, 100)$ – the steady state is strongly eductively stable.

This example provides a striking contrast to the coordination problems we have demonstrated for the RBC model with infinitely-lived agents.

8 Conclusions

The difficulties of expectational coordination can be ascertained from two sides, the "eductive" one and the "evolutive" one. In both cases, farsighted agents are sensitive to the whole path of expectations and long-run expectations significantly matter. Long-run concerns do unsurprisingly influence present decisions and future plans. But the sensitivity to expectations of long-run plans envisaged today is extreme. And this is at the heart of the impossibility of strong "eductive" stability. Indeed, in the eductive framework there is negative short-run feedback, which *may* be destabilizing, and positive long-run feedback, which *will be* destabilizing. That is, an expectation that aggregate K will persistently exceed $\bar{K}$ will lead agents to reduce their capital in the coming period, which, if the effect is strong enough, can be destabilizing. But, in addition, the optimal dynamic plans of agents call for them to eventually accumulate capital in excess of the conjectured level of aggregate K. Hypothetical Common Knowledge of the equilibrium cannot trigger Common Knowledge of the equilibrium, whatever the specific characteristics of the economy, an extreme form of expectational instability which has no counterpart in previously studied models.[29] If evolutive learning is incorporated into the model, so that expectations evolve adaptively over time, these two sources of instability remain pivotal. If the adaptation parameter is large then unstable overshooting can arise in the short-run, while if the adaptation rate is small then low-frequency swings over the medium run will necessarily generate instability.

[29] We do not claim however that the difficulty occurs in every model with infinitely lived agents. For example, it would not occur in the world of Lucas (1978), where the assets returns do not depend upon "extrinsic" uncertainty, but only upon intrinsic uncertainty.

Appendix

Proof of Lemma 1: These results follow from the envelope theorem. The Lagrangian is given by

$$L=\sum_{t\geq 0}\beta^t\left(U(c_t)+\lambda_t\left(\sum_{s\geq 0}R^e_{st}\left(q^e_{t+s}-c_{t+s}\right)+\left(1+r^e_t\right)k_t(\omega)\right)\right),$$

where

$$R_{st}=\prod_{i=1}^{s}(1+r_{t+i})^{-1},$$

and $R^e_{st}=\prod_{i=1}^{s}(1+r^e_{t+i})^{-1}$ is the point expectation of R_{st} given $\tilde{K}^e(\omega)$, and similarly for q^e_{t+s} and r^e_t. The first result follows easily from

$$\frac{\partial L}{\partial k_0(\omega)}=\lambda_0(1+r_0):$$

since $K_0=\bar{K}$ it follows that $(1+r_0)\beta=1$; and since λ_0 is the time zero marginal utility of wealth, $\lambda_0=U'(c_0)$.

To obtain the second result, notice that because production has constant returns to scale, it follows that $\frac{\partial q_0}{\partial K_0}+\frac{\partial r_0}{\partial K_0}K_0=0$. Now simply compute

$$\frac{\partial L}{\partial K_0}=\lambda_0\left(\frac{\partial q_0}{\partial K_0}+\frac{\partial r_0}{\partial K_0}K_0\right).$$

The final result obtains as follows:

$$\begin{aligned}\frac{\partial L}{\partial K^e_T(\omega)} &= \sum_{t=0}^{T-1}\beta^t\lambda_t\left(\frac{\partial}{\partial K^e_T(\omega)}\sum_{s\geq 0}R^e_{st}\left(q^e_{t+s}-c_{t+s}\right)\right)\\ &\quad+\frac{\partial}{\partial K^e_T(\omega)}\beta^T\lambda_T\left(q^e_T+(1+r^e_T)k_T(\omega)\right).\end{aligned}$$

We may compute

$$\frac{\partial}{\partial K^e_T(\omega)}\left(q^e_T+(1+r^e_T)k_T(\omega)\right)=0.$$

Also, for $t\leq T-1$, we have

$$\begin{aligned}\frac{\partial}{\partial K^e_T(\omega)}R^e_{st} &= \begin{cases}-\beta^{s+1}f'' & t+s\geq T\\ 0 & \text{else}\end{cases}\\ \frac{\partial}{\partial K^e_T(\omega)}q^e_{s+t} &= \begin{cases}-\bar{K}f'' & t+s=T\\ 0 & \text{else}\end{cases}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial}{\partial K_T^e(\omega)}\sum_{s\geq 0}R_{st}^e\left(q_{t+s}^e-c_{t+s}\right) &= -\beta^{T-t}\bar{K}f''-(\bar{q}-\bar{c})\,f''\beta\sum_{s\geq T-t}\beta^s \\ &= \bar{K}f''\left(-\beta^{T-t}+\beta\bar{r}\beta^{T-t}\sum_{s\geq 0}\beta^s\right) \\ &= \bar{K}f''\beta^{T-t}\left(\frac{\beta}{1-\beta}\bar{r}-1\right)=0. \blacksquare\end{aligned}$$

Proof of Lemma 2: The individual Euler equation (3) implies

$$c_t(\omega)=c_0(\omega)\left(\beta^{t/\sigma}\textstyle\prod_{s=1}^{t}(1+r_s^e(\omega))^{1/\sigma}\right).$$

Taking logs gives

$$\log(c_t(\omega))=\log(c_0(\omega))+(t/\sigma)\log\beta+(1/\sigma)\sum_{s=1}^{t}\log(1+r_s^e(\omega)).$$

Taking small changes, as argued above, we have

$$\frac{dc_t(\omega)}{\bar{C}}=\frac{dc_0(\omega)}{\bar{C}}+\frac{\beta}{\sigma}\left(\sum_{s=1}^{t}dr_s^e(\omega)\right)$$

Using the welfare corollary we obtain

$$0=\frac{1}{\bar{C}}\sum_{t\geq 0}\beta^t dc_t(\omega)=\frac{1}{1-\beta}\frac{dc_0(\omega)}{\bar{C}}+(\frac{\beta}{(1-\beta)\sigma})\left(\sum_{s=1}^{+\infty}\beta^s dr_s^e(\omega)\right),$$

where we have used the identity $\sum_{t=1}^{\infty}\beta^t\sum_{s=1}^{t}dr_s^e(\omega)=(1-\beta)^{-1}\beta\sum_{s=1}^{\infty}\beta^s dr_s^e(\omega)$. The result follows. $\blacksquare$

Discussion and statement of Lemma A1

Let $dK_1(d\tilde{K}^e)$ be the deviation of aggregate capital in period one given an arbitrary beliefs profile $d\tilde{K}^e$. Weak eductive stability implies that if $d\tilde{K}^e$ satisfies B1 then $dK_1(d\tilde{K}^e)\in(-\varepsilon,\varepsilon)\equiv I(\varepsilon)$. To establish conditions sufficient for weak eductive stability, it is useful to identify the beliefs profile(s) $d\tilde{K}^e$ that satisfy B1 and that maximize the magnitude of $dK_1(d\tilde{K}^e)$. Some additional notion is helpful: denote by $d\tilde{S}^e(\alpha)$ an agent specific constant beliefs time path: $dK_t^e(\omega)=\alpha$ for all times t. Abusing notation slightly, we will also use $d\tilde{S}^e(\alpha)$ to indicate a constant and homogeneous beliefs profile: $dK_t^e(\omega)=\alpha$ for all agents ω and times t.

Lemma A1: Let $d\tilde{K}^e$ be any beliefs profile consistent with B1. Then

$$dK_1\left(d\tilde{S}^e(\varepsilon)\right) \leq dK_1\left(d\tilde{K}^e\right) \leq dK_1\left(d\tilde{S}^e(-\varepsilon)\right).$$

The proof follows fairly straightforwardly from Lemma 2.

Constructive proof of Theorem 5: We consider $dK_0 = 0$ and $dK_0^e(\omega) = e$, with $e = \pm\varepsilon$, for all ω. This implies $dK_1 = -\xi e$. The dynamics of the system are then given by (14), which can equivalently be written as

$$dK_{t+2} = (2 - \alpha(1+\xi))dK_{t+1} - (1-\alpha)dK_t$$

with $dK_0 = 0$ and $dK_1 = -\xi e$. The eigenvalues of the dynamics are complex if $\alpha(1+\xi)^2 < 4\xi$, and therefore for $\alpha > 0$ sufficiently small and are given by

$$\lambda, \bar{\lambda} = \frac{1}{2}\left\{2 - \alpha(1+\xi) \pm i\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}\right\}$$

or

$$\begin{aligned} \lambda, \bar{\lambda} &= r(\cos(\theta) \pm i\sin(\theta)), \text{ where} \\ r^2 &= 1 - \alpha \text{ and} \\ r\sin\theta &= \frac{1}{2}\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}. \end{aligned}$$

When the roots are complex the solution meeting the initial conditions is given by

$$dK_t = -\frac{\xi e}{r\sin\theta} r^t \sin(\theta t).$$

At $t = T = \frac{\pi}{2\theta}$ we have

$$\begin{aligned} dK_T &= -\frac{2\xi e}{\sqrt{\alpha}\sqrt{4\xi - \alpha(1+\xi)^2}}(1-\alpha)^{\pi/(4\theta)}, \text{ where} \\ \theta &= \sin^{-1}\sqrt{\frac{4\xi\alpha - \alpha^2(1+\xi)^2}{4(1-\alpha)}}. \end{aligned}$$

Taking the limit as $\alpha > 0$ tends to zero it can be verified that

$$\lim_{\alpha \to 0} dK_{T(\theta(\alpha))} = \pm\infty$$

where the sign is opposite to the sign of e. It follows that for $\alpha > 0$ sufficiently small we have $|dK_t| > \varepsilon$ for values of t near $T(\theta(\alpha))$, and hence the equilibrium is not robustly B1-stable under adaptive learning. ■

References

[1] Benhabib, J. and C. Bull (1988), Conjectural equilibrium bounds, in Nonlinear Dynamics and Economics and Social Sciences, ed. M. Galeotti, L. Geronazzo and F. Gori. Pitagori Editrice Bologna.

[2] Brock, W.A. and C. Hommes (1997), A rational route to randomness, Econometrica 65, 1059–1095.

[3] Desgranges, G. and S. Ghosal (2010), P-stable equilibrium: definition and some properties, Warwick economic research paper No. 952.

[4] Diamond, P.A. (1965), National debt in a neoclassical growth model, American Economic Review 55, 1126-1150.

[5] Eusepi, S. and B. Preston (2008), Expectations, learning and business cycle fluctuations, NBER Working Paper 14181.

[6] Evans, G.W. and R. Guesnerie (1993), Rationalizability, strong rationality and expectational stability, Games and Economic Behavior 5, 632-646.

[7] Evans, G.W. and R. Guesnerie (2005), Coordination on saddle-path solutions: the eductive viewpoint – linear multivariate models, Journal of Economic Theory 124, 202-229.

[8] Evans, G.W. and S. Honkapohja (1995), Local convergence of recursive learning to steady states and cycles in stochastic nonlinear models," Econometrica 63, 195-206.

[9] Evans, G.W. and S. Honkapohja (2001), Learning and expectations in macroeconomics. Princeton University Press, Princeton, New Jersey.

[10] Evans, G.W. and S. Honkapohja (2003), Expectations and the stability problem for optimal monetary policies, Review of Economic Studies 70, 807–824.

[11] Evans, G.W., S. Honkapohja and K. Mitra (2009), Anticipated fiscal policy and adaptive learning, Journal of Monetary Economics 56, 930-953.

[12] Evans, G.W. and B. McGough (2009), Learning to optimize, mimeo.

[13] Gauthier, S. and R. Guesnerie (2004), Comparing Expectational Stability criteria in dynamical models: a preparatory overview, Delta DP 2004-9.

[14] Guesnerie, R. (2002), Anchoring economic predictions in common knowledge, Econometrica 70, 439-480.

[15] Guesnerie, R. (2005), Assessing rational expectations 2: "eductive" stability in economics. Cambridge MA: MIT Press.

[16] Guesnerie, R. and Jara-Moroni, P. (2011), Expectational coordination in simple economic contexts: concepts and analysis with emphasis on strategic substitutabilities, forthcoming Economic Theory.

[17] Guesnerie, R. and M. Woodford (1991), Stability of cycles with adaptive learning rules, in Equilibrium Theory and Applications, Proceedings of the Sixth International Symposium in Economic Theory and Econometrics, ed. W. Barnett et al., Cambridge University Press, Cambridge UK, 111-134.

[18] Hommes, C. and F. Wagener (2010), Does eductive stability imply evolutive stability?, Journal of Economic Behavior and Control 75, 25-39.

[19] Howitt, P. (1992), Interest rate control and nonconvergence to rational expectations, Journal of Political Economy 100, 776-800.

[20] Kydland, F.E. and E.C. Prescott (1982), Time to build and aggregate fluctuations, Econometrica 50, 1345-1370.

[21] Long, J.B. and C.I. Plosser (1983), Real business cycles, Journal of Political Economy 91, 39-69.

[22] Lucas, Jr., Robert E. (1978), Asset Prices in an Exchange Economy, Econometrica 46, 1429-1445.

[23] Marcet, A. and T.J. Sargent (1989), Convergence of least squares learning mechanisms in self-referential linear stochastic models, Journal of Economic Theory 48, 337-368.

[24] Matsui, A. and D. Oyama (2006), Rationalizable foresight dynamics, Games and Economic Behavior 56, 299-322.

[25] Prescott, E.C. (1986), Theory ahead of business cycle measurement, Carnegie-Rochester Conference Series on Public Policy 25 (Autumn), 11-44.

[26] Preston, B (2006), Adaptive learning, forecast-based instrument rules and monetary policy, Journal of Monetary Economics 53, 507-535.

[27] Sargent, T.J. (1993), Bounded rationality in macroeconomics, Oxford University Press, Oxford UK.

[28] Woodford, M. (1990), Learning to believe in sunspots, Econometrica 58, 277-307.