The English Auction, Rushes, and a Sealed Bid Efficient Auction*

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Abstract

We analyze an auction setting that fits well privatizations, takeover contests, and procurement auctions where one incumbent has a superior information regarding a common value component of the asset on sale. We show that a simple two stage sealed bid mechanism used in reality outperforms the English auction in terms of efficiency. Indeed, the former mechanism implements the (second best) efficient allocation whereas the latter does not. This is a consequence of *rushes* along the equilibrium path of the English auction and the usual uniformly random tie breaking rule.

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1 Introduction

In this paper, we show that the open ascending auction, also referred to as the English auction, that it is usually proposed by academics for allocating privatized assets does not maximize social expected surplus in the environment in which privatizations usually take place. Instead, the multistage sealed bid auction used in some real-life privatizations, for instance ENI's, see Caffarelli (1998), and Telebras', see Dutra and Menezes (2002), is (second best) efficient.

A key element of these environments is that there is one bidder, *the incumbent*, with private information about a component affecting all the other bidders' valuations. An illustrative example is franchise bidding for serving a privatized service whose market size is privately known by the incumbent. It is well known that the first best efficient allocation is not implementable under the natural assumption that entrants' face higher setup costs but lower marginal cost than the incumbent, see for instance Hernando-Veciana and Michelucci (2011). We show that the second best may require allocating to the most efficient entrant. However, the English auction does not always implement this allocation because entrants *rush* if the incumbent quits at a low price denoting a small market size.

The mechanism we propose to deal with this problem is a simple two stage sealed mechanism. The auction is won by the bidder who provides the highest bid across the two stages and the price is determined by the highest bid of the opponents. Only the two highest bids qualify to the second stage and bidders can (upward) revise their first stage bid upon the knowledge of the bids and the identity of the non qualifying bids.

Our two stage mechanism is built to exploit two facts. First, the use of a sealed phase in the first stage avoid the occurrence of a rush and enables the mechanism to always select to the second stage the most efficient of the less informed bidders. Second, for the second stage we make use of the result in Hernando-Veciana and Michelucci (2011) that the English auction and thus the SPA is second best efficient when there are only two bidders [1] From the last observation it follows that we can run the second stage either in a sealed or an open manner. In the latter case, our mechanism can be seen as a hybrid auction where the order of the sealed bid and open auction is reversed relatively to the Anglo-Dutch auction that has been advocated by Klemperer (2002) among others.[2]

It is worth stressing that we believe there is a novel insight to be taken out of this paper that goes

[1] It is well known that SPA and English auction are strategically equivalent for two bidders.

[2] Dutra and Menezes (2002) documents that a variant of this type of format has been used in the privatization of Telebras System, the Brazilian telecom. They provide a setting where such auction outperforms any standard auction in terms of revenues.

beyond the specific application with an incumbent that we have illustrated. Our main point is that in some environments the continuous flow of information such as the one that bidders receive in an open ascending auction can impede the smooth aggregation of information and therefore negatively impact efficiency. Thus, the optimality of a multi stage mechanism follows as a way to break in an optimal manner the flow of information. We elaborate more on the generality of the limits of open formats to optimally aggregate information in our concluding remarks.

We contribute to two different literatures. To the one on efficient auctions, and in particular to the papers that look at the efficiency properties of the English auction, by providing an impossibility result regarding the implementation of the second best efficient allocation. To the literature on multi stage auctions, by providing a two stage format that implements the second best.

Some scholars, see especially Klemperer (2002), have pointed out that two stage formats might be used to create an hybrid format that can combine desirable features of open formats (efficiency) and sealed bid formats (better against collusion and to stimulate entry).[3] On a different spirit, Ye (2007), and Boone and Goeree (2009) have motivated the use of two stage formats in applications that involve considerable risk for the bidders such as privatizations, takeover contests from the observation that indicative bidding is sometimes used in the first stage. Indicative bids are non binding bids that express bidders interest and initial appraisal for the object.Typically the lowest bids do not qualify for the second stage in which bids are binding.

Ye (2007) model assumes a symmetric setting in which bidders have some preliminary information about the value of the asset for sale and need to incur in some costs in order to obtain further information. Indicative bidding in a setting in which entry costs are high might be useful to ensure entry in the second stage. In fact, the indicative bidding stage enables a bidder to test the level of the competition and assess if she has good enough chances to justify paying the high cost. From a theoretical point of view, however, Ye (2007) finds that there is no symmetric equilibrium in the game with indicative bidding, which implies inefficient entry. Thus, he proposes a two stage auctions with binding bids that does better. Kagel, Pevnitskaya, and Ye (2008) test experimentally this model and find that in the lab indicative bidding does no worse in terms of efficiency than the supposedly superior mechanism, while it does better in other dimensions.

Boone and Goeree (2009) focus on a setting with private and common values, and where one insider has an informational advantage over a common value component that can take two values

[3]For other papers that look at hybrid mechanisms, see Dutra and Menezes (2002) and Levin and Ye (2008).

representing a bad and a good state of the world. They argue that such environment is typical of many privatizations and procurement auctions and that it exposes the less informed bidders to a substantial winner's curse that can depress revenues. They focus on revenues and compare a proxy variant of the English auction, with the Second Price Auction (SPA), and with a two stage mechanism which uses indicative bidding in the first stage to select the n-1 bidders that qualify to the second stage. They show that the latter format is an optimal mechanism if the common value is large relatively to the private value one so as to guarantee full separation between the valuations that can span from the good and the bad state. The intuition for the optimality of indicative bidding is that if bids are not binding in the fist stage the less informed bidders can safely bid their expected value without conditioning on winning at such price. Given such more aggressive bidding, it can never be optimal for the insider to qualify whenever the state is bad, and thus the uninformed bidders can infer the state of the world from the knowledge of the non qualifying bid and safely revise their bids accordingly in the binding stage. However, the authors show that a bubbling equilibrium in which first stage bids are non informative also exists. Boone, Chen, Goeree, and Polydoro (2009) experimental test shows that in the lab the latter equilibrium prevails and that the English auction outperforms both in terms of revenues and efficiency indicative bidding.

The closest paper to ours is Boone and Goeree (2009). However, we differentiate from that paper on several dimensions. First, we look for the optimal mechanism in terms of efficiency rather than revenues. Second, we analyze the standard version of the English auction as our benchmark mechanism rather then a proxy auction. Third, their setting represents only a specific example of our more general model. Finally, but importantly, in our mechanism bids are binding, and so bubbling equilibria are absent. Another related paper is Perry, Wolfstetter, and Zamir (2000), which analyzes the same sealed format as ours to show that in the Milgrom-Weber setting it generates as much revenues as the English auction.

The remaining of the paper is organized as follows: Section 3 defines the environment we model; Section 4 characterizes the most efficient allocation subject to incentive compatibility constraints; Section 5 characterize the equilibrium of the English auction and points out that rushes emerge in equilibrium; Section 6 presents our efficient Second Price Sealed bid Auction (SSPA); finally, Section 7 concludes.

2 The Model

One unit of an indivisible good is put up for sale to $n+1$ bidders with quasilinear preferences in money. Bidder 1, the *incumbent*, puts monetary value $\hat{v}(s_1)$ in getting the good and each bidder $i \neq 1$, the *entrants*, puts monetary value $v(s_i, s_1)$, where s_j, $j \in \{1, 2, ...\}$, denotes Bidder j's private information. We assume that each s_j is equal to the realization of an independent random variable, with distribution F_j, density f_j and support normalized to be $[0, 1]$.

Along this paper, the equilibrium concept is sequential equilibrium in non-weakly dominated strategies. Note that the assumption that the value of bidder i is not a function of the signal of the other entrants $j \neq i$ is only to ease the exposition, as long as we preserve symmetry all proofs follows with trivial adaptations.

3 Second Best Allocations

In this section, we characterize the maximum expected social surplus that can be achieved in the equilibrium of a selling mechanism.

An *allocation* is a measurable function p from the set of types $[0, 1]^{n+1}$ into the $(n+1)$-dimensional simplex $\Delta(n+1)$ such that $p_i(s)$ denotes the probability of allocating the good to Bidder i when the vector of types is s. An allocation is *implementable*[4] if it is the equilibrium outcome of a selling mechanism and it is *second best* if it maximizes the expected social surplus:

$$\int_{[0,1]^{n+1}} \left(p_1(s)\hat{v}(s_1) + \sum_{i \neq 1} p_i(s) v(s_i, s_1) \right) dF_1(s_1) dF_{-1}(s_{-1}), \tag{1}$$

subject to implementability, where $F_{-1}(s_{-1})$ stands for $\prod_{j \neq 1} F_j(s_j)$.

We characterize the second best allocation with an auxiliary function $\phi : [0, 1] \to [0, 1]$. This is defined by the minimum solution to the following maximization problem:

$$\max_{s_1^* \in [0,1]} \int_0^{s_1^*} \left(v(s, s_1) - \hat{v}(s_1) \right) dF_1(s_1). \tag{2}$$

for any $s \in [0, 1]$. We refer to the allocation associated to ϕ, to an allocation function that assigns the good to the entrant with highest type $s_{(1)}$ if $\phi(s_{(1)}) \geq s_1$ and otherwise to the incumbent. Note that since the integrand that defines ϕ is increasing in s so it is ϕ, and hence the allocation associated.

[4] See Hernando-Veciana and Michelucci (2011) for a formal definition.

The analysis of Hernando-Veciana and Michelucci (2011) implies here that for the case of only one entrant, the allocation associated to ϕ maximizes the expected social surplus amongst the allocations that are deterministic and monotone in 1. This in particular means that the allocation associated to ϕ is second best if there is a second best allocation that it is deterministic and monotone in 1. Next proposition shows that this is indeed the case and show that the result also holds true when there are more than one entrant.

Proposition 1. *The allocation associated to ϕ is second best.*

Proof. Implementability requires that the incentive compatibility constraints of all the bidders are verified. We study a relaxed version of the problem that defines the second best in which we only impose the incentive compatibility constraint of the incumbent. Later, we shall show that the solution to this auxiliary problem also verifies the entrants' incentive compatibility constraints and thus solves our original problem.

One can deduce, as in Myerson's (1981), that a necessary and sufficient condition for an allocation p to be incentive compatible for the incumbent is that:

$$Q_1(s_1, p) \equiv \int_{[0,1]^n} p(s_1, s_{-1}) dF_{-1}(s_{-1}) \text{ is increasing in } s_1. \tag{3}$$

Our auxiliary problem is the maximization of (1) subject to (3). We solve this auxiliary problem in two stages: first, we maximize with respect to p subject to the constraint $Q(., p) = \hat{Q}(.)$ for any function $\hat{Q} : [0,1] \to [0,1]$ increasing function, and second, we choose the optimal $\hat{Q}(.)$ increasing. The first stage is thus:

$$\max_{p:[0,1]^{n+1} \to \Delta(n+1)} \int_0^1 \hat{v}(s_1)\hat{Q}(s_1)dF_1(s_1) + \int_{[0,1]^{n+1}} \left(\sum_{i\neq 1} p_i(s)v(s_i, s_1)\right) dF_1(s_1)dF_{-1}(s_{-1})$$
$$\text{s.t. } \int_{[0,1]^n} \sum_{i\neq 1} p_i(s) dF_{-1}(s_{-1}) = 1 - \hat{Q}(s_1) \,\forall s_1 \in [0,1].$$

where $\hat{Q} : [0,1] \to [0,1]$ is increasing. The Lagrangian of this problem is:

$$\int_{[0,1]^{n+1}} \left(\sum_{i\neq 1} p_i(s)v(s_i, s_1)\right) dF_1(s_1)dF_{-1}(s_{-1}) - \int_0^1 \lambda(s_1) \left(\int_{[0,1]^n} \sum_{i\neq 1} p_i(s)dF_{-1}(s_{-1}) - (1 - \hat{Q}(s_1))\right) dF_1(s_1),$$

where λ is the Lagrange multiplier. Reorganizing the Lagrangian we get:

$$\int_{[0,1]^{n+1}} \left(\sum_{i\neq 1} p_i(s)\left(v(s_i, s_1) - \lambda(s_1)\right)\right) dF_1(s_1)dF_{-1}(s_{-1}) + \int_0^1 \lambda(s_1)(1 - \hat{Q}(s_1))dF_1(s_1).$$

Any optimizer to this problem verifies that $p_i(s) = 1$ if $v(s_i, s_1) > \lambda(s_1)$ and $s_i > \max_{j\neq 1,i}\{s_j\}$, and $p_i(s) = 0$ if either $v(s_i, s_1) < \lambda(s_1)$ or $s_i < \max_{j\neq 1,i}\{s_j\}$. This solution can be characterized with the function $\psi : [0, 1] \to [0, 1]$ implicitly defined by $v(\psi(s_1), s_1) = \lambda(s_1)$. The constraint of our problem requires that $\hat{Q}_1(s_1) = F_{(1)}(\psi(s_1)) \equiv \prod_{j\neq 1} F_j(\psi(s_1))$, which also determines ψ and implies that ψ is increasing. Thus, the allocation that solves the first stage is the allocation associated to an increasing function $\hat{\phi}$ that verifies $\hat{Q}_1(\hat{\phi}(s)) = F_{(1)}(s)$. Consequently, the second stage maximization after substituting the optimal value of the first stage can be formulated as:

$$\max_{\hat{\phi} \text{ incr.}} \int_0^1 \int_0^{\hat{\phi}(s_{(1)})} \big(v(s_{(1)}, s_1) - \hat{v}(s_1)\big)\, dF_1(s_1) dF_{(1)}(s_{(1)}).$$

Our function ϕ solves this problem and hence our auxiliary problem. It remains to be shown that the allocation associated is implementable, and in particular that it verifies the entrants' incentive compatibility constraints. We postpone this proof as it is a direct consequence of Proposition 6. ■

4 The English Auction

In this section, we analyze the English auction. First, we prove that the second best allocation cannot usually be implemented with an English auction. Second, we focus on a particular framework to illustrate the equilibrium of the English auction when it does not implement the second best and discuss the potential efficiency losses.

We assume the model of the English auction described by Krishna (2003). This auction model is a variation of the Japanese auction proposed by Milgrom and Weber (1982) in which the identity of the bidders is observable. In our analysis, it is important that we make the tie-breaking rule explicit. We shall assume the good is allocated with equal probability among the bidders that tie.[5]

[5]In our auction, the price increases continuously until one bidder or more quit. Then, the price is stopped and the following algorithm is repeated: (1) If there are no more active bidders, the good is allocated with equal probability among the bidders that last quitted at the current price. Otherwise, (2) the identity of the bidders that still remain active is announced. (3) After the announcement, bidders that still remain active declare independently and simultaneously whether they quit. If no bidder quits, the price is increased again from the current level. If some bidder quits, we go to (1). We conjecture that any other tie-breaking rule that does not condition on the bidders' types would imply similar results.

4.1 On the impossibility of a Second Best implementation

We say that the *first best allocates to the incumbent* if $v(s_{(1)}, s_1) < \hat{v}(s_1)$ for $s_{(1)}$ the highest of the entrant's types. Similarly, we say that *the second best allocates to one of the entrants* if $\phi(s_{(1)}) > s_1$.

Proposition 2. *If there is an open set of types for which the first best allocates to the incumbent and the second best to one of the entrants, then there is no equilibrium of the English auction in non-weakly dominated strategies that implements the second best.*

Proof. Our proof makes use of four features that must be verified along the equilibrium path of any equilibrium implementing the second best:

(i) Bidder 1 remains active until the price reaches her value $\hat{v}(s_1)$.

(ii) In any information set in which the incumbent has already quit, say at price p, an entrant with type s_i quits if the current price is greater or equal than $v(s_i, \hat{v}^{-1}(p))$.

(iii) In any information set in which the only remaining bidder is the incumbent, our entrant quits if the current price is greater or equal than $\hat{v}(\phi(s_i))$.

(iv) In information sets in which the incumbent and at least two entrants are active, an entrant with type s_i quits at a price not higher than $\hat{v}(\underline{s}_1(s_i))$, where $\underline{s}_1(s_i)$ is the infimum of the set of the incumbent's types s_1 such that (s_1, s_i) is in the interior of the set of types for which the first best allocates to the incumbent if non empty, and $\underline{s}(s_i) \equiv 1$ otherwise.

(i) follows from the fact that the unique weakly dominant strategy for the incumbent is to bid her value, as in a private value auction. (ii) is a straightforward consequence of (i) and rationality, and (iii) of the characterization of the second best and (i). (iv) follows from features (i) and (iii) and two facts: that the second best only allocates to the entrant with highest type and that the tie breaking rule allocates with the same probability among all the bidders that tie.

To prove that there is no equilibrium that implements the second best, we show that there exists an open set of types of an entrant that have strict incentives to deviate when all the bidders play a strategy profile that implements the second best and verifies (i)-(iv). This set contains any entrant's type s_i for which there exists another type $s_i' > s_i$ such that $\underline{s}_1(\tilde{s}) < \phi(s_i)$ for any $\tilde{s} \in (s_i, s_i')$. This set is non-empty as a consequence of the conditions of the proposition. The profitable deviation is to play his corresponding strategy but for type s_i' rather than s_i until either (a) the incumbent quits, or

(b) all the other entrants quit. If (a) occurs, the deviation is to play as in (ii), whereas if (b) occurs, the deviation is to play as in (iii).

To see why the deviation is strictly profitable, we first note that our entrant wins in the same cases (and pays the same prices) with the original strategy as with the deviation if the highest of the other entrants' types, say $y_{(1)}$, is less than s_i. We also note that our entrant loses with both the original strategy and the deviation if $y_{(1)} \geq s'_i$. Both statements hold true as a consequence of the fact that the second best is monotonic (it allocates to an entrant's type, it also allocates to a higher type, and if it does not allocate to a type, it also does not allocate to a lower type) and that after (a) and (b) our entrant's play verifies (ii) and (iii), respectively, with the original strategy and with the deviation.

Suppose now that $y_{(1)} \in (s_i, s'_i)$. The fact that the second best only allocates to the entrant with highest type means that our entrant loses when playing the original strategy. We shall show that he wins with positive expected profits when she plays the deviation. To see why, note that (iv) means that all the other entrants quit before the incumbent if the incumbent's type is greater than $\underline{s}_1(y_{(1)})$). Since $\underline{s}_1(y_{(1)}) < \phi(s_i)$, because $y_{(1)} \in (s_i, s'_i)$, and since the deviation verifies (iii), our entrant's expected payoffs in the deviation when $y_{(1)} \in (s_i, s'_i)$ are equal to:

$$\int_{\underline{s}_1(y_{(1)})}^{\phi(s_i)} (v(s_i, s_1) - \hat{v}(s_1))dF_1(s_1),$$

which is strictly positive, as desired, by definition of ϕ. ■

4.2 Equilibrium Rushes, and Efficiency Losses

Next, we illustrate the properties of the equilibrium of the English auction when the English auction is not second best efficient. To simplify the equilibrium description, we introduce some additional assumptions: (I) all the entrants' types follow the same distribution F with density f; (II) there are only three bidders; (III) $v(s, s_1) - \hat{v}(s_1)$ is strictly increasing in s_1; and (IV) $\int_0^1 (v(0, s_1) - \hat{v}(s_1))\, dF_1(s_1) > 0$. (III) and (IV) imply that $\phi(s) = 1$ for any $s \in [0, 1]$. We also assume that (V) $v(0, 0) - \hat{v}(0) < 0$ so that the conditions of Proposition 2 are met. Our characterization can be extended to the more general setting of the rest of the paper when all the entrants' types follow the same distribution.

As we have already argued, the incumbent finds it weakly dominant to bid her value. Besides, an entrant with type s_i finds it optimal to quit if the incumbent has already quit, say at price p, and the current price is greater or equal than $v(s_i, \hat{v}^{-1}(p))$. Finally, it can easily be deduced from the definition of ϕ that an entrant with type s_i finds it optimal to remain active until price $\hat{v}(\phi(s_i))$

(recall that $\phi(s_i) = 1$) if the other entrant has already quit at a lower price. Note that in any equilibrium that verifies the previous conditions an entrant with type $s < \rho(s_1)$ quits immediately after the incumbent if the incumbent's type is equal to s_1 for $\rho(s_1)$ the minimum entrant's type $\tilde{s}$ for which $v(\tilde{s}, s_1) - \hat{v}(s_1) \geq 0$.

We propose an equilibrium that verifies the conditions in the former paragraph. It only remains to characterize the entrants' equilibrium bids in information sets in which all the bidders are still active. We propose that an entrant with type s bids in these information sets $\hat{v}(\gamma(s))$ where γ is equal to the solution to the differential equation in (4) with initial condition $\gamma(0) = 0$ until it reaches 1 and then equal to one.

$$\gamma'(s) = \frac{f(s) \int_{\gamma(s)}^{1} (v(s, s_1) - \hat{v}(s_1))\, dF_1(s_1)}{f_1(\gamma(s)) \frac{F(\rho(\gamma(s))) - F(s)}{2} (\hat{v}(\gamma(s)) - v(s, \gamma(s)))} \tag{4}$$

This differential equation ensures that the entrant does not have marginal incentives to increase her bid. To see why, note that this deviation only matters if it allows the bidder to outbid the lower of the other bidders's bids (and thus, avoids being the first bidder quitting), i.e. in the marginal event of breaking a tie with the lower bid of the other bidders. Conditional on this event, two things may happen. First, it is the other entrant who submits this lower bid, which occurs with conditional probability $\frac{f(s) \frac{1 - F_1(\gamma(s))}{\gamma'(s)}}{f(s) \frac{1 - F_1(\gamma(s))}{\gamma'(s)} + (1 - F(s)) f_1(\gamma(s))}$. In this case, our entrant competes in the last stage against the incumbent and makes conditional expected profits of $\int_{\gamma(s)}^{1} (v(0, s_1) - \hat{v}(s_1)) \frac{dF_1(s_1)}{1 - F_1(\gamma(s))}$. Second, it is the incumbent who submits the lower bid, which occurs with conditional probability $\frac{(f_1(\gamma(s))(1 - F(s))}{\frac{1 - F_1(\gamma(s))}{\gamma'(s)} f(s) + (1 - F(s)) f_1(\gamma(s))}$. In this case, our entrant gets negative payoffs equal to $\frac{1}{2}(v(s, \gamma(s)) - \hat{v}(\gamma(s))$ if both entrants quit simultaneously, which occurs with conditional probability $\frac{F(\rho(\gamma(s))) - F(s)}{1 - F(s)}$ if $s < \rho(\gamma(s))$.

Proposition 3. *The proposed strategies are an equilibrium of the English auction under assumptions (I)-(V).*

Proof. To prove that the proposed strategies we start remarking that it is optimal for an entrant to remain active until price $\hat{v}(1)$ in any information set after the other entrant quits. Besides, it is optimal to quit immediately after the incumbent. The former is a direct consequence of $\phi(s) = 1$ and the latter of $\rho(\gamma(s)) > s$ whenever $\gamma(s) < 1$, as implied by the definition of γ.

Consequently, the expected utility of an entrant with type s that bids b in an information set in which all the bidders are still active is equal to:

$$\int_0^{\hat{b}^{-1}(b)}\int_{\gamma(\tilde{s})}^1 (v(s,s_1)-\hat{v}(s_1))\,dF_1(s_1)dF(\tilde{s}) + \int_0^{\hat{v}^{-1}(b)}\int_{\min\{\gamma^{-1}(s_1),\rho(s_1)\}}^{\rho(s_1)} \frac{1}{2}\,(v(s,s_1)-\hat{v}(s_1))\,dF(\tilde{s})dF_1(s_1).$$

assuming the other two bidders follow the proposed strategies. The differential of this expression with respect to b is increasing in s and it is equal to zero at $b = \hat{v}(\gamma(s))$ by definition of γ. Thus, our entrant does not have incentives to deviate. ■

In our equilibrium, there are *rushes*, i.e. a bidder quitting prompts the remaining bidders to quit immediately. This occurs when the incumbent quits first and both entrants quit immediately after the incumbent, i.e. when both entrants' types are greater than $\gamma^{-1}(s_1)$ and smaller than $\rho(s_1)$ respectively, for s_1 the incumbent's type. The proposition below illustrates that we can even construct examples where the extent of the problem is so severe that the probability of a rush tends to one.

Proposition 4. *Suppose that* $\hat{v}(s_1) = s_1 + 1$ *and* $v(s,s_1) = s + \frac{\alpha}{\delta - s_1}$ *where* $\alpha > 0$ *and* $\delta \equiv \frac{1}{1-e^{-\frac{1}{\alpha^2}}}$, *and that* F *and* F_1 *are uniform. In this case, the probability of a rush occurring tends to one as* α *goes to zero. In the limit, the good is allocated to each entrant with equal probability independently of their types.*

Proof. To prove the proposition we show that $\rho(s_1)$ tends to one and $\gamma^{-1}(s_1)$ to zero. The former can be deduced from the definition of ρ and the fact that $v(s,s_1) - \hat{v}(s_1) = s + \frac{\alpha}{\frac{1}{1-e^{-\frac{1}{\alpha^2}}} - s_1} - s_1 - 1$, and $\lim_{\alpha\to 0} \frac{\alpha}{\frac{1}{1-e^{-\frac{1}{\alpha^2}}} - s_1} = 0$ for any $s_1 \in (0,1)$.

To prove that $\gamma^{-1}(s_1)$ tends to zero for any $s_1 \in (0,1)$, it is sufficient to show that $\gamma(s)$ tends to one for any $s \in (0,1]$. To prove so, we argue by contradiction. Suppose that there exists an $s \in (0,1]$ for which $\gamma(s)$ remains bounded away from one as α tends to zero. Note that by definition γ verifies:

$$\gamma(s) = \int_0^s \frac{\int_{\gamma(\tilde{s})}^1 \left(\tilde{s} + \frac{\alpha}{\frac{1}{1-e^{-\frac{1}{\alpha^2}}} - s_1} - s_1 - 1\right) ds_1}{\frac{\rho(\gamma(\tilde{s}))-\tilde{s}}{2}(\gamma(\tilde{s})+1) - \tilde{s} - \frac{\alpha}{\frac{1}{1-e^{-\frac{1}{\alpha^2}}} - \gamma(\tilde{s})}} d\tilde{s}, \tag{5}$$

where recall that $\gamma(\tilde{s})$ also remains bounded away from one for any $\tilde{s} \in [0,s]$ because γ is increasing. The denominator in Equation (5) tends to $\frac{1-\tilde{s}}{2}(\gamma(\tilde{s})+1) - \tilde{s} > 0$ as α tends to zero, and the numerator diverges to infinity, which contradicts that $\gamma(s)$ is bounded away from one. ■

Quite surprisingly, the first price auction does pretty well in this setup. Indeed, it implements the second best if one additional condition is met.

Remark 1. *Under assumptions (I), (III) and (IV), the first price auction implements the second best if:*

$$\int_0^1 v(0, s_1) dF_1(s_1) \geq \hat{v}(1).$$

Proof. It is an equilibrium of the first price auction that the incumbent bids her value and all the entrants bid as in a private value auction in which each entrant's private value is equal to $\int_0^1 v(s, s_1) dF_1(s_1)$. This is because the incumbent does not have incentives to deviate given that she looses with the proposed strategy and there is no price below her value at which she can win. Besides, entrants' do not have incentives to deviate to a bid larger than $\int_0^1 v(0, s_1) dF_1(s_1)$ because of the same reasons as in a first price private value auction, and below, because they make average expected profits with the proposed strategies conditional on winning and any bid below $\int_0^1 v(0, s_1) dF_1(s_1)$ loses with probability one. The proposed equilibrium implements the second best because the winner of the auction is the entrant with highest type and this is the second best efficient allocation under (III) and (IV). ■

5 A Sequential Sealed Bid Auction

In this section, we show that the sequential sealed bid auction analyzed by Perry, Wolfstetter, and Zamir (2000) has an equilibrium that implements the second best allocation. This mechanism has two stages. In the first one, all bidders submit a bid. In the second stage, the bids of all the bidders but the two highest bids, *the top bidders*, are made public and the top bidders are allowed to revise their bids upwards. The good is allocated to the bidder submitting the highest (final) bid at a price equal to the second highest (final) bid.

Our proposed equilibrium strategy for the incumbent is that she bids her value $\hat{v}(s_1)$ in the first stage and that she does no revise her bid in the second stage. This is her unique weakly dominant strategy as it can be deduced using the same arguments as in an open ascending auction with private values. To construct our proposed strategy for the entrants, we introduce an auxiliary function. This function is defined by the condition of no marginal incentives to change the first stage bid, as we argue below. In particular, let $\psi : [0, 1] \to [0, \phi(s)]$ be implicitly defined by:[6]

$$\int_0^{\psi(s)} \left(v(s, s_1) - \hat{v}(\psi(s))\right)^- dF_1(s_1) + \int_{\psi(s)}^{\phi(s)} \left(v(s, s_1) - \hat{v}(s_1)\right) dF_1(s_1) = 0. \tag{6}$$

[6]We let $(a)^-$ be equal to zero if $a > 0$ and otherwise equal to a. Similarly, we let $(a)^+$ be equal to a if $a > 0$ and otherwise equal to 0.

That ψ is well defined follows from the fact that the left hand side is positive at $\psi(s) = 0$ by definition of ϕ, negative at $\psi(s) = \phi(s)$, and it is a continuous strictly decreasing function in $\psi(s)$. Besides, that the left hand side is strictly increasing in s means that $\psi(s)$ is a strictly increasing function.

Our proposed equilibrium strategy for the entrants is that a type s bids $b(s) \equiv \hat{v}(\psi(s))$ in the first stage and, in the second state, he revises his bid upwards to $\hat{v}(\phi(s))$ if the incumbent is also a top bidder and to $\max\{b(s), v(s, s_1)\}$, otherwise, where s_1 is the type of the incumbent consistent with the incumbent's weakly dominant strategy and the price p at which she quit, i.e. $s_1 = \hat{v}^{-1}(p)$.

When all the bidders follow the proposed strategies, an entrant has no incentives to change marginally her first stage bid. This is because such a change is payoff relevant only when our entrant is tying in the first stage with either the other entrant with highest type or the incumbent, and the conditional expected payoffs are equal to zero in both cases. In the latter case, this is true because the marginal change does not vary the conditions in which our entrant wins: whenever it is the incumbent who submits the highest of the other bidders' bids and the incumbent's type is less than $\phi(s)$, for s our entrant's type. To see why this is also true in the former case distinguish whether the incumbent's type s_1 is greater than $\psi(s)$. If this is the case, the marginal increase makes our entrant top bidder together with the incumbent and hence gets utility $v(s, s_1) - \hat{v}(s_1)$ whenever $s_1 < \phi(s)$. If the incumbent's type s_1 is less than $\psi(s)$, our entrant and the entrant with highest type are the top bidders and a marginal increase of our entrant's bid makes her bid the highest. Thus, the marginal increase makes our entrant win when $v(s, s_1) < \hat{v}(\psi)$ with utility $v(s, s_1) - \hat{v}(\psi(s))$. Otherwise, he makes zero profits in any case since the entrant with highest bid also has a type s. Our argument that the conditional expected payoffs are equal to zero thus follows from the fact that ψ verifies Equation (6).

Next proposition shows that the proposed strategies are indeed an equilibrium.

Proposition 5. *The proposed strategies are an equilibrium.*

Proof. In this proof, we assume that all the bidders but one entrant with an arbitrary type s follows the proposed strategies and show that this entrant does not have incentives to deviate. Along the proof, we denote by s_1 the incumbent's type and by $y_{(k)}$ the k-th highest type of the other entrants.

We first compute the optimal bid in the second stage for our entrant given that she submitted an arbitrary bid p in the first stage. We use this analysis to argue, first, that the entrant does not have incentives to deviate in the second stage and, second, to provide an upper bound on the payoffs after a deviation in the first stage.

Consider, first, the case in which the other top bidder is the incumbent. The increase in our entrant's expected payoffs when she increases her first stage bid p to a bid $\hat{v}(s_1^*)$ are equal to:

$$\int_{\hat{v}^{-1}(p)}^{s_1^*} (v(s, s_1) - \hat{v}(s_1)) \; dF_1(s_1). \tag{7}$$

This is because by increasing the bid, our entrant can win additionally when the incumbent bids in $(p, \hat{v}(s_1^*))$, i.e. $s_1 \in (\hat{v}^{-1}(p), s_1^*)$, and in this case our entrant pays the incumbent's bid $\hat{v}(s_1)$. The above payoffs plus the constant (in s_1^*) $\int_0^{v^{-1}(p)} (v(s, s_1) - \hat{v}(s_1)) \; dF_1(s_1)$ are equal to the objective function of the problem that defines ϕ in Equation (2). This means that whenever the constraint $s_1^* > \hat{v}^{-1}(p)$ does not bind, $s_1^* = \phi(s)$ characterizes our entrant's optimal bid. Since this is the solution implemented by our proposed strategy and the constraint is not binding in this case because $\hat{v}(\psi(s)) \leq \hat{v}(\phi(s))$, our entrant does not have incentives to deviate as desired. The constraint is also not binding after a downward deviation in the first stage and hence it is also optimal to choose $s_1^* = \phi(s)$ in this case. The constraint can be binding after an upward deviation in the first stage, and then it may be optimal to pick an s_1^* larger than $\phi(s)$. We denote by $\hat{\phi}(s, p)$ the optimal s_1^* in this case. Consequently, when $p \leq b(s)$ our entrant wins if and and only if $s_1 < \phi(s)$, and when $p > b(s)$ our entrant wins if and only if $s_1 < \hat{\phi}(s, p)$.

Consider, now, the case in which the other top bidder is another entrant. Then, the incumbent's type s_1 gets revealed by her first stage bid and hence we can conclude from the usual arguments in private value second price auctions that our entrant finds it optimal to bid $\max\{p, v(s, s_1)\}$. This the bid prescribed by our proposed strategy, and hence our entrant cannot improve with a deviation. This also means that both in equilibrium and after a first stage deviation followed by optimal play, the entrant that wins is the one submitting the higher bid in the first stage unless the other top bidder finds it profitable to outbid the former. Thus, our entrant wins if she is the top bidder with higher bid in the first stage and $s > y_{(1)}$ and she losses if she is not the top bidder with higher bid in the first stage and $s < y_{(1)}$.

Next, we show that our entrant does not have incentives to deviate upward to bid $p > b(s)$ in the first stage assuming the optimal play described above for the second stage. We compute bounds on the expected increase in payoffs after the deviation distinguishing two cases:

(a) If $s_1 \geq \psi(y_{(1)})$, our entrant can only be top bidder with the incumbent. Thus, it pays price $\hat{v}(s_1)$ if she wins and the only difference between bidding p or $b(s)$ occurs when either (i) $b(y_{(1)}) \in (b(s), p)$ and $s_1 < \hat{\phi}(s, p)$; or (ii) $b(y_{(1)}) < b(s)$ and $s_1 \in (\phi(s_1), \hat{\phi}(s, p))$. If either $b(y_{(1)}) \geq p$ or

$s_1 \geq \hat{\phi}(s,p)$, our entrant loses with both $b(s)$ and p, as it is outbid by another entrant or the incumbent, respectively, whereas if $b(y_{(1)}) < b(s)$ and $s_1 < \phi(s)$, our entrant wins with both $b(s)$ and p. In case (i) and (ii), however, our entrant wins with the bid p but losses with $b(s)$. In case (i), this is because our entrant is a top bidder with p but not with $b(s)$ and in case (ii) because our entrant outbids the incumbent in the second stage when she bid p in the first stage but not when she bid $b(s)$ in the first stage. Since case (ii) makes deviations less profitable because $\int_{\phi(s)}^{\hat{\phi}(s,p)} (v(s,s_1) - \hat{v}(s_1))\, dF_1(s_1) \leq 0$ by definition of ϕ, the expected increase in payoffs that corresponds to the case $s_1 \geq \psi(y_{(1)})$ is no more than:

$$\int_s^{b^{-1}(p)} \int_0^{\hat{\phi}(s,p)} (v(s,s_1) - \hat{v}(s_1))\, dF_1(s_1) dF(y_{(1)})^{n-1} \tag{8}$$

$$\leq \int_s^{b^{-1}(p)} \int_0^{\hat{\phi}(s,p)} \big(v(y_{(1)},s_1) - \hat{v}(s_1)\big)\, dF_1(s_1) dF(y_{(1)})^{n-1} \tag{9}$$

$$\leq \int_s^{b^{-1}(p)} \int_0^{\phi(y_{(1)})} \big(v(y_{(1)},s_1) - \hat{v}(s_1)\big)\, dF_1(s_1) dF(y_{(1)})^{n-1}, \tag{10}$$

where we have used the monotonicity of v in the first inequality and the definition of ϕ in the second inequality.

(b) If $s_1 < \psi(y_{(1)})$, our entrant can only be top bidder with the entrant with type $y_{(1)}$. In this case, the only difference between bidding p or $b(s)$ occurs when $b(y_{(1)}) \in (b(s), p)$. If $b(y_{(1)}) < b(s)$ our entrant wins with both $b(s)$ and p whereas if $b(y_{(1)}) > p$, our entrant loses with both $b(s)$ and p. The former because our entrant is the top bidder with highest bid in the first stage with either p and $b(s)$ and $s > y_{(1)}$, and the latter because our entrant is not the top bidder with highest bid with either p or $b(s)$ and $y_{(1)} > s$. If $b(y_{(1)}) \in (b(s), p)$ our entrant is the top bidder with highest bid if she bids p but she is not when she bids $b(s)$. Since $y_{(1)} > s$, this means that our entrant does not with bid $b(s)$ but she wins with bid p if the other top bidder does not find it profitable to win, i.e. $v(y_{(1)}, s_1) \leq \max\{p, v(s,s_1)\}$. In this case, our entrant makes profits $v(s,s_1) - \max\{v(y_{(1)},s_1), \hat{v}(\psi(y_{(1)}))\}$ which are are less than:

$$v(y_{(1)},s_1) - \max\{v(y_{(1)},s_1), \hat{v}(\psi(y_{(1)}))\} = \big(v(y_{(1)},s_1) - \hat{v}(\psi(y_{(1)}))\big)^{-}.$$

Hence, the expected increase in payoffs that corresponds to the case $s_1 < \psi(y_{(1)})$ is no more than:

$$\int_s^{b^{-1}(p)} \int_0^{\psi(y_{(1)})} (v(y_{(1)}), s_1) - \hat{v}(\psi(y_{(1)})))^{-} dF_1(s_1) dF(y_{(1)})^{n-1}. \tag{11}$$

We can thus conclude that our entrant does not have incentives to deviate because the sum of Equation (10) and (11) is equal to zero by definition of ψ.

Finally, we follow a similar approach for downward deviations in the first stage, $p < b(s)$:

(a) If $s_1 \geq \psi(y_{(1)})$, our entrant can only be top bidder with the incumbent. Thus, it pays $\hat{v}(s_1)$ if she wins and the only difference between bidding p or $b(s)$ occurs when $b(y_{(1)}) \in (p, b(s))$ and $s_1 < \phi(s)$. If $b(y_{(1)}) \leq p$, our entrant wins if and only if $\hat{v}(s_1) < \hat{v}(\phi(s))$ with both p and $b(s)$, as in either case is a top bidder that bids $\hat{v}(\phi(s))$. Besides, if either $b(y_{(1)}) \geq b(s)$ or $s_1 > \phi(s)$, our entrant losses with both p and $b(s)$ because either she is not a top bidder or she is outbid by the incumbent, respectively. However, $b(y_{(1)}) \in (p, b(s))$ means that our entrant is a top bidder with $b(s)$ but not with p, and $s_1 < \phi(s)$ means that whenever the entrant is a top bidder, she outbids the incumbent in the second stage. Thus, the expected decrease in payoffs after the deviation is equal to:

$$\int_{b^{-1}(p)}^{s} \int_{0}^{\phi(s)} \left(v(s, s_1) - \hat{v}(s_1)\right) dF_1(s_1) dF(y_{(1)})^{n-1} \tag{12}$$

$$\leq \int_{b^{-1}(p)}^{s} \int_{0}^{\phi(s)} \left(v(y_{(1)}, s_1) - \hat{v}(s_1)\right) dF_1(s_1) dF(y_{(1)})^{n-1}, \tag{13}$$

where we have used the monotonicity of v.

(b) If $s_1 < \psi(y_{(1)})$, our entrant can only be top bidder with the entrant with type $y_{(1)}$. In this case, the only difference between bidding p or $b(s)$ occurs when $b(y_{(1)}) \in (p, b(s))$. If $b(y_{(1)}) < p$ our entrant wins with both $b(s)$ and p because she is in either case the top bidder with higher bid and $y_{(1)} < s$. Besides, if $b(y_{(1)}) > b(s)$, our entrant loses with both $b(s)$ and p because she is not the top bidder with higher and $s < y_{(1)}$. However, if $b(y_{(1)}) \in (p, b(s))$, our entrant always wins with $b(s)$ but with p only if she is a top bidder, i.e. when both $\hat{v}(s_1)$ and $b(y_{(2)})$ are less than p, and if our entrant finds it profitable to outbid the entrant with type $y_{(1)}$, i.e. when $v(s, s_1) - \max\{v(y_{(1)}, s_1), \hat{v}(\psi(y_{(1)}))\} \geq 0$. In any case, our entrant gets $v(s, s_1) - \max\{v(y_{(1)}, s_1), \hat{v}(\psi(y_{(1)}))\}$ when she wins. Consequently, the expected decrease in payoffs after the deviation conditional on $y_{(1)}$ and s_1 is equal to:

$$\begin{aligned}
v(s,s_1) - \max\{v(y_{(1)},s_1),\hat{v}(\psi(y_{(1)}))\} - \left(\frac{F(b(p)^{-1})}{F(y_{(1)})}\right)^{n-2} (v(s,s_1) - \max\{v(y_{(1)},s_1),\hat{v}(\psi(y_{(1)}))\})^{+} \\
\geq v(s,s_1) - \max\{v(y_{(1)},s_1),\hat{v}(\psi(y_{(1)}))\} - (v(s,s_1) - \max\{v(y_{(1)},s_1),\hat{v}(\psi(y_{(1)}))\})^{+} \\
= \left(v(s,s_1) - \max\{v(y_{(1)},s_1),\hat{v}(\psi(y_{(1)}))\}\right)^{-} \\
= \left(v(s,s_1) - \hat{v}(\psi(y_{(1)}))\right)^{-} \\
\geq \left(v(y_{(1)},s_1) - \hat{v}(\psi(y_{(1)}))\right)^{-}
\end{aligned}$$

which we integrate with respect to $y_{(1)}$ and s_1 to get:

$$\int_{\hat{b}^{-1}(p)}^{s} \int_{0}^{\psi(y_{(1)})} (v(y_{(1)},s_1) - (\hat{v}(\psi(y_{(1)})))^{-} dF_1(s_1)) dF(y_{(1)})^{n-1}. \tag{14}$$

We can thus conclude that our entrant does not have incentives to deviate because the sum of Equation (13) and (14) is equal to zero by definition of ψ. ■

Finally, we conclude our argument with the following result.

Proposition 6. *If all bidders play our proposed equilibrium, the second best allocation associated to ϕ is implemented.*

Proof. Suppose first that $\phi(s_{(1)}) > s_1$ where $s_{(1)} \equiv \max\{s_i\}_{i\neq 1}$ and suppose that the two top bidders are entrants. Then, the entrant with highest type wins as required by the second best because the entrants' bid functions are strictly increasing in both stages. Suppose now that $\phi(s_{(1)}) > s_1$ and that the top bidders are the incumbent and an entrant. Then, this entrant is the entrant with highest type because their bid function in the first stage is strictly increasing. Besides, this entrant outbids the incumbent because the entrant bids $\hat{v}(\phi(s_{(1)}))$ in the second stage and the incumbent $\hat{v}(s_1)$. Suppose now that $\phi(s_{(1)}) < s_1$. Then, since bid functions are strictly increasing and $\psi(s) \leq \phi(s)$ the incumbent and the entrant with highest type are the top bidders but the incumbent wins because its bid $\hat{v}(s_1)$ is larger than the entrant's bid in the second stage $\hat{v}(\phi(s_{(1)}))$. ■

6 Concluding Remarks

The present paper points out a novel source of inefficiency that is characteristic of the English auction and lies in the impossibility for its random tie breaking rule to resolve rushes that happen with strictly positive probability in equilibrium. We show what the most efficient allocation is in this setting and we provide a solution to the problem by identifying a very simple two stage sealed bid format that is efficient. Interestingly, this mechanism has already been used in practice for important privatization such as the one of ENI, see Caffarelli (1998). The more general implications of our work is that the continuos flow of new information that is present in many markets can be detrimental for efficiency if it can lead to rushes. We have shown what the optimal solution is for the setting we were interested in, which fits many important application such as privatizations, procurement auctions, takeovers. It remains an interesting open question what the optimal mechanism looks like in general. Our findings suggest that such format is likely to be some variant of a multi stage SPA that breaks optimally the flow of information.

We would like to remark that not only the generality of our findings is likely to go beyond the setting analyzed here, but also that the efficiency loss due to rushes can be even more severe in more complex environments. Regarding the first observation, our environment applies in general to situations where bidders (or more generally agents in a market) are active at prices where given the uncertainty about some common value information still to be aggregated they might suffer a loss if being awarded the object at such price. In particular, this goes beyond the type of incumbent/entrants situations presented here and in Hernando-Veciana and Michelucci (2011). To see why consider a fully symmetric setting where there are n bidders with value $v(s_i, s)$, where $s \in \{0, 1\}$ represents an exogenous information that might become available during the auction/market process. A way to model this given that in the English auction the price is raised at a continuous pace, and that possible values span from $v(0, 0)$ to $v(1, 1)$ is to say that the exact realization of s is equally likely to be revealed at any point on such support. It is easy to see how such setting would give rise to rushes in equilibrium: for reasons similar to the ones in our model the prospect of aggregating the good information $s = 1$ justifies the risk of being active at prices at which if $s = 0$ is revealed the bidder would make a loss if allocated the object. Regarding the second observation at the beginning of this paragraph, note that even though we defined rushes as situations in which all active bidders quit simultaneously, in general we have a loss of information and thus efficiency even if some bidders are still active. That is, rushes are likely to be relevant with a higher frequency than they are in our model. To see the reason for this second point notice that if an entrant value is also a function of the other entrants types, the fact that a subset of them with different willingness to pay exit at the same price means that their exact signal cannot be aggregated by the remaining bidders. If these latter bidders are not symmetric this can translate into severe inefficiencies. To conclude note that in this paper we have focused on efficiency because we had in mind situations (as privatization) where this objective is very important. We leave for further research to investigate how our two stages sealed bid format ranks in terms of revenues in comparison to the English auction and to the benchmark of the optimal auction.

References

Boone, J., R. Chen, J. K. Goeree, and A. Polydoro (2009): "Risky Procurement with an Insider Bidder," *Experimental Economics*, 12(4), 417–436.

Boone, J., and J. K. Goeree (2009): "Optimal Privatizations using Qualifying Auctions," *The Economic Journal*, 119, 277–297.

Caffarelli, F. V. (1998): "The ENI group Multiple Round Auction Procedure," *Rivista Di Politica Economica*, 88(12), 109–140.

Dutra, J., and F. Menezes (2002): "Hybrid Auctions," *Economics Letters*, 77(3), 301–307.

Hernando-Veciana, A., and F. Michelucci (2011): "Second Best Efficiency and The English Auction," *Games and Economic Behavior*, forthcoming.

Kagel, J., S. Pevnitskaya, and L. Ye (2008): "Indicative Bidding: An Experimental Analysis," *Games and Economic Behavior*, 62, 697–721.

Klemperer, P. (2002): "What really matters in Auction Design," *Journal of Economic Perspectives*, 16(1), 169–189.

Krishna, V. (2003): "Asymmetric English Auctions," *Journal of Economic Theory*, 112, 261–288.

Levin, D., and L. Ye (2008): "Hybrid Auctions Revisited," *Economics Letters*, 99(3), 591–594.

Milgrom, P., and R. Weber (1982): "A Theory of Auctions and Competitive Bidding," *Econometrica*, 50, 1089–1122.

Myerson, R. B. (1981): "Optimal Auction Design," *Mathematics of Operation Research*, 6(1), 58–73.

Perry, M., E. Wolfstetter, and S. Zamir (2000): "A Sealed Bid Auction that matches the English Auction," *Games and Economic Behavior*, 33(2), 265–273.

Ye, L. (2007): "Indicative bidding and a theory of two-stage auctions," *Games and Economic Behavior*, 58, 181–207.