Time Allocation and Task Juggling*

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Abstract

A single worker is assigned a stream of projects over time. We provide a tractable theoretical model in which the worker allocates her time among different projects. When the worker works on too many projects at the same time, the output rate decreases and the time it takes to complete each project grows. We call this phenomenon "task juggling," and we argue that this phenomenon is pervasive in the workplace. We show that task juggling is a strategic substitute of worker effort. We then present an augmented model, in which task juggling is the result of lobbying by clients, or co-workers, each of whom seeks to get the worker to apply effort to his project ahead of the others'. We also study how the worker should be incentivized, when the worker can multitask across projects of different complexity: We extend the model to allow for switching costs, where the worker forgets projects which are infrequently worked on. We also model environments, such as triage, where task juggling can in fact be optimal.

1 Introduction

This paper studies the way in which a worker allocates time across different projects, or equivalently, effort across different projects through time. We study, in particular, the phenomenon of *task juggling* (frequently called multitasking), whereby a worker switches from one project to another "too frequently."

Task juggling is a first-order feature in many workplaces. Using time diaries and observational techniques, the managerial literature on Time Use documents that knowledge workers

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(engineers, consultants, etc.) frequently carry out a project in short incremental steps, each of which is interleaved with bits of work on other projects. For example, in a seminal study of software engineers Perlow (1999) reports that

> "a large proportion of the time spent uninterrupted on individual activities was spent in very short blocks of time, sandwiched between interactive activities. Seventy-five percent of the blocks of time spent uninterrupted on individual activities were one hour or less in length, and, of those blocks of time, 60 percent were a half an hour or less in length."

Similarly, in their study of information consultants Gonzalez and Mark (2005, p. 151) report that

> "the information workers that we studied engaged in an average of about 12 working spheres per day. [...] The continuous engagement with each working sphere before switching was very short, as the average working sphere segment lasted about 10.5 minutes."

The fact that much work is carried out in short, interrupted segments is, in itself, a descriptively important feature of the workplace. But what causes these interruptions? The Time Use literature points to the "interdependent workplace," meaning an environment in which other workers can (and do) ask/demand immediate attention to joint projects which may distract the worker from her more urgent tasks. One of the workers interviewed by Gonzalez and Mark (2005, p. 152) puts it this way:

> "Sometimes you just get going into something and they [call] you and you have to drop everything and go and do something else for a while [...] it's almost like you are weaving through, it is like, you know, a river, and you are just kind of like: "Oh these things just keep getting in your way", and you are just like: "get out of my way" and then you finally get through some of the other tasks and then you kind of get back, get back along the stream, your tasks [...]."

The literature on Human Scheduling, instead, attributes task juggling to the cognitive limitations of individual human schedulers. Crawford and Wiers (2001, p. 34), for example, write:

> "One way in which human schedulers try to reduce the complexity of the scheduling problem is by simplification [...]. However, a simplified scheduling model leads to the oversimplification of the real system to be scheduled, and this in turn creates unfeasible or sub-optimal schedules."

The physiological constraints on scheduling ability are explored in the medical literature.[1] The popular press, however, has already rendered its verdict: scheduling is a challenge for many workers for reasons both internal and external to the worker. Popular literature books such as Covey (1989) and Allen (2001) exort (and attempt to help) the reader to prioritize better. In *The Myth of Multitasking: How "Doing It All" Gets Nothing Done,* we find a list of suggestion designed to help people reduce multitasking on the job. The first two are:

> – Resists making active [e.g., self-initiated] switches.
> – Minimize all passive [e.g., other-initiated] switches.

(Cited from Crenshaw 2008, p. 89).

1.1 Effects of task juggling on productivity

We are interested in task juggling insofar as it affects productivity. The next example illustrates a productivity loss which is mechanical, and inherent to task juggling.

Example 1 *Consider a worker who is assigned two independent projects, A and B, each requiring 10 days of undivided attention to complete. If she juggles both projects, for example working on A on odd days and on B on even days, the average duration of the two projects is equal to 19.5 days. If instead she focuses on each projects in turn, she completes A on the 10-th day and then takes the next ten days to complete B. In the second case, the average duration of both projects from the time of assignment is 15 days. Note that under the second work schedule projects B does* not *take longer to complete, while A is completed much faster; in other words, avoiding task juggling results in a Pareto-improvement across projects durations.*

The example shows that a worker who juggles too many projects takes longer to complete each of them, than if she handled projects sequentially. The latter procedure corresponds to the "greedy algorithm," which is widely studied in the operations research literature.[2]

In addition to this mechanical slowdown, there may be "human" effects related to interruption: the worker may forget what she was doing before being interrupted, which impacts the speed or quality of the worker's output. Or the worker may enjoy the variety and, conceivably, work harder if tasks are alternated.

[1] See, e.g., Morris *et al.* (1993) and Baker *et al.* (1996).

[2] The name "greedy" refers to prioritizing those projects which are closest to completion (which project A is after day 1).

1.2 Need for a model of task juggling

That task juggling must decrease productivity is well known from the operations research literature. But that literature has a *normative approach*: it tells us that workers *shouldn't* juggle and should do "greedy" instead. And yet the evidence overwhelmingly shows that workers actually do juggle. If we see this behavior in the workplace, and it is prevalent, then as economists we want to understand it. Accordingly, we are interested in the following *positive questions*:

1. How much do workers juggle, if at all? (That is, establish an appropriate metric on juggling which can be taken to data)
2. How large, quantitatively, are the productivity consequences of juggling? (Spell out the relationship between juggling and productivity, for given effort and ability of the worker)
3. Who/what makes workers juggle? And when workers are made to juggle, do they do it in the specific way that our model assumes?
4. If workers can be prevented from juggling by giving them strong incentives based on productivity, what measure of productivity should these incentives depend on?
5. How does task juggling impact the workers' incentives to exert effort?

Many of these questions have to do with incentives and, therefore, are properly in the domain of economics as opposed to (classical) operations research. To answer these questions we need a model where workers behave contrary to the prescriptions of operations research, i.e., they juggle tasks. A useful model must be richer than the "toy model" in Example 1, because we want the model to match time series/panel data patterns like the ones presented in Section 1.3 just below. It must also be tractable, because we want the model to answer questions 1-5 above. This paper provides such a model.

1.3 Illustrative application

For an example of the kind of empirical settings that our model is relevant for, consider Figure 1. This figure refers to the production process of a panel of Italian labor law judges in the Milan labor court.[3] Adjudication takes place in a series of distinct steps (involving motions, hearings, etc.) The interleaving of these steps across different trials raises the

[3] For details on this data see Coviello et al. (2010).

possibility that a judge may, as in Example 1, work on too large a number of cases at a point in time.

The figure reveals a correlation between task juggling and the effectiveness of the production process. The stock of **Active Cases** (cases which have received a first hearing but are not yet closed) proxies for task juggling: it grows at a steady pace, from about 150 in the year 2000 to more than 300 in 2005. At the same time, **Completion Time** (defined as the number of days elapsing between the date in which the first hearing is held and the date the case is completed) is also increasing over time. The increase in completion time is *prima facie* evidence of a slowdown in productivity, because it takes judges longer and longer to work through a given case.[4,5] In sum, Figure 1 suggests that task juggling and productivity are inversely related.

The possibility that task juggling may significantly reduce judicial productivity, which in Italy is a major policy concern, should make task juggling an important policy issue.[6] Of course, the relevance of task juggling extends much beyond the Italian case.

1.4 Outline of the paper

As a first step toward more complex models, in this paper we focus on *a single worker* who faces time allocation issues. In Section 3 we model a production process which may feature task juggling. Formally, the model is summarized in a system of four functional equations (1) through (4). Finding a solution to this system represents an original mathematical contribution which is offered in Theorem 1. Based on this solution, we demonstrate that effort and task juggling are *strategic substitutes*. This means that anything that makes workers juggle more tasks will also, indirectly, reduce the worker's incentives to exert effort.

Section 4 addresses the incentives that generate task juggling. We model a lobbying game in which the worker allocates effort under pressure by her co-workers, superiors, or clients. This model is inspired by the idea of "interdependent workplace" discussed in the introduction. We fully characterize the equilibrium of the lobbying game and show that, no matter how low

[4] Notice that the growth in completion time is a measure of productivity which need not mechanically reflect judge overload. Even if a judge is assigned too many cases, she could refrain from holding the hearings on a just-assigned case until her stock of active cases gets below a threshold. This is a "greedy" policy like the one described in Example 1. If she did that, she would have few active cases and low completion times, as well as lower durations.

[5] Coviello *et al.* (2010) relies on instrumental variables to establish that the association is in fact causal: an increase in task juggling *causes* an increased duration of cases.

[6] In Italy in 2009, civil trials lasted on average 960 days in the court of first instance, and an additional 1,509 days in court of appeals (if appealed). Such durations place Italy at n. 88 in the world in "speed of enforcing contracts" as measured by the Doing Business survey of the World Bank—behind Mongolia, the Bahamas, and Zambia.

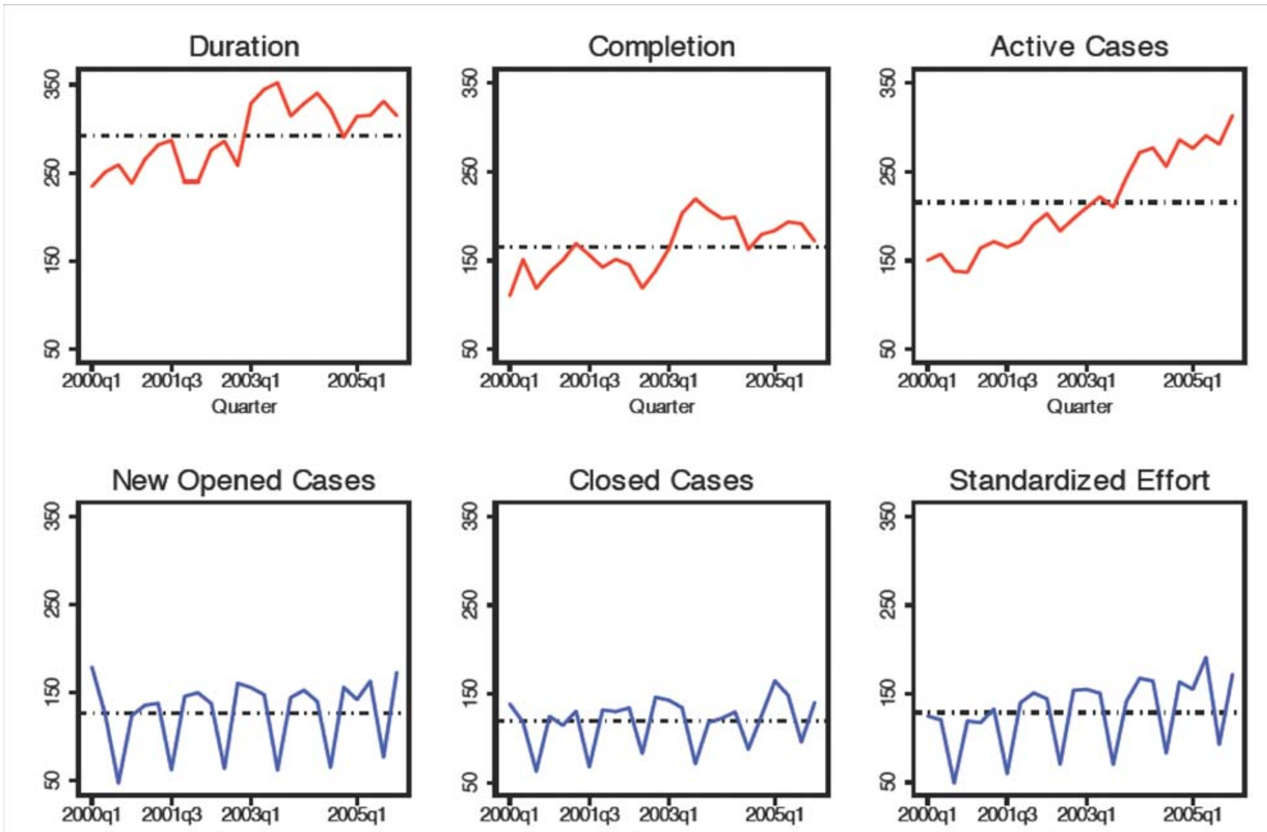

Figure 1: Statistics on the productivity of labor judges. See Coviello *et al.* (2010).

the cost of lobbying, in equilibrium there will be lobbying, which will induce task juggling. This model provides a microfoundation of task juggling. We also show that, when worker effort is non-contractible, more intense lobbying which makes workers juggle more tasks will also, indirectly, reduce the worker's incentives to exert effort. This indirect, strategic effect compounds the direct effect of task juggling.

One might argue that task juggling is the consequence of soft incentives: give workers sharp enough incentives based on productivity, and task juggling will go away. In Section 5 we explore this idea. We analyze the different options available to a principal who needs to incentivize the worker. We assume that, in response to the strong incentives, the worker is no longer task-juggling. We explore, however, another problems which can arise due to strong incentives: multitasking (in the sense of Holmostrom and Milgrom 1991). If the worker can strategically elect to work on projects of different complexity, then there is a risk that the worker will focus only on the easy/quick projects. We show that if the worker is compensated based on aggregate output irrespective of the complexity of the project, then that worker will totally ignore complex projects and focus on easy ones. Under this strategy, complex projects never get done. If instead the worker is penalized based on the average duration of assigned projects, then we show that a worker will work on cases of all complexities, and in fact will devote more than proportional effort to complex cases.

In Section 6 we analyze several extension. Section 6.1 adds to the model the possibility that interruptions may make the worker forget, or conversely, that interruptions may stave off boredom. We study how the production function is amended in the presence of these effects. In Section 6.2 we analyze a model in which value is created when a project completes early intermediate goals, as is the case for triage in medical care, for example. In this case task juggling could be *efficient*. We also explore variable-speed strategies, that is, strategies that calibrate the amount of effort devoted to a case according to its stage of completion. In Section 6.3 we deal with the case in which projects have different degrees of complexity, but they cannot be treated disparately. An example is judicial trials, which may settle at an unknown time. If the judge cannot (perfectly) distinguish trials by their likelihood of settling early, then she will not be able to treat these cases disparately. Other extensions are explored as well.

2 Related Literature

What we call task juggling is viewed as an aberration in the queuing literature. The focus of the queuing literature is to provide algorithms ("greedy"-type algorithms, usually) that prevent task juggling. As we discussed in the introduction, we believe that this particular aberration is worth studying because it arises empirically, arguably as a predictable result of incentives. From the technical viewpoint, our model also departs from the queuing literature because that literature focuses on giving algorithms that keep queing systems *stable*, that is, sufficient conditions under which queues can't ever get unacceptably or infinitely long.[7] Our model is by nature unstable because the arrival rate exceeds the capacity of the worker (in our notation, $\alpha > \eta/X$). We believe that there is merit in going beyond stable queuing systems because stability requires the serving facility to be idle at least a fraction of their time, which is counterfactual in many environments (judges always have a backlog of cases that they should be working on, for example). Finally, our paper is distinct from most of the queuing literature in that the study of the incentives such as the ones we examine is largely absent from that literature.

In the economics literature, Radner and Rothschild (1975) discuss task prioritization by a

[7] An exception to the focus on stability is Dai and Weiss (1996), who do study the evolution of an unstable queing network.

single worker. They give conditions under which no element of a multidimensional controlled Brownian motion ever falls below zero. The control represents a worker's (limited) effort being allocated among several tasks, and the dimensions of the Brownian motion represent the satisfaction levels with which each task is performed. Although broadly similar in its subject matter, that paper is actually quite different from the present one. Among other differences, it focuses on optimality and features no discussion of incentives.

Task juggling is studied in the sociological/management literature on time use (see Perlow 1999 for a good example and a review of the literature). This literature uses time logs and observations to document the patterns of uninterrupted work time, and the causes of the interruptions. This literature identifies "interdependent work" as the source of interruptions. The "lobbying by clients" model presented in Section 4 captures this effect. There is also a small literature devoted to measuring the *disruption cost* of interruptions, i.e., the additional time to reorient back to an interrupted task after the interruption is handled (see e.g. Mark *et al.* 2008, who review the literature). We introduce this cost in Section 6.1. At a more popular level, there is large time management culture which focuses on the dynamics of distraction and on "getting things done" (see e.g. Covey 1989, Allen 2001).[8]

The managerial "firefighting" literature (see Bohn 2000, Repenning 2001) documents the phenomenon whereby an organization focuses resources on unanticipated flaws in almost-completed projects (firefighting), and in so doing starves projects at earlier development stages of necessary resources, which in turn ensures that these projects will later require more firefighting, etc. This phenomenon is specular to the one we study because in our model the inefficiency is caused by too few, not too many, resources devoted to late-stage projects.

Dewatripont *et al.* (1999) provide a model in which expanding the number of projects a worker works on will indirectly reduce the worker's incentives to exert effort. We get the same effect in Proposition 4. In their setup, the effect results from the worker's incentives to exert effort in order to signal his ability. Clearly, that effect is quite different than the one analyzed in this paper.

3 The Production Process

In this section we introduce a dynamic production process which incorporates the possibility of multitasking in a very simple way. Imagine a worker who is assigned a stream of project over time. Assuming the worker cannot deal with all the projects instantaneously (a reasonable assumption), then the worker has to choose how to deal with the excess. We assume

[8]For a review of the academic literature on this subject see Bellotti *et. al.* (2004).

that, as cases are progressively assigned to the worker at rate α, she puts them in a queue of inactive cases. The worker draws from this queue at rate ν. When a case is drawn from the queue it is "put in production" along with all other already active cases. Finally, we assume that the worker's attention is divided in a perfectly equal fashion among all active cases, in a process that parallels the "perfect task juggle" in Example 1. This modeling strategy allows us to span the range between much task juggling (ν large, approaching α) and no task juggling, close to "greedy" (ν low).

We will derive an exact formula for the production function which, given an effort rate, a degree of complexity of projects, and a level of task juggling, yields an output rate. Having an exact formula for the production function will allow us later to study strategic behavior pertaining to task juggling. Of note, in this section we abstract from the possibility that multitasking might cause the worker to forget; this additional effect of multitasking will be introduced in Section 6.1.

3.1 The Model

The model lives in continuous time, starting from $t = 0$. At time 0 the worker has no projects. Projects are assigned at rate α. There is a continuum of projects.

Each project takes X steps to complete. A project is characterized, at any point in time, by its degree of completion $x \in [0, X]$, which measures how far away the project is from being completed. We call a project **completed** when $x = 0$. Note that, because x is a continuous variable, we are assuming that there is a continuum of steps for each project. X can be interpreted as measuring the complexity of the project, or the worker's ability.

As soon as the worker starts working on a project, we say that the project becomes **active**. The project stops being active when it is completed. At any time t, the worker has A_t active projects, in various degrees of completion. The distribution $\varphi_t(x)$ denotes the mass of projects which are exactly x steps away from being done. By definition, the number of active projects at time t is

$$A_t = \int_0^X \varphi_t(x)\, dx \tag{1}$$

We assume that all active projects are moved towards completion at a rate η_t/A_t, where η_t is the rate at which effort is exerted. Informally, this means that in the time interval between t and $t + \Delta$, the worker's work shaves off approximately $(\eta_t/A_t)\,\Delta$ steps from *each* active project.[9] This formulation captures the idea that the worker divides a fixed amount of working hours equally among all projects active at time t. This procedure means that the

[9] Note that this formulation requires $A_t > 0$.

worker is working "in parallel." If all active projects proceed at the same speed, then after Δ has elapsed, the distribution $\varphi_t(x)$ is translated horizontally to the left (refer to Figure 2), and so for Δ "small enough" we can write intuitively

$$\varphi_{t+\Delta}\left(x - \frac{\eta_t}{A_t}\Delta\right) = \varphi_t(x).$$

To express this condition rigorously, bring $\varphi_t(x)$ to the right-hand side, divide by Δ and let $\Delta \to 0$ to get

$$\frac{\partial \varphi_t(x)}{\partial t} - \frac{\partial \varphi_t(x)}{\partial x}\frac{\eta_t}{A_t} = 0. \tag{2}$$

This partial differential equation embodies the assumption of perfectly parallel work on the active cases.

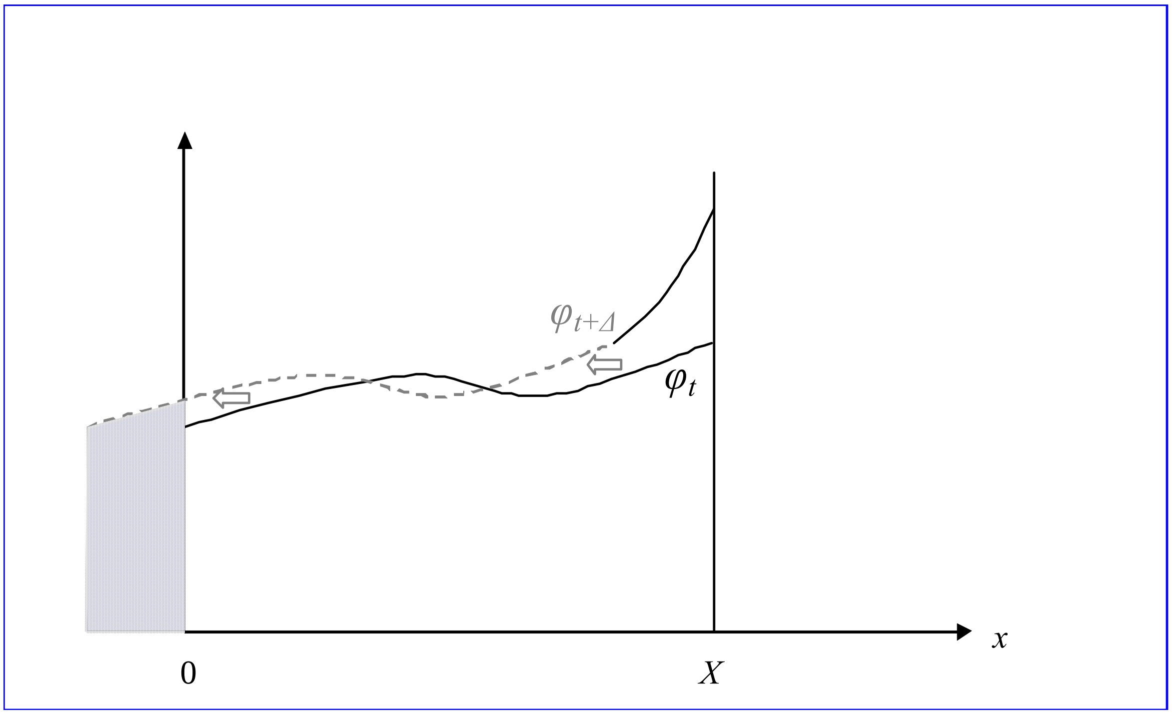

Figure 2: The function φ_t is traslated horizontally to the left as time passes. Newly opened cases are added to the right. The grey mass of cases to the left of zero are completed.

The projects that fall below 0 (grey mass in Figure 2) are the ones that get completed within the interval Δ. These are the projects whose x at t is smaller than $\frac{\eta_t}{A_t}\Delta$. Therefore, the mass of output between t and $t+\Delta$ is approximately

$$\int_0^{\frac{\eta_t}{A_t}\Delta} \varphi_t(x)\,dx.$$

To get the **output rate** ω_t, divide this expression by Δ and let $\Delta \to 0$ to get

$$\omega_t = \lim_{\Delta\to 0} \frac{1}{\Delta} \int_0^{\frac{\eta_t}{A_t}\Delta} \varphi_t(x)\,dx = \frac{\eta_t}{A_t}\varphi_t(0)\,. \tag{3}$$

The worker is not required to open projects as soon as they are assigned. Rather, we allow the worker to open new projects at a rate ν_t. A larger ν_t will, ceteris paribus, mean more task juggling—more projects being worked on simultaneously. This ν_t is seen either as a choice on the part of the worker, or as determined by lobbying, or else imposed by some regulation. For Δ small, the change in the mass of projects active at t is approximately

$$A_{t+\Delta} - A_t = \nu_t \cdot \Delta - \omega_t \cdot \Delta.$$

Divide both sides by Δ and let $\Delta \to 0$ to get the formally correct expression

$$\frac{\partial A_t}{\partial t} = \nu_t - \omega_t. \tag{4}$$

Graphically, the mass of newly opened projects is squeezed in at the back of the queue in Figure 2, just to the left of X, in whatever space is vacated on the horizontal axis by the progress made in Δ on the pre-existing open projects.

This completes the description of the production process. In the model, two variables are interpreted (for now) as exogenously given: η_t and ν_t. The first describes *how much* the worker works, the second *how* she works—how many projects she keeps open at the same time. These two variables will determine, through the process described mathematically by equations (1) through (4), the key variable of interest, the output rate ω_t. This variable, in turn, will determine the duration of a project and its completion time.[10] Our first major task is to uncover the law through which η_t and ν_t determine ω_t. We turn to this next.

[10] The two (endogenous) functions A_t and $\varphi_t(x)$ are, perhaps, of merely instrumental interest: they describe the state of the worker's docket at any point in time—how many projects he has open, and the degree of completeness of each.

3.2 Derivation and Characterization of the Production Function

Definition 1 *Fix X. We say that input and effort rates ν_t, η_t* ***generate*** *output rate ω_t if the quintuple of positive real functions $[\nu_t, \eta_t, \varphi_t(x), A_t, \omega_t]_{\substack{t\in(0,\infty)\\ x\in[0,X]}}$ satisfies (1), (2), (3) and (4).*

The next theorem identifies the law through which ν_t and η_t determines ω_t. Implicitly, then the theorem identifies the production function. The result restricts attention to the case in which ν_t and η_t are constant and equal to ν and η respectively.

Theorem 1 ***(production function)*** *The pair of constant functions $[\nu_t = \nu, \eta_t = \eta]$ generate $\omega_t \equiv \omega$ if the triple ν, η, ω solves*

$$(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega). \tag{5}$$

Proof. We start by guessing a functional form for $\varphi_t(x)$ and A_t. Let

$$\varphi_t^*(x) = \frac{(\nu - \omega)}{\eta}\omega\ t\ e^{\frac{\nu-\omega}{\eta}x},$$

and

$$A_t^* = (\nu - \omega)\ t.$$

One can verify directly that for any K, λ, the pair $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x},\ A_t = \lambda t$ solves (2) above. Moreover, for any λ the triple $\varphi_t(x) = Kte^{\frac{\lambda}{\eta}x},\ A_t = \lambda t,\ \omega_t$ satisfies (3) if and only if $K = \frac{\lambda}{\eta}\omega_t$, which implies $\omega_t = \omega$. Finally, the triple ν_t, A_t, ω satisfies (4) if and only if $\lambda = \nu_t - \omega$, which implies $\nu_t = \nu$. This shows that, for any ν, ω, the quadruple $[\nu, \varphi_t^*(x), A_t^*, \omega]$ satisfies all the equalities except (1). However, we do not yet know which values of ν and ω are compatible with each other along a growth path. We now show that the pair $\varphi_t^*(x) = Kte^{\frac{\lambda}{\eta}x}, A_t^* = \lambda t$ solves (1) if and only if $X\frac{\nu-\omega}{\eta} = \log(\nu) - \log(\omega)$. Condition (1) reads

$$A_t^* = \int_0^X \varphi_t^*(x)\ dx.$$

Substituting $\varphi_t^*(x)$ and A_t^* yields

$$\begin{aligned}\lambda t &= \int_0^X Kte^{\frac{\lambda}{\eta}x}\ dx \\ &= \frac{\eta}{\lambda}Kt\ \left[e^{\frac{\lambda}{\eta}X} - 1\right].\end{aligned}$$

Now substitute for $K = \frac{\lambda}{\eta}\omega$ and $\lambda = \nu - \omega$ and rearrange to get

$$\frac{\nu}{\omega} = e^{\frac{(\nu-\omega)}{\eta}X}.$$

Taking logs yields

$$\log(\nu) - \log(\omega) = X\frac{(\nu-\omega)}{\eta}.$$

Therefore, Theorem 1 is proved. ■

Equation (5) implicitly yields the production function we are seeking. A convenient feature of the production function is that the output rate is constant (this is actually a subtle result, as we discuss on page 30 in the appendix). We will now be studying the properties of the production function.

Before we start, however, an observation. The functions $\varphi_t^*(x), A_t^*$ identified in Theorem 1 are only well defined if the input rate ν exceeds the output rate ω. Expressed in terms of primitives, this condition is equivalent to $\nu \geq \eta/X$. (This equivalence is proved in Appendix A.1.) The threshold η/X represents the "greedy" input rate, the smallest input rate at which the worker is never idle. So our analysis is restricted to input rates such that the worker is never idle. (See Appendix A.1 for what happens when $\nu < \eta/X$). From now on, we implicitly maintain this "non-idleness" assumption.

Proposition 1 ***(comparative statics on the production function)*** *For each pair $(\nu, \eta/X)$ denote by $\Omega(\nu;\eta/X)$ the unique $\omega < \nu$ that is generated by ν, η through (5). Then we have:*

a) $\Omega(\nu;\eta/X)$ is decreasing in ν.

b) $\Omega(\nu;\eta/X)$ is increasing in η/X.

c) $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta} < 0$, which means that ν and η are strategic substitutes in $\Omega(\nu;\eta/X)$.

d) The function $\Omega(\cdot;\cdot)$ is homogeneous of degree 1.

e) $\Omega(\eta/X;\eta/X) = \eta/X$.

Proof. See the Appendix. ■

Part a) captures the effect of task juggling: increasing the input rate ν reduces output. Therefore setting ν as small as possible, provided that the worker is not idle, produces the maximum feasible output rate. Maximal output is therefore achieved when $\nu = \eta/X$. In that case, part e) shows that the output rate equals η/X. This policy corresponds to the "greedy algorithm," and gives rise to a steady state which is analyzed in Proposition 10.

Part b) simply says that if a worker works more then the output rate is larger.

Part c) deals with the complementarity of inputs in the production of the output rate. It says that the returns to effort decrease when ν increases. Intuitively, this is because A_t is larger and so an increase in effort needs to be spread over a greater number of projects.

The parameter r in part d) can be interpreted as governing the pace at which the system operates. Setting $r > 1$ means that the entire system is working at a faster pace: per unit of time, we have more input, more effort, and more output, all in the same proportion. We build on this interpretation in Section 6.5.

Part e) identifies the "greedy" rate of input. Given η and X, that rate is $\nu = \eta/X$. At this rate, output is $\omega = \eta/X$, the highest achievable output rate (given effort and ability).

We now define two measures of durations.

Definition 2 *For a project assigned at t we define the* ***duration*** *D_t as the time which elapses between t and the completion of the project. For a project opened at t (and thus assigned at a time before t), we define* ***completion time*** *C_t the time which elapses between t and the completion of the project.*

The next result translates results about output into results about durations. The main message is that task juggling increases durations.

Proposition 2 *(a) Fix ω, ν, η. Then $C_t = \frac{(\nu-\omega)}{\omega}t$ and $D_t = \frac{(\alpha-\omega)}{\omega}t$.*

(b) Fix η, and let ω be generated by $[\nu, \eta]$. Then C_t and D_t are increasing in ν.

Proof. See Appendix A.1. ■

4 Strategic Determination of Degree of Task Juggling, and Endogenous Effort

In the previous sections we have assumed that ν_t, the exogenous input rate, is constant through time and, furthermore, that it exceeds the duration-minimizing "greedy" rate η/X. We have not discussed how such a ν_t might come about. In this section we "micro-found" ν_t by introducing a game in which the input rate is determined endogenously as an equilibrium phenomenon. In this game ν_t will in fact turn out to be constant through time, and to exceed η/X. Therefore, this section microfounds the time-use behavior which was assumed to be exogenous in the previous section.

The basic setup is that each project is "owned" by a different co-worker, supervisor, or *client* who in each instant can lobby the worker to devote a fraction of effort to his project, regardless of its order of assignment. The private benefit of lobbying is that the client avoids its project waiting unopened and gets the worker working on it immediately. The social cost of lobbying is that the worker distributes her effort among more projects. This will increase the number of active projects, which slows down all projects. This externality, which is not internalized by the lobbyists, gives rise to an inefficiency.

Clients are not allowed to use money to lobby; rather, the cost of lobbying per unit of time is assumed to be fixed exogenously. We interpret this fixed cost as a sort of cost of supervision, the cost of stopping by and asking "how are we doing on my project?" or of exerting other kinds of pressures. We believe this formulation best captures the process that goes on within organizations, where monetary bribes are not allowed. Also, this type of lobbying process might take place after several principals have signed separate contracts with an agent, for example after several homeowners have contracted for the services of a single building contractor and now each is pushing and cajoling the contractor to finish her home first.

The model is as follows. The worker's effort η is constant through time and fixed exogenously (we will relax the second assumption later). Lobbying is modeled as a technology whereby, at any instant t, a client can pay $\kappa \cdot \Delta$ and force activity on his project during the interval $(t, t+\Delta)$. Activity on the project means that the project moves forward by $(\eta/A_t) \cdot \Delta$. The rate κ is interpreted as the per-unit of time cost of lobbying. If κ is not paid then the project sits idle at some x until either lobbying is restarted or the never-lobbied projects of its vintage (those assigned at the same time) *catch up* to x, at which time the project becomes active again and stays active without any need of, or benefit from, further lobbying. In every instant, $\underline{\nu}$ "never lobbied" projects are opened, in the order they were assigned. Once a never-lobbied project is opened, it forever remains active whether or not it is lobbied. The rate $\underline{\nu}$ represents the input rate that would prevail in the absence of any lobbying by the clients.[11] Here A_t denotes the mass of all projects active in instant t and it is composed of the two type of projects: all those that are lobbied in that instant, and some that are not.[12]

[11] One could be concerned that in equilibrium there might not be enough never-lobbied projects to open, and that therefore it would be more precise to state that in every instant the worker opens the minimum of $\underline{\nu}$ never-lobbied cases and the balance of the never lobbied projects. However, we will see that in equilibrium the balance of never-lobbied projects never falls below $\underline{\nu}$.

[12] Under these rules, for a case that has been lobbied in the past, two scenarios are possible in instant t. First, the case may have been "caught up" by the never-lobbied cases of its own assignement vintage; in other words, the case was lobbied in the past, but then the lobbying lapsed and the case is now at the same stage of advancement (same x) as its never-lobbied assignment vintage. Such a case is worked on without the need for further lobbying and proceeds at speed η/A_t. The second scenario is that the case has not been caught up at time t. In this scenario the case is worked on in the interval Δ and makes $\eta\Delta/A_t$ progress if $\kappa\Delta$ is spent; otherwise, the case does not proceed.

We assume that clients minimize B times the duration of their project, from assignment to completion, plus κ times the time spent lobbying. B represents the rate of loss experienced by a client whose project is not completed. We assume no discounting for simplicity.

Since our goal is to explain why lobbying makes the input rate ν inefficiently large, let's tie our hands by stipulating that the input rate of never-lobbied projects $\underline{\nu}$ is "low," that is, it belongs to the interval $\left[0, \frac{\eta}{X}\right]$. This choice of baseline ensures that any slowdown in the output rate cannot be attributed to an excessively large $\underline{\nu}$.

Projects are indexed by the time τ they are assigned and by an index a that runs across the set of the α projects assigned at time τ. We now introduce the notion of lobbying strategy and lobbying equilibrium.

Definition 3 *A* ***lobbying strategy*** *for project* (a, τ) *is a measurable indicator function* $S_{a\tau}(t)$ *defined on the interval* $[\tau, \infty)$ *which takes value 1 if project* a *is lobbied in instant* t*, and is zero otherwise. A* ***lobbying equilibrium*** *is a set of strategies such that, for each project* (a, τ)*, the strategy* $S_{a\tau}(t)$ *minimizes* κ *times the time spent lobbying plus* B *times the project's duration.*

Equilibrium strategies could potentially be quite unwieldy, featuring complex patterns of activity interspersed with periods of no lobbying. Lemma 3 in Appendix A.2 characterizes equilibrium strategies, achieving considerable simplification. Based on that result, we conjecture (and show existence below) of simple equilibria in which a time-invariant fraction z of the α newly assigned projects is never lobbied, and the remaining fraction $(1 - z)\alpha$ is lobbied immediately upon assignment and then continuously until they are done. We will call these equilibria **constant-growth lobbying equilibria**. Note that the definition of constant-growth lobbying equilibrium does not restrict the strategy space.

If players adopt the strategies of a constant-growth lobbying equilibrium, the input rate $\nu(z)$ is determined by z via the identity

$$\nu(z) = \underline{\nu} + (1 - z)\alpha.$$

The percentage of lobbyists $(1 - z^*)$, and hence the input rate $\nu(z^*)$, are determined in equilibrium.

The equilibrium construction is delicate. In every instant each client has a choice to lobby or not, and so in equilibrium each client has to opt to follow the equilibrium prescription. Moreover, every newly assigned client must be indifferent between lobbying and not. The cost of lobbying is proportional to the time the project is expected to require lobbying, which is the time that active projects take to get done. The drawback of not lobbying is the additional delay incurred from not "skipping the line."

Proposition 3 *Suppose* $\alpha > \frac{\eta}{X}$*. Then, for any* $\underline{\nu}$ *and any cost of lobbying* κ*,*

a) a constant-growth lobbying equilibrium exists;

b) in any constant-growth lobbying equilibrium $\nu(z^*) > \frac{\eta}{X}$*, i.e., the input rate exceeds the duration-minimizing one;*

c) the constant-growth lobbying equilibrium is unique;

d) the fraction $(1 - z^*)$ *of projects that are lobbied in equilibrium is increasing in* $\frac{\alpha}{\underline{\nu}}$ *and* $\frac{\eta}{X}$*, and decreasing in* $\frac{\kappa}{B}$*;*

e) the equilibrium input rate $\nu(z^*)$ *is decreasing in* $\frac{\kappa}{B}$ *and increasing in* $\frac{\alpha}{\underline{\nu}}$ *and* $\frac{\eta}{X}$*.*

Proof. See the Appendix. ■

Part a) can be viewed as providing a microfoundation for the behavioral assumption of constant ν_t which was maintained through Section 3. What was previously a behavioral assumption about the worker is now the outcome of lobbying equilibrium in which, in principle, ν_t need not be constant.

Part b) of the proposition says that, no matter how large the cost of lobbying, input rates will always exceed the "greedy" rate, and so we will have task juggling. The intuition is clear: if input rates were efficient, say $\nu \leq \eta/X$, then completion time would be zero. This means that the cost of lobbying would be zero and, also, that a project which is lobbied would be completed instantaneously. Therefore lobbying is a dominant strategy, which would give rise to an input rate $\nu = \alpha > \eta/X$. Thus an equilibrium input rate ν cannot be smaller than η/X.

Part e) of the proposition says that if a worker is less susceptible to lobbying, which we can model as κ being larger, then the worker will have a smaller input rate and a larger output rate. Moreover, there is more lobbying when the assignment rate is larger, which is intuitive because then the time spent waiting for one's project to be opened becomes larger. Finally, harder working workers and easier projects will give rise to more lobbying. Intuitively, this is because then the completion time gets shorter relative to the duration of a non-lobbied project.

A few words of comment on the causes of inefficiency. The source of a slowdown in output is that, if an additional project is lobbied, that project is able to obtain a small fraction of effort, taking it away from other active projects. In this respect, our model is analogous to models of common resource extraction ("common pool" models) where utilizers cannot be excluded from the pool. We think this is a natural modeling assumption in many cases.

Finally we turn to the case in which η is chosen by the worker, rather than being exogenously given. Suppose η is determined as the solution to the problem

$$\max_{\eta} \Omega \left(\nu \left(z^{*} \right) ; \frac{\eta}{X} \right) - c \left(\eta \right) . \tag{6}$$

According to this formulation, the worker chooses η by trading off the output rate (increasing in η) against a cost of effort $c(\eta)$. Note that since z^* is taken as given in problem (6), the worker does *not* behave as a Stackelberg leader. This assumption reflects the idea that the worker cannot commit to maintain a given level of effort regardless of lobbying.

Definition 4 *A* ***lobbying equilibrium with endogenous effort*** *is a lobbying equilibrium in which effort η^* solves (6).*

To ensure that the equilibrium effort level is greater than zero and smaller than αX we assume $c'(0) = 0$, and $c'(\alpha X) = \infty$.

Proposition 4 *Consider a lobbying equilibrium with endogenous effort. If κ increases, then the input rate decreases and the worker's effort increases.*

Proof. See the Appendix. ■

This proposition highlights another dimension of inefficiency associated with lobbying. Not only does lobbying slow down projects, but it also induces the worker to slack off. The intuition behind this result lies in the "strategic substitutes" property stated in Proposition 1 c).

5 Incentives and Multitasking

Proper scheduling may require mental effort.[13] If personal/mental costs are what prevents "proper" scheduling, then the solution would seem to be to give the worker sufficiently sharp incentives based on productivity. Once the worker's objective is tightly aligned with the principal's, then the worker will overcome her mental costs and schedule properly. The argument is sound, we believe, but it ignores a key problem: if the worker is steeply rewarded for output, then she may have a tendency to focus on easy/quick projects at the expense

[13] One source of effort is that workers face projects which are heterogeneous: some are easier/quicker, others are harder and require more effort. Judicial proceedings, for example, differ widely in the judicial effort required. In this case, most any type of deliberate scheduling (greedy, for one) will requires treating different cases differently. And so proper scheduling requires triaging these cases, which is mentally costly for the worker.

of complex ones. This may be problematic for the principal. For example, in Italy there is a concern that an excessive focus on productivity may lead some judges to not work on complex trials.

In this section we study the problems which arise from steeply tying the worker's incentives to productivity. We do away with scheduling inefficiencies, because we want to take at face value the idea that they are solved by incentives. The challenge, instead, is that the worker can strategically direct her effort to projects of different complexity, but the incentive scheme cannot condition on the complexity of the projects. Instead, the incentive scheme can only condition on "aggregate productivity." This is a "multitasking" setup (in the sense of Holmstrom and Milgrom 1991) which we believe arises frequently, for example when the worker is an "expert" vis a vis a less-expert principal, or when the complexity of the projects is not verifiable to a third party and so incentive contracts based on such information cannot be enforced.

We compare two simple incentive schemes: one based on the aggregate output rate; the other based on the average duration of assigned projects. We find that compensating the worker based on the aggregate output rate leads to severe multitasking problems (in the sense of Holmstrom and Milgrom 1991) where the worker totally focuses his effort on those projects requiring the fewest steps and totally neglects to work on the other projects. In contrast, penalizing large average durations leads to a more "Rawlsian" behavior whereby the worker focuses relatively more effort on projects requiring more steps.

The model is as follows. In each instant, the worker is assigned α_1 projects that will take X_1 tasks to complete, α_2 projects that will take X_2 tasks to complete, ... , up to α_N projects that will take X_N tasks to complete. Without loss of generality we set $X_i < X_{i+1}$. In this setup the vector $\{\alpha_i, X_i\}_i$ fully describes how many projects are assigned of which length. The worker chooses the rate ν_i at which to open projects of type i, and the effort η_i to devote to each type of project.

A worker who is rewarded based on aggregate output maximizes

$$\int_0^\infty e^{-\rho t} \left[w \sum_{i=1}^{N} \omega_i - c\left(\sum_{i=1}^{N} \eta_i \right) \right] \, dt, \tag{7}$$

where ρ represents the worker's discount factor, w is a parameter that captures the magnitude of the incentives, ω_i is the output rate of projects of type $i, c(\cdot)$ is a convex cost of effort, and $\sum_{j=1}^{N} \eta_j$ represents the total effort exerted by the worker. A worker who is penalized linearly based on the average duration of his projects maximizes

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^{N}\alpha_i D_{it} - c\left(\sum_{i=1}^{N}\eta_i\right)\right]\ dt, \tag{8}$$

where D_{it} represents the duration of a project of type i assigned at t.

Proposition 5 *(a) Suppose the worker is rewarded based on aggregate output rates. There there exists a threshold $\overline{X}$ such that the worker will immediately open all assigned projects requiring fewer than $\overline{X}$ steps to complete and immediately complete them. All projects requiring more than $\overline{X}$ steps are never worked on.*

(b) A worker who is penalized linearly based on the average duration of her projects will devote positive effort to all projects, even those that take many steps to complete. Moreover, $(\alpha_j)^2 X_j > (\alpha_i)^2 X_i$ implies that either $\eta_j^* > \eta_i^*$ or else $\eta_j^* = \alpha_j X_j$. In other words, ceteris paribus the worker will work more on projects requiring more steps to complete, and on those assigned at a higher rate.

Proof. See the Appendix. ■

Part (a) says that if incentives are tied to the output rate, then the worker will only work on the easiest (quickest) projects and will completely ignore the complex ones. Part (b) is understood more easily by setting $\alpha_j = \alpha_i$; then the statement says that the worker works *more* on the more complex (time-consuming) project j, unless she is already devoting to that project all the effort it can absorb (that is, unless η_j^*/X_j already equals α_j). In other words, more complex projects are prioritized. Now, allow α_j to differ from α_i; then ceteris paribus the proposition says that projects which there are more of are allocated more total effort.

6 Extensions and Discussion

In this section we consider several extensions of the main model.

6.1 Costs and Benefits of Interruptions: Forgetful Worker

In this section we deal with the case in which, as completion time grows and any open project is worked on less and less frequently per unit of time, the worker progressively forgets about the details of each individual project. Thus, every time the worker picks up a project again, she needs to spend some additional effort to "remind herself" of where she left off before she can make progress.

We model this phenomenon by assuming that in the time interval between t and $t+\Delta$, the worker's effort shaves off approximately $\frac{\eta}{A_t+F_t}\Delta$ steps from each active project. The factor $F_t > 0$ captures a "forgetfulness penalty." We assume that F_t becomes larger over time; its exact form of will be specified later. The presence of forgetfulness requires amending equations (2) and (3) from Section 3. The two amended equations read

$$\frac{\partial \varphi_t(x)}{\partial t} - \frac{\partial \varphi_t(x)}{\partial x} \frac{\eta}{A_t + F_t} = 0, \tag{9}$$

and

$$\omega_t = \frac{\eta}{A_t + F_t} \varphi_t(0). \tag{10}$$

Equations (1) and (4) remain unchanged.

Definition 5 *Fix the function F_t. We say that the pair ν, η generate ω if the quintuple $[\nu, \eta, \varphi_t(x), A_t, \omega]$ satisfies conditions (1), (4), (9) and (10).*

Now let us specify F_t. We want to capture the notion of "time elapsed between the accomplishment of two consecutive steps," even though in our model steps are continuous and so strictly speaking there are no two consecutive steps. In our model, instead, we can think about the time that elapses between the accomplishment of given percentiles of completion, say between 20% and 30% of completion. A large completion time C_t corresponds to bigger stretches of time elapsing between the achievement of any two percentiles of completion, so we assume that the "forgetfulness penalty" F_t is proportional to the completion time C_t according to a factor of proportionality f. Formally, we assume

$$\begin{aligned} F_t &= f \cdot C_t \\ &= f \, \frac{\nu - \omega}{\omega} \, t. \end{aligned}$$

where the second equality follows from Proposition 2, and the third is simply a definition of the real number F. Note that, by making F_t proportional to C_t, we have made F_t endogenous.

Proposition 6 *Suppose the worker is forgetful. Then:*

(a) *The pair ν, η generates ω if the following equation is satisfied:*

$$\log(\nu) - \log(\omega) = \frac{X}{\eta} \left(\nu - \omega + f \, \frac{\nu - \omega}{\omega} \right). \tag{11}$$

(b) If $\nu > \frac{\eta}{X}$ and the worker is forgetful then the output rate is smaller than in the one described in Section 3.2.

Proof. See the Appendix. ■

6.2 Triage, Optimal Multitasking, and Variable-Speed Work Strategies

In most of the paper we assume that the objective function is to maximize the output rate or, equivalently, minimize durations. Under this assumption the optimal input rate is the lowest possible compatible with no worker idleness, namely the "greedy" rate η/X. In this section we explore a different hypothesis: that some private or social value may be generated when projects clear *intermediate goals*. So we consider the possibility that value may accrue when a project is merely being opened, or when it gets half-done, etc. One example is a judge who may issue preliminary injunctions early on in the trial, which might increase social welfare. Another example is triage in medical care, where examining the patient immediately produces social value.

When weight is placed on clearing intermediate goals, the optimal input rate need no longer equal η/X. To see this, consider an extreme case in which value is generated *only* when a project is opened (e.g., completion does not matter); in this case, the optimal input rate is clearly the largest possible, $\nu = \alpha$. This observation suggests that if we care about clearing intermediate goals, high input rates help (and so the "greedy" rate is no longer optimal). Proposition 11 in Appendix A.5 identifies conditions under which this can actually be stated as a formal result: the more we care about clearing early intermediate goals, the higher the optimal input rate ν. Thus Proposition 11 lays the foundations for a theory optimally chosen input rates that might account for rates exceeding η/X.

When the objective function places value on clearing intermediate goals, moreover, it may be expedient to a adopt a strategy that is more sophisticated than the one where all open projects receive the same fraction of the worker's attention and all proceed at the same speed. To see why, consider a situation in which all projects are identical, and half the value is realized when projects are opened, the other half when they are completed. If it is possible to move projects along at different rates, then the optimal strategy is the following: all assigned projects are opened immediately, thus setting the input rate equal to α; but, as soon as projects clear X, they are placed in a queue from which projects are retrieved at rate η/X. As soon as projects are retrieved, they are worked on continuously until completion. This "stop and go" strategy achieves an average value of $\alpha \cdot (1/2) + \frac{\eta}{X} \cdot (1/2)$. Since $\alpha \geq \nu$ and $\frac{\eta}{X} \geq \Omega(\nu; \eta/X)$ (with at least one inequality holding strictly), this average value is larger than $\nu \cdot (1/2) + \Omega(\nu; \eta/X) \cdot (1/2)$, which is the average rate that can be achieved with an input rate of ν and equal treatment of projects.

In Appendix A.5 we extend our analysis to *variable speed strategies* which include as special cases the "stop and go" strategy mentioned above. These strategies involve allocating different amounts of effort to different projects, depending on the project's closeness to being done. Since these variable-speed strategies are more flexible than "regular" strategies, they are obviously going to do better if we care about intermediate goals. A number of formal results are provided in Appendix A.5 relating "regular" strategies with variable speed strategies.

6.3 Heterogeneous, Equally Treated Cases

Let us now consider the situation in which projects are heterogeneous in the number of tasks they take to complete. We still assume "equal treatment," however, in the sense that once opened, all projects proceed at the same speed according to equation (2). Such equal treatment will arise if, for example, the worker cannot distinguish which projects take fewer tasks to complete, as may be the case for legal cases that settle *unexpectedly* during trial. Another reason why equal treatment may prevail is that disparate treatment of projects may not be legal.

In this section we provide exact formulas that characterize how the input/output ratio varies across projects with different X's. To build some intuition for the results that follow, consider two projects which are opened at the same time: project 1 taking X_1 tasks to complete, and project 2 taking X_2. If $X_1 < X_2$, then we should expect the output rate of projects of type 1 to be larger, *relative to their input rate*, compared to projects of type 2. To get some intuition, consider the polar case in which type-1 projects take so few tasks to complete that $X_1 \approx 0$. Then $\omega_1 \approx \nu_1$, and thus the ratio of input to output rates approaches its theoretical maximum.

The model is as follows. Fix the worker's effort level at η. In each instant, the worker opens ν_1 projects that will take X_1 tasks to complete, ν_2 projects that will take X_2 tasks to complete, ... , up to ν_N projects that will take X_N tasks to complete. We allow for the possibility that $N = \infty$, in which case the set of different types of projects is countable.[14] In this setup the vector $\{\nu_i, X_i\}_i$ fully describes how many projects are opened of which length. For every open project, in the time interval between t and $t + \Delta$, the worker's work shaves

[14] The analysis in this section generalizes immediately to the case in which the set of possible types of cases has the power of the continuum.

off approximately

$$\frac{\eta}{A_t}\Delta,$$

where A_t represents the mass of all projects open at time t.

The next proposition provides an exact characterization of the output rates as a function of the input rates and of the characteristics of the project.

Proposition 7 *Fix η and a constellation of $\{\nu_i, X_i\}_i$. If $\eta < \sum_{i=1}^{N} \nu_i X_i$ then completion time is positive for all projects, and there exists a constant $K < 1$ such that $\omega_i = \nu_i \cdot K^{X_i}$ for all i.*

If $\eta > \sum_{i=1}^{N} \nu_i X_i$ completion time is zero for all projects and $\frac{\nu_i}{\omega_i} = 1$ for all i.

This proposition informs us about the *relative* magnitudes of the input/output ratios $\frac{\nu_i}{\omega_i}$ and $\frac{\nu_j}{\omega_j}$. According to the proposition, $X_i < X_j$ implies $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$, that is, projects that take more tasks to complete have a worse input/output rate ratio. Note that the constant K in the proposition is an unspecified function of effort and of the entire vector $\{\nu_i, X_i\}$. Therefore Proposition 7 should not be construed as informing us about the *level* of any particular ω_i.

6.4 Time To Build

In certain contexts, there may be technological limitations on how fast a project can be completed. For example, a judge may need to allow the lawyers time, between two successive hearings, to produce certain evidence and to evaluate and respond to the evidence produced by the adversary. For an academic researcher, before being able to produce a final draft, time may be required to absorb one's intermediate findings and "put the puzzle together". And so on. The common thread in all these these examples is that one cannot complete a project under a certain time threshold, no matter how large the effort. Let us call this threshold $\underline{T}$. We introduce the "time to build" constraint in our model via the constraint

$$\underline{C}_t = \max\{C_t, \underline{T}\},$$

where $\underline{C}_t$ denotes the completion time for a case started at t in the model with a time-to-build constraint, and $C_t = t(\nu - \omega)/\omega$ is the completion time in a model without that constraint. One way to think about this constraint is to imagine that the system evolves exactly like in the case without constraint, except that cases that were completed in less than $\underline{T}$ are "held back" and not "released" until $\underline{T}$ has elapsed from the time they were opened. Note that, if $\nu > \omega$ then C_t grows linearly with time and so after a certain time we have $\underline{C}_t = C_t$. Thus in our formulation the time-to-build constraint stops binding after a certain time because, as shown in Section 3.2, eventually projects take long enough to complete. In particular, the completion times for all projects started after a $\widehat{t}$ such that $C_{\widehat{t}} = \underline{T}$ are unaffected by the time-to-build constraint. One can easily verify that

$$\widehat{t} = \frac{\omega}{(\nu - \omega)}\underline{T}.$$

The output rate will, after a certain time $\widehat{\widehat{t}}$, coincide with the one in Section 3.2, which we denoted by ω. The value of $\widehat{\widehat{t}}$ is given by

$$\widehat{\widehat{t}} = \widehat{t} + C_{\widehat{t}} = \frac{\nu}{(\nu - \omega)}\underline{T}.$$

For $t < \underline{T}$, the output rate is zero which is the time-to-build effect. For $t \in \left[\widehat{t}, \widehat{\widehat{t}}\right]$, the output rate will be higher than ω and decreasing.

6.5 Non-Stationary Input and Effort Rates

In this section we relax the assumption of time-invariant input and effort rates, $\nu_t = \nu$ and $\eta_t = \eta$. This is to accommodate data patterns like the one seen in Figure 1, where both series **New Opened Cases** (corresponding to ν_t) and **Standardized Effort** (corresponding to η_t/X) trend up, and moreover they are characterized by the same sharp seasonality (judges do not work much in August).

We analyze the special case in which input and effort rates, while not stationary, evolve at the same pace. In this case there is an easy argument that makes all our previous theory applicable almost directly. The idea is that, if input and effort rates evolve at the same pace, then we can construct an artificial time scale under which these two are time-invariant. We may call this time scale "worker time," as opposed to the conventional "calendar time." So, for example, when there is a (calendar) time interval in which input and output rate are higher than average, we allow worker time to "speed up" in this interval such that the input and effort rate *per unit of worker time* are no higher than average. Since under this artificial time scale the input and effort rates are constant, then the previous theory ensures that the output rate is constant *in worker time*. Finally, we recover the (non-stationary) output rate *in calendar time* by inverting the transformation operated by worker time. Using this technique we obtain the following proposition.

Proposition 8 *Suppose ν_t, η_t are not stationary but they vary at the same rate, that is,*

η_t/ν_t is stationary and equal to η_0/ν_0. Then the (non-stationary) output rate ω_t equals $\omega_0 \cdot (\nu_t/\nu_0)$, where $\omega_0 = \Omega(\nu_0, \eta_0/X)$.

Proof. See the Appendix. ■

7 Conclusion

Task juggling is prevalent in the workplace. We have developed a theory of a worker who deals with overload by choosing how many projects to work on simultaneously. Working on too many projects at the same time reduces the worker's output, for given effort and ability. We have investigated an "interdependent workplace" environment which will lead the worker to behave in this inefficient way. Moreover, we have shown that task juggling and effort are strategic substitutes, suggesting that when effort is not contractible, whatever worsens task juggling will also indirectly decrease effort.

Giving the worker powerful incentives based on productivity may well decrease task juggling, but it raises the spectre of "multitasking" problems in the sense of Holmstrom and Milgrom (1991). We have studied two simple incentive schemes from this perspective. In addition, we have studied several extensions of the basic framework, including one in which task juggling can in fact be efficient if there is social value being created by clearing intermediate goals.

A noteworthy feature of our model is that, unlike queuing theory, we look at an environment in which the worker is never idle We do this for the purpose of realism: judges in Italy are never in a situation of no backlog, and the same is true of many other workers.[15] An implication of this assumption is that in our model the worker's backlog and duration grow without bound as time goes on. We do not think of this trend as realistic in the long run, as long as the workforce can costlessly be expanded. But we interpret the model as a description of short and medium run congestion effects, such as those portrayed in Figure 1.

We view the single-worker model presented here as a building block for future research of two types. First, empirical work, which might take advantage of increasingly available workplace micro-data to quantitatively evaluate the inefficiencies caused by task juggling, and to perform counterfactual calculations. In our companion paper (Coviello et al 2010) we use this framework and a sample of Italian judges to guide an empirical analysis that estimates the causal effect of an exogenously induced increase in parallel working. We find that judges do juggle tasks, and that the slowdown in output resulting from task juggling is large.

[15] Public health care systems in many countries, for example, also have permanent queues for various treatments.

Second, we foresee the possibility of theoretical work extending this analysis to a multi-worker hierarchical worplace.

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Appendices: Not For Publication

A Proofs and Technical Results

A.1 Characterization of the Production Process

The function $\varphi_t^*(x)$ is exponential in x and multiplicative in t, as depicted in Figure 3. As $t \to 0$ the function $\varphi_t^* : [0, X] \to \mathbb{R}$ converges to zero uniformly. As t grows, the function φ_t^* grows multiplicatively in t. Growth in t reflects a progressive increase in the number of active cases, that is, growing task juggling over time.

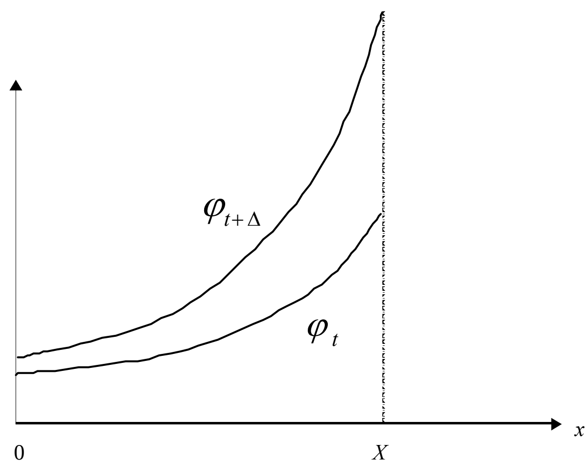

Figure 3: Distribution of active cases, by number of steps away from being done. On the growth path it is exponential.

Growth in task juggling also explains why the function $\varphi_t(x)$ is exponential in x. This is because, when the worker juggles an increasing number of projects over time, projects proceed at a progressively slower pace (that pace is η/A_t, and remember that A_t grows linearly with t). As projects grind along more and more slowly, the constant rate ν of newly inputed cases must squeeze in the progressively smaller "empty segment" available near X. This effect accounts for the exponential shape of $\varphi_t(x)$. Yet, remarkably, despite these complex dynamics the output rate is constant through time. This remarkable property of the output rate results from two opposite effects offsetting each other: on the one hand, cases move through at progressively slower rates, which tends to progressively reduce the output rate.

On the other hand, the mass of cases that are almost done increases with time (this is because $\varphi_t(0)$ grows with t), which tends to progressively increase the output rate. These two effects exactly offset each other along a constant growth path, and thus the output rate is time-invariant.

Theorem 1 goes a long way towards characterizing a constant growth path, but there is still some work to do. We need to characterize the relationship that links ν, η and ω along a growth path or, said differently, we need to understand what level of output is possible given certain input and effort rates. According to Theorem 1, the relationship between ν, η and ω along a growth path is given by equation (5). Define

$$h(y) = \frac{X}{\eta} y - \log(y).$$

Then equation (5) reads

$$h(\nu) = h(\omega).$$

The next lemma characterize the function $h(\cdot)$.

Lemma 1 *The function $h(y)$ is strictly convex on $(0, \infty)$, converges to infinity at $y = 0$ and $y = +\infty$, and it has its minimum at $y = \eta/X$.*

Proof. One can easily verify that $h(0) = +\infty = h(\infty), h'(y) = \frac{X}{\eta} - \frac{1}{y}$, and finally $h''(y) = \frac{1}{y^2}$. ∎

Figure 4 depicts $h(y)$. For a particular level of ν, the ω that solves equation (5) is represented graphically as the point on the horizontal axis that achieves the same level of the function h. But not all solutions to equation (5) can be part of a growth path. Which solutions are consistent with a growth path is described in the next proposition.

Proposition 9 *ν, η and ω are related by (5) if and only if $\nu > \frac{\eta}{X}$. In that case, the ω generated by the pair $[\nu, \eta]$ is the unique solution that is smaller than $\frac{\eta}{X}$ to the equation $h(\nu) = h(\omega)$.*

Proof. The solution $\omega = \nu$ to equation (5) is not acceptable because then $A_t^* \equiv 0$ and (3) is not well-defined. Nor can we accept solutions where $\omega > \nu$, for then $\varphi_t^*(x)$ and A_t^* would be negative and thus the quadruple identified in Theorem 1 would not meet the definition of a growth path. So we need to find solutions with $\omega < \nu$. This implies $\nu > \frac{\eta}{X}$. The rest of the Proposition follows immediately from Theorem 1. ∎

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle; we will discuss this interpretation at the end of this section. Proposition 9

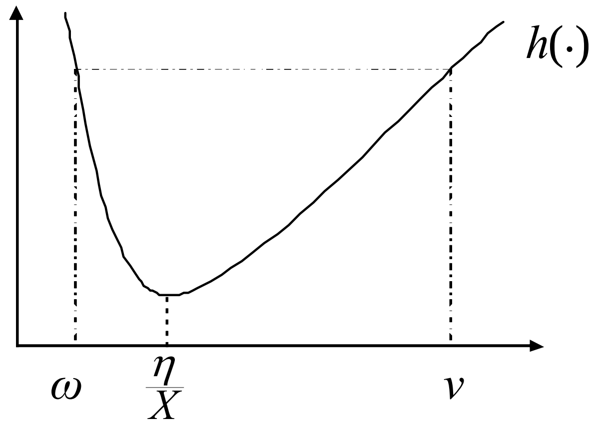

Figure 4: Relationship between input and output rates on a growth path.

shows how to construct the entire growth path associated with any pair (ν, η). Given a constant input rate $\nu > \frac{\eta}{X}$, one can uniquely identify the corresponding output rate $\omega < \frac{\eta}{X}$ which solves $h(\nu) = h(\omega)$. Then the triple (ν, η, ω) is plugged into the expressions for $\varphi_t^*(x)$ and A_t^* to obtain a full characterization of the growth path.

Proposition 9 shows that our solution only makes sense if the input rate is sufficiently large. What happens otherwise? Then the worker can solve projects faster than she opens them, and in that case our model predicts $A_t \equiv 0$. In this case we do not have a model of task juggling, but rather one of "undercommitment." We conclude this section by analyzing this case. In the analysis we allow for an "initial condition" $A_0 \geq 0$, a possibly positive mass of cases active at time zero. (This hypothesis deviates from our assumption that at $t = 0$ the mass of active cases is zero.) The next proposition shows that if $\nu < \frac{\eta}{X}$ then A_t shrinks over time, and if $\nu = \frac{\eta}{X}$ then A_t is constant.

Proposition 10 ***(steady-state and shrink paths)*** *If $\nu = \frac{\eta}{X}$ then there are a continuum of steady-state paths, indexed by the mass of projects active at time zero, A_0. In each of these steady states $A_t \equiv A_0$, the output rate is equal to η/X, and the duration of projects is increasing in A_0.*

If $\nu < \frac{\eta}{X}$ then whatever the value of A_0, after a transition period it will be $A_t \equiv 0$ and, from then on, the duration of projects will be zero and the output rate will be equal to ν.

Proof. Let's start with the case $\nu < \frac{\eta}{X}$. In this case the setup of the model described in Section 3 is no longer applicable, since that setup implicitly required that $A_t > 0$, which now cannot be guaranteed. In fact, if we start at time 0 with $A_0 > 0$ and open projects at rate

$\nu < \frac{\eta}{X}$, we expect $\omega_t > \nu$, and so we are on a temporary "shrink path" where over time A_t will shrink down to zero. After A_t hits zero, the worker completes projects instantaneously as soon as they are opened, and the system settles into a long-run path with $\omega_t = \nu < \frac{\eta}{X}$, and $A_t = C_t = D_t = 0$. In this long-run steady state, increasing ν increases ω contrary to Proposition 1.

In the case $\nu = \frac{\eta}{X}$, let us conjecture $\nu = \omega$ and so by (4) we have $A_t = A$. Fix any $A_0 > 0$. Note that this requires assuming an initial load of projects at time zero. Then (3) reads

$$\omega = \frac{\eta}{A_0}\varphi_t(0),$$

whence for all $t > 0$

$$\varphi_t(0) = \frac{A_0}{\eta}\omega. \tag{12}$$

Now, by definition we have that for all $x > 0$ we have $\varphi_t(0) = \varphi_\tau(x)$ for some $\tau < t$. This observation, together with (12), implies

$$\varphi_\tau(x) = \frac{A_0}{\eta}\omega \text{ for all } x, \tau.$$

Then (1) reads

$$A_0 = \int_0^X \varphi_t(x)\, dx = \frac{X}{\eta}A_0\omega.$$

Note that this equality reduces to the identity $\omega = \eta/X$, which yields no new information. This means that any A_0 is compatible with the steady state path when $\nu = \eta/X$. Whatever is the initial condition of open projects A_0, choosing $\nu = \eta/X$ will exactly perpetuate that mass of open projects.

The completion time of a newly opened project is the interval of time it takes the worker to process the A_0 projects that have precedence over it. We are looking for the time interval C_t it takes for a worker to complete A_0 projects. At a completion rate ω, C_t solves

$$\begin{aligned} A_0 &= \int_t^{t+C_t} \omega \; ds \\ &= \omega C_t = \frac{\eta}{X}C_t, \end{aligned}$$

whence the completion time of a newly activated project is $C_t = \frac{A_0}{\eta}X$, which is increasing in A_0. Given an arrival rate α, a project assigned at t finds

$$A_0 + \alpha t - \omega t$$

projects in front of it. The duration of a project assigned at t is the time it takes to complete these projects given an output rate ω. Thus the duration of a project assigned at t is also increasing in A_0. ■

The threshold η/X can be interpreted as the minimum input rate compatible with the worker not being idle given that the worker exerts effort at rate η. To understand this interpretation, fix effort η and observe that if $\nu' < \eta/X$ then there exists a smaller effort rate η' such that $\eta'/X = \nu' \geq \omega'$ (the inequality is true because cannot be more cases being completed than there are coming in). This means that if the input rate ν' falls below η/X then the worker could achieve the same level of output ω' by exerting effort at the lower rate η'. This is equivalent to saying that the worker is idle at rate $\eta - \eta'$.

Proof of Proposition 1 a), b).

Proof. a) There are three types of solutions to the equation $h(\nu) = h(\omega)$. The first one is $\nu = \omega$. This solution is not compatible with the analysis we have carried out because then $A_t = 0$. Then there are two kinds of solutions, one where $\nu < \frac{\eta}{X} < \omega$, which is not acceptable for then $A_t < 0$. The remaining kind of solution is $\nu > \frac{\eta}{X} > \omega$. Under this restriction, the shape of $h(\cdot)$ guarantees the required property.

b) Fix ν, and consider two values $\eta > \eta'$ with associated ω and ω'. The output rates ω and ω' solve

$$h(\omega; \eta/X) = h(\nu; \eta/X)$$
$$h(\omega'; \eta'/X) = h(\nu; \eta'/X).$$

Combining these equalities yields

$$h(\omega'; \eta'/X) - h(\omega; \eta/X) = h(\nu; \eta'/X) - h(\nu; \eta/X). \tag{13}$$

Now, an easy to verify property of $h(y; \eta/X)$ that, for any $y_1 < y_2$,

$$h(y_1; \eta'/X) - h(y_1; \eta/X) < h(y_2; \eta'/X) - h(y_2; \eta/X).$$

Setting $y_1 = \omega, y_2 = \nu$, and combining with (13) gives

$$h(\omega; \eta'/X) - h(\omega; \eta/X) < h(\omega'; \eta'/X) - h(\omega; \eta/X)$$
$$h(\omega; \eta'/X) < h(\omega'; \eta'/X) \tag{14}$$

Now, remember that $\omega' < \eta'/X$. Then either $\omega > \eta'/X$, in which project $\omega > \omega'$ and there is nothing to prove, or else $\omega < \eta'/X$. In this project both ω and ω' lie on the decreasing portion of the function $h(\cdot; \eta'/X)$. Then equation (14) yields $\omega > \omega'$. ■

Next we prove a technical lemma that is necessary to prove Proposition 1 c).

Lemma 2 *Take any triple* $\left(\nu,\omega,\frac{\eta}{X}\right)$ *where* $\omega=\Omega\left(\nu;\eta/X\right)$. *Then* $\left|\nu-\frac{\eta}{X}\right|>\left|\omega-\frac{\eta}{X}\right|$. *That is, along a growth path the* ***actual output*** *rate is closer to the* ***efficient output*** *rate than is the input rate.*

Proof. For any $\nu>\frac{\eta}{X}>\omega$ we can write

$$h\left(v\right)=h\left(\frac{\eta}{X}\right)+\int_{0}^{\nu-\frac{\eta}{X}}h'\left(\frac{\eta}{X}+s\right)ds \tag{15}$$
$$h\left(\frac{\eta}{X}\right)=h\left(\omega\right)+\int_{0}^{\frac{\eta}{X}-\omega}h'\left(\omega+r\right)dr.$$

Make the change of variable $r=-\omega+\frac{\eta}{X}-s$ in the second equation, and one gets

$$\begin{aligned}h\left(\frac{\eta}{X}\right)&=h\left(\omega\right)-\int_{-\omega+\frac{\eta}{X}}^{0}h'\left(\frac{\eta}{X}-s\right)ds\\&=h\left(\omega\right)+\int_{0}^{-\omega+\frac{\eta}{X}}h'\left(\frac{\eta}{X}-s\right)ds.\end{aligned}$$

Substitute into equation (15) to get

$$h\left(v\right)=h\left(\omega\right)+\int_{0}^{-\omega+\frac{\eta}{X}}h'\left(\frac{\eta}{X}-s\right)ds+\int_{0}^{\nu-\frac{\eta}{X}}h'\left(\frac{\eta}{X}+s\right)ds.$$

Since the triple $\left(\nu,\omega,\frac{\eta}{X}\right)$ solves (5), it follows that $h\left(v\right)=h\left(\omega\right)$ and so we may rewrite equation (15) once more as

$$\int_{0}^{\frac{\eta}{X}-\omega}-h'\left(\frac{\eta}{X}-s\right)ds=\int_{0}^{\nu-\frac{\eta}{X}}h'\left(\frac{\eta}{X}+s\right)ds \tag{16}$$

Now, from the proof of Lemma 1 we have $h'\left(y\right)=\frac{X}{\eta}-\frac{1}{y}$ and so

$$\begin{aligned}h'\left(\frac{\eta}{X}+s\right)&=\frac{X}{\eta}-\frac{1}{\frac{\eta}{X}+s}=\frac{X}{\eta}\left(1-\frac{1}{1+s\frac{X}{\eta}}\right)=\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1+s\frac{X}{\eta}}\right)\\h'\left(\frac{\eta}{X}-s\right)&=\frac{X}{\eta}-\frac{1}{\frac{\eta}{X}-s}=\frac{X}{\eta}\left(1-\frac{1}{1-s\frac{X}{\eta}}\right)=-\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1-s\frac{X}{\eta}}\right)\end{aligned}$$

for any s such that $h'\left(\frac{\eta}{X}-s\right)$ is well defined, that is, $s<\frac{\eta}{X}$. If in addition $s>0$ then

$$h'\left(\frac{\eta}{X}+s\right)=\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1+s\frac{X}{\eta}}\right)<\frac{X}{\eta}\left(\frac{s\frac{X}{\eta}}{1-s\frac{X}{\eta}}\right)=-h'\left(\frac{\eta}{X}-s\right). \tag{17}$$

Now let us turn to equation (16) and let us suppose, by contradiction, that $\nu-\frac{\eta}{X}<\frac{\eta}{X}-\omega$. We may then rewrite that equation as

$$\int_0^{\frac{\eta}{X}-\omega}-h'\left(\frac{\eta}{X}-s\right)ds-\int_0^{\nu-\frac{\eta}{X}}h'\left(\frac{\eta}{X}+s\right)ds=0$$

$$\int_0^{\nu-\frac{\eta}{X}}\left[-h'\left(\frac{\eta}{X}-s\right)-h'\left(\frac{\eta}{X}+s\right)\right]ds+\int_{\nu-\frac{\eta}{X}}^{\frac{\eta}{X}-\omega}-h'\left(\frac{\eta}{X}-s\right)ds=0$$

The range of s in the above equation is at most $\left(0,\frac{\eta}{X}-\omega\right)\subset\left(0,\frac{\eta}{X}\right)$, and therefore (17) applies. This guarantees that the first integral is strictly positive. The second integral is strictly positive as well. Hence the equation cannot be verified. We therefore contradict our assumption that $\nu-\frac{\eta}{X}<\frac{\eta}{X}-\omega$. ■

Proof of Proposition 1 c)

Proof. Equation (5) reads

$$(\nu-\Omega(\nu;\eta/X))=\frac{\eta}{X}\left[\log(\nu)-\log(\Omega(\nu;\eta/X))\right]. \tag{18}$$

Fix ν and differentiate both sides of (18) with respect to η to get

$$-\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}=\frac{1}{X}\left[\log(\nu)-\log(\Omega(\nu;\eta/X))\right]-\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}\frac{\partial(\Omega(\nu;\eta/X))}{\partial\eta}.$$

Rearranging we get

$$\begin{aligned}\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]&=\frac{1}{X}\left[\log(\nu)-\log(\Omega(\nu;\eta/X))\right] &(19)\\ &=\frac{1}{\eta}\left(\nu-\Omega(\nu;\eta/X)\right), &(20)\end{aligned}$$

where the second equation susbtitutes from (18). Now, fix η and differentiate (19) with respect to ν. This yields

$$\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]-\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\frac{\eta}{X}\frac{1}{(\Omega(\nu;\eta/X))^2}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}$$
$$=\frac{1}{X}\left[\frac{1}{\nu}-\frac{1}{\Omega(\nu;\eta/X)}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\right],$$

which can be rewritten as

$$\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]=\frac{1}{X}\left[\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left(\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}\eta\frac{1}{\Omega(\nu;\eta/X)}-1\right)\right]. \tag{21}$$

The term in brackets on the left-hand side is positive, so $\frac{\partial^2\Omega(\nu;\eta/X)}{\partial\eta\partial\nu}$ has the same sign as the term in brackets on the right hand side of (21). We need to sign this term. To this end, substitute for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\eta}$ from (20) so that the term in brackets on the right hand side of (21) reads

$$\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left[\frac{(\nu-\Omega(\nu;\eta/X))}{\left(\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right)}\frac{1}{\Omega(\nu;\eta/X)}-1\right]$$
$$=\frac{1}{\nu}+\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\frac{1}{\Omega(\nu;\eta/X)}\left[\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right]. \tag{22}$$

Now, to get an expression for $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}$, fix η and differentiate both sides of (18) with respect to ν to get

$$1-\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}=\frac{\eta}{X}\left[\frac{1}{\nu}-\frac{1}{\Omega(\nu;\eta/X)}\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\right]$$
$$\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}\left[\frac{\eta}{X}\frac{1}{\Omega(\nu;\eta/X)}-1\right]=\frac{\eta}{X\nu}-1$$
$$\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu}=\frac{\Omega(\nu;\eta/X)}{\nu}\frac{\frac{\eta}{X}-\nu}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}.$$

Substituting into (22) yields

$$\frac{1}{\nu}-\frac{1}{\nu}\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right)^2$$
$$=\frac{1}{\nu}\left[1-\left(\frac{\nu-\frac{\eta}{X}}{\frac{\eta}{X}-\Omega(\nu;\eta/X)}\right)^2\right]. \tag{23}$$

By Lemma 2,

$$\frac{\nu - \frac{\eta}{X}}{\frac{\eta}{X} - \Omega\left(\nu; \eta/X\right)} > 1$$

and so equation (23) is negative. Thus the right hand side of (21) is negative, which implies $\frac{\partial\Omega(\nu;\eta/X)}{\partial\nu\partial\eta} < 0$. ■

Proof of Proposition 1 d),e)

Proof. d) Suppose the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). We need to show that for any scalar $r > 0$, the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ also solves (5). Write

$$\begin{aligned} r\frac{\eta}{X}\left[\log\left(r\nu\right) - \log\left(r\omega\right)\right] &= r\frac{\eta}{X}\left[\log\left(\nu\right) - \log\left(\omega\right)\right] \\ &= r\left(\nu - \omega\right) = \left(r\nu - r\omega\right). \end{aligned}$$

where the second equality follows because the triple $\left(\nu, \omega, \frac{\eta}{X}\right)$ solves (5). The equality between the first and the last element in this chain of equalities shows that the triple $\left(r\nu, r\omega, r\frac{\eta}{X}\right)$ solves (5).

e) Immediate from inspection of Figure 4. ■

Proof of Proposition 2.

Proof. (a) The completion time C_t of a project started at t is the time that it takes all the projects in front of it to clear. These projects are A_t, and given an output rate ω that duration is given by the solution to the following equation

$$\int_t^{t+C_t} \omega \; ds = A_t,$$

which equals

$$\omega C_t = \left(\nu - \omega\right) t.$$

Solving for C_t yields the desired expression. Let us now turn to duration. Given an arrival rate α, a project assigned at t finds

$$\alpha t - \omega t$$

projects in front of it. Given an output rate of ω, these projects will take

$$D_t = \frac{\left(\alpha - \omega\right)}{\omega} t$$

to complete. This is the duration of a project assigned at t.

(b) From part (a) we have

$$C_t = \left(\frac{\nu}{\Omega\left(\nu; \eta/X\right)} - 1 \right) t = \left(\frac{1}{\Omega\left(1; \eta/\nu X\right)} - 1 \right) t,$$

where the second equality follows from Proposition 1 d). From Proposition 1 (b) we have that Ω is increasing in its second argument, whence increasing ν decreases $\Omega\left(1; \eta/\nu X\right)$ and increases C_t.

As for duration, from part (a) we have

$$D_t = \left(\frac{\alpha}{\Omega\left(\nu; \eta/X\right)} - 1 \right) t.$$

From Proposition 1 (a) we have that Ω is decreasing in its first argument, whence increasing ν decreases $\Omega\left(\nu; \eta/X\right)$ and increases D_t. ■

A.2 Proofs for Section 4

The next lemma suggests that we should look for equilibria in which clients play just two simple strategies.

Lemma 3 *In any lobbying equilibrium in which the number of active projects grows, two strategies payoff-dominate all others: strategy* $\mathbf{1}(\cdot)$ *which denotes immediate and perpetual lobbying starting from time of assignment, and strategy* $\mathbf{0}(\cdot)$ *which denotes never lobbying.*

Proof. We prove that any strategy $S_\tau(\cdot)$ (typically displaying "intermittent" lobbying) is dominated either by strategy $\mathbf{0}(\cdot)$ or by strategy $\mathbf{1}(\cdot)$. Let us show this next. First, if $S_\tau(\cdot)$ is caught up, then it is dominated by the strategy $\mathbf{0}(\cdot)$ which achieves the same completion date at a lower lobbying cost. This is because after a strategy is caught up, it cannot go any faster than its assignment vintage. Suppose then that $S_\tau(\cdot)$ is not caught up.

Denote

$$\chi(t) = \int_\tau^t S_\tau(s)\ ds$$

where by construction $\chi(\cdot)$ is non-decreasing, $\chi(\tau) = 0$ and $\chi(t) \leq t - \tau$. The function $\chi(t)$ can be interpreted as a measure representing how much activity has occurred on the project between τ and t or, equivalently, the state of advancement of the project. When strategy S_τ

is employed, the project's advancement at time t is given by

$$x_S(t) = X - \int_\tau^t \dot{x}_S(r)\ dr$$
$$= X - \int_\tau^t \frac{\eta}{A_r}\ d\chi(r).$$

Denote by T the time at which the project is done, that is, T is the smallest value that solves $x_S(T) = 0$. Create a new strategy $\widetilde{S}(t)$ which equals 1 for $t \in [\tau, \tau + \chi(T)]$ and 0 for $t > \tau + \chi(T)$. Then we have

$$\begin{aligned}
0 &= x_S(T) \\
&= X - \int_\tau^T \frac{\eta}{A_r}\ d\chi(r) \\
&= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_{\chi^{-1}(y-\tau)}}\ dy \\
&\geq X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y}\ dy \\
&= X - \int_\tau^{\tau+\chi(T)} \frac{\eta}{A_y}\widetilde{S}_\tau(y)\ dy = x_{\widetilde{S}}(\tau + \chi(T))
\end{aligned}$$

where the third equality reflects a change of variable $y = \tau + \chi(r)$, and the inequality follows because $\chi(y) \leq y - \tau$, hence $\chi^{-1}(y - \tau) \geq y$ and $A_{\chi^{-1}(y-\tau)} \geq A_y$. The inequality shows that strategy S is just done at time T, whereas strategy $\widetilde{S}$ is more than done already by time $\tau + \chi(T) \leq T$. This means that the duration under strategy $\widetilde{S}$ is smaller than that under strategy S. . Denote by $\widetilde{T} \leq \tau + \chi(T)$ the time strategy $\widetilde{S}$ is done. Let us now turn to lobbying expenditures. Strategy S's lobbying expenditure is given by $\kappa\chi(T)$. Strategy $\widetilde{S}$'s lobbying expenditure is given by $\kappa\left(\widetilde{T} - \tau\right)$. Since $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$'s lobbying expenditure is smaller than strategy S's.

Summing up, we have shown that duration and lobbying expenditure are smaller under strategy $\widetilde{S}$ than under strategy S. Thus strategy $\widetilde{S}$ dominates S. Notice that, since under $\widetilde{S}$ a project ends at $\widetilde{T} \leq \tau + \chi(T)$, strategy $\widetilde{S}$ is payoff-equivalent to strategy $\mathbf{1}(\cdot)$. Thus strategy S is dominated by strategy $\mathbf{1}(\cdot)$. ■

The intuition behind Lemma 3 is the following. Lobbying "buys advancement" at the speed of η/A_t. If it is profitable to lobby at the assignment of the project, then it makes no sense to have interludes of no lobbying. During those interludes the project does not advance, but the mass of active projects A_t keeps growing, making lobbying (once it is restarted) less productive. Of course, even taking Lemma 3 into account, lobbying equilibria could potentially be very complex because of the possibility of non-constant growth equilibria in which the input rate is not constant through time.

Proof of Proposition 3

Proof. a) We show that there is a time-invariant z such that the value at the time of assignment of two players who follow the two different equilibrium strategies (lobby and not) are the same. The lobbyist's value at the time of assignment for a project assigned at τ, assuming the project is lobbied from assignment through to completion, is $(-\kappa - B)\, C_\tau$ where C_τ is the completion time of a project started at τ. Substituting for C_t from Proposition 2, the value is given by

$$VL_\tau (z) = (-\kappa - B) \quad \left[\frac{\nu (z)}{\Omega (\nu (z) \, ; \eta / X)} - 1 \right] \tau.$$

The value of the non-lobbyist at the time of assignment for a project assigned at τ, assuming that she never lobbies, is computed as follows. First, the fraction of non-lobbyist projects inputed in each instant is given by $\frac{\underline{\nu}}{\nu(z)}$, and consequently the output rate is made up of a fraction $\frac{\underline{\nu}}{\nu(z)}$ of non-lobbyist projects. Thus, a project assigned at τ finds

$$z\alpha\tau - \frac{\underline{\nu}}{\nu (z)} \Omega (\nu (z) \, ; \eta / X) \, \tau$$

non-completed projects in front of it. These projects are completed at rate $\frac{\underline{\nu}}{\nu(z)} \Omega (\nu (z) \, ; \eta / X)$, so it takes

$$\left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)} \Omega (\nu (z) \, ; \eta / X)} - 1 \right] \tau$$

before all non-lobbied projects assigned before τ are completed. Therefore the value of a non-lobbyist at the time of assignment, assuming that she never lobbies in the future, is

$$VN_\tau (z) = -B \left[\frac{z\alpha}{\frac{\underline{\nu}}{\nu(z)} \Omega (\nu (z) \, ; \eta / X)} - 1 \right] \tau$$

In an equilibrium with lobbyists and non-lobbyists, z^* solves $VL_\tau (z^*) = VN_\tau (z^*)$, or

$$(-\kappa - B) \quad \left[\frac{\nu (z^*)}{\Omega (\nu (z^*) \, ; \eta / X)} - 1 \right] = -B \left[\frac{\alpha}{\underline{\nu}} z^* \frac{\nu (z^*)}{\Omega (\nu (z^*) \, ; \eta / X)} - 1 \right] \tag{24}$$

It is important to note that condition is independent of τ. Thus, if a z^* exists that verifies equation (24), this z^* will be time-invariant, consistent with the definition of constant-growth lobbying equilibrium. We conclude the proof by showing that at least one z^* exists that verifies equation (24) and it lies between $\frac{\underline{\nu}}{\alpha}$ and $\frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)$.

The lowest possible value of z^* is $\frac{\underline{\nu}}{\alpha}$. If z falls below this level, there are not enough non-lobbyists to fill $\underline{\nu}$, and then non-lobbied projects get started immediately. Formally, in this project the expression in brackets on the RHS of (24) is no greater than the brackets on the LHS, whence $VN_\tau(z) > VL_\tau(z)$. So $z \leq \frac{\underline{\nu}}{\alpha}$ is not consistent with equilibrium. The highest possible value of z^* is that for which $\nu(z^*) = \eta/X$. At this level the LHS of (24) is zero, and so $VN_\tau(z) < VL_\tau(z)$. Intuitively, if z^* were any higher, then $\nu(z^*) < \eta/X$ and then completion times would be zero, and then anyone who lobbyied could do so at zero cost. Thus such z cannot be part of the equilibrium in which not everyone lobbies. To find an expression for this bound, write $\eta/X = \nu^* = \underline{\nu} + (1-z)\alpha$, and solving for z yields $z = \frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)$. We have shown that on the lower bound of the interval $z \in \left(\frac{\underline{\nu}}{\alpha}, \frac{\underline{\nu}}{\alpha}+\frac{1}{\alpha}\left(\alpha-\frac{\eta}{X}\right)\right)$ we have $VN_\tau(z) > VL_\tau(z)$, and on the upper bound $VN_\tau(z) < VL_\tau(z)$. Since the two functions $VN_\tau(z)$ and $VL_\tau(z)$ are continuous in z over the interval, they must cross at least once. Any crossing is consistent with an equilibrium.

b) Suppose not, so that $\nu^* \leq \frac{\eta}{X}$. Then $\alpha > \nu^*$, and so a project assigned at τ finds a backlog of $(\alpha-\nu^*)\tau$ unopened projects in front of it. Since projects are opened at rate ν^*, the time it takes the last project in the backlog to be opened is

$$\frac{(\alpha-\nu^*)}{\nu^*}\tau.$$

This expression, which we will call the unopened duration, is positive and grows linearly with τ. This time can be eliminated by lobbying from assignment time all the way through completion, at a total cost that is proportional to completion time. Proposition 10 proves that when $\nu^* \leq \frac{\eta}{X}$ completion time is stationary, i.e., it is the same for projects opened at any τ. Therefore, the strict best response of all projects assigned after a certain $\widehat{\tau}$ is to lobby all the way through completion time, in order to eliminate the unopened duration which exceeds lobbying costs. But then for every $t > \widehat{\tau}$ not lobbying cannot be equally profitable as lobbying. Therefore we have shown that if $\nu^* \leq \frac{\eta}{X}$, a positive mass cannot be not lobbying after $\widehat{\tau}$. Yet the construction requires that in any instant $\alpha - \nu^*$ projects are not lobbied, and this mass is positive because by assumption $\alpha > \frac{\eta}{X} \geq \nu^*$. Contradiction.

c) The equilibrium z^* solves (24), which can be rearranged as

$$\frac{\kappa+B}{B}\left[\frac{\nu(z)}{\Omega(\nu(z);\eta/X)}-1\right] = \left[\frac{\alpha}{\underline{\nu}}z\frac{\nu(z)}{\Omega(\nu(z);\eta/X)}-1\right]$$

and rewritten as

$$\left[\frac{\kappa+B}{B}-\frac{\alpha}{\underline{\nu}}z\right]\frac{\nu(z)}{\Omega(\nu(z);\eta/X)} = \left[\frac{\kappa+B}{B}-1\right]. \tag{25}$$

The LHS in (25) is the product of two positive and decreasing functions of z, and therefore it is decreasing in z. The RHS does not depend on z. Therefore equation (25) admits a unique solution z^*.

d) Rewrite slightly (25) as

$$H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right) = \left[\frac{\kappa}{B} + 1 - \frac{\alpha}{\underline{\nu}} z\right] \frac{\nu(z)}{\Omega(\nu(z); \eta/X)} = \frac{\kappa}{B}. \tag{26}$$

The function $H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right)$ is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$, so increasing $\frac{\alpha}{\underline{\nu}}$ or $\frac{\eta}{X}$ results in a downward shift of the function. Since the function is decreasing in z, shifting the function downward results in a shift to the left of the intersection point between the function and the constant line $\frac{\kappa}{B}$. Thus z^* is decreasing in $\frac{\alpha}{\underline{\nu}}$ and $\frac{\eta}{X}$.

The function $H\left(z; \frac{\kappa}{B}, \frac{\alpha}{\underline{\nu}}\right)$ is increasing in $\frac{\kappa}{B}$, and increasing $\frac{\kappa}{B}$ by δ results in an upward shift of $\delta \frac{\nu(z)}{\Omega(\nu(z);\eta/X)} > 1$ in the function. So increasing $\frac{\kappa}{B}$ results in the function shifting upward by more than $\frac{\kappa}{B}$. So, start from a given $\frac{\kappa}{B}$ and focus on the resulting equilibrium z^*, which is the z at which the function H attains height $\frac{\kappa}{B}$. Then increase $\frac{\kappa}{B}$. At z^*, the function H moves up by more than $\frac{\kappa}{B}$. This means that z^* is to the left of the new equilibrium. Thus z^* is increasing in $\frac{\kappa}{B}$.

e) Follows directly from d) and the definition $\nu(z) = \underline{\nu} + (1 - z)\alpha$. ■

Proof of Proposition 4

Proof. Suppose by contradiction that, as κ increases to $\widehat{\kappa}$, we have $\widehat{z} < z^*$. Then by definition we have $\nu(\widehat{z}) > \nu(z^*)$. Since by Proposition 1 c) $\frac{\partial \Omega(\nu;\eta/X)}{\partial \nu \partial \eta} < 0$, it follows from problem (6) that $\widehat{\eta} < \eta^*$. Then $\Omega(\nu(\widehat{z}); \widehat{\eta}/X) < \Omega(\nu(z^*); \eta^*/X)$, and then the entire LHS of equation (26) becomes larger. Since the RHS stays unchanged, equation (26) can no longer be satisfied, and so we do not have an equilibrium. Therefore it must be that as κ increases to $\widehat{\kappa}$, the input rate decreases. It then follows from problem (6) that the worker's effort increases. ■

A.3 Proofs for Section 5

Proof of Proposition 5

Proof. (a) The square bracket in the integral of (7) is time-invariant, so it can be factored out of the integral. Then maximizing (7) is equivalent to solving

$$\max_{\{\nu_i,\eta_i\}} \; w\sum_{i=1}^{N}\left[\Omega\left(\nu_i;\eta_i/X_i\right)\right] \; - c\left(\sum_{j=1}^{N}\eta_j\right),$$

Fix any constellation of $(\eta_1,...,\eta_N)$ (not necessarily optimal). Given that constellation, the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, because this choice achives the maximal feasible output rate given η_i which is $\Omega\left(\eta_i/X_i;\eta_i/X_i\right) = \frac{\eta_i}{X_i}$ We can therefore rewrite the worker's problem as

$$\max_{\eta_i} \; w\sum_{i=1}^{N}\frac{\eta_i}{X_i} - c\left(\sum_{j=1}^{N}\eta_j\right).$$

The maximand is concave and so the first order conditions identify a maximum. They read

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \leq 0 \text{ for all } i \text{ such that } \eta_i^* = 0,$$

$$\frac{w}{X_k} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) = 0 \text{ for the unique } k \text{ such that } \eta_k^* \in \left(0,\alpha_k X_k\right), \text{ and}$$

$$\frac{w}{X_i} - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \geq 0 \text{ for all } i \text{ such that } \eta_i^* = \alpha_i X_i.$$

These conditions imply that the worker will set η_i at its maximum on projects with the lowest X_i. All other projects are ignored.

(b) Substitute from Proposition 2 into (8) to get

$$\int_0^\infty e^{-\rho t}\left[-w\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right)t - c\left(\sum_{i=1}^{N}\eta_i\right)\right]\,dt$$
$$= -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i}{\Omega\left(\nu_i;\eta_i/X_i\right)}-1\right) - \frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right),$$

where we have used the identities $\int_0^\infty e^{-\rho t}t\,dt = -\frac{1}{\rho}\left(t+\frac{1}{\rho}\right)e^{-\rho t}\Big|_0^\infty = 1/\rho^2$ and $\int_0^\infty e^{-\rho t}\,dt = 1/\rho$. Like in part (a), the worker will optimally chose the input rate $\nu_i = \frac{\eta_i}{X_i}$, so the maximization problem simplifies to

$$\max_{\{\eta_i\}} -\frac{w}{\rho^2}\sum_{i=1}^{N}\alpha_i\left(\frac{\alpha_i X_i}{\eta_i}-1\right) - \frac{1}{\rho}c\left(\sum_{i=1}^{N}\eta_i\right).$$

The maximand is concave in each η_i and so the first order conditions identify an interior maximum. They read

$$\frac{w}{\rho}(\alpha_i)^2 X_i \left[\frac{1}{\eta_i^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \leq 0 \text{ for all } i \text{ such that } \eta_i^* = 0,$$

$$\frac{w}{\rho}(\alpha_k)^2 X_k \left[\frac{1}{\eta_k^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) = 0 \text{ for all } k \text{ such that } \eta_k^* \in (0, \alpha_k X_k)\,, \text{ and}$$

$$\frac{w}{\rho}(\alpha_i)^2 X_i \left[\frac{1}{\eta_i^*}\right]^2 - c'\left(\sum_{j=1}^{N}\eta_j^*\right) \geq 0 \text{ for all } i \text{ such that } \eta_i^* = \alpha_i X_i.$$

From these conditions we see that the project $\eta_i^* = 0$ is not possible for any i. Then, either $\eta_i^* = \alpha_i X_i$ for all i, or else there is an i such that η_i^* is interior, that is, $\eta_i^* < \alpha_i X_i$. If η_i^* is interior then for all j such that $(\alpha_j)^2 X_j > (\alpha_i)^2 X_i$ we have

$$\begin{aligned} c'\left(\sum_{j=1}^{N}\eta_j^*\right) &= \frac{w}{\rho}(\alpha_i)^2 X_i \left[\frac{1}{\eta_i^*}\right]^2 \\ &< \frac{w}{\rho}(\alpha_j)^2 X_j \left[\frac{1}{\eta_i^*}\right]^2 \\ &\leq \frac{w}{\rho}(\alpha_j)^2 X_j \left[\frac{1}{\eta_j^*}\right]^2, \end{aligned}$$

where the last inequality holds only for $\eta_j^* \leq \eta_i^*$. Thus if $\eta_j^* \leq \eta_i^*$ then derivative of the objective function with respect to η_j is positive at η_j^*. By the first order conditions this means $\eta_j^* = \alpha_j X_j$. Summing up, we have shown that if $(\alpha_j)^2 X_j > (\alpha_i)^2 X_i$ then either $\eta_j^* > \eta_i^*$ or else $\eta_j^* = \alpha_j X_j$. ■

A.4 Proofs for Section 6.1

Define

$$F = f\,\frac{\nu - \omega}{\omega}\,t,$$

so that $F_t = F \cdot t$. Also define

$$\varphi_t^{**}(x) = \frac{(\nu - \omega) + F}{\eta}\omega\; t\; e^{\frac{(\nu-\omega)+F}{\eta}x},$$

and recall the previous definition

$$A_t^* = (\nu - \omega)\;\; t.$$

Proof of Proposition 6

Proof. (a) Condition (9) is verified because

$$\begin{aligned}
&\frac{\partial\varphi_t^{**}(x)}{\partial t}-\frac{\partial\varphi_t^{**}(x)}{\partial x}\frac{\eta}{A_t+F_t}\\
&=\frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x}-\frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\frac{(\nu-\omega)+F}{\eta}\frac{\eta}{(\nu-\omega+F)\,t}\\
&=\frac{(\nu-\omega)+F}{\eta}\omega\ e^{\frac{(\nu-\omega)+F}{\eta}x}-\frac{(\nu-\omega)+F}{\eta}\omega\ \ e^{\frac{(\nu-\omega)+F}{\eta}x}\\
&=0.
\end{aligned}$$

Condition (10) is verified because

$$\begin{aligned}
\omega&=\frac{\eta}{A_t+F_t}\varphi_t^{**}(0)\\
&=\frac{\eta}{(\nu-\omega+F)\,t}\frac{(\nu-\omega)+F}{\eta}\omega\ t\\
&=\omega.
\end{aligned}$$

Condition (4) can be verified immediately.

Condition (1) reads

$$A_t^*=\int_0^X\varphi_t^*(x)\ dx.$$

Substituting for $\varphi_t^{**}(x)$ and A_t^* yields

$$\begin{aligned}
(\nu-\omega)\,t&=\int_0^X\frac{(\nu-\omega)+F}{\eta}\omega\ t\ e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx\\
&=\frac{(\nu-\omega)+F}{\eta}\omega\ t\int_0^X\ e^{\frac{(\nu-\omega)+F}{\eta}x}\ dx\\
&=\frac{(\nu-\omega)+F}{\eta}\omega\ t\ \ \frac{\eta}{(\nu-\omega)+F}\ e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X}\\
&=\omega\ t\ \ \ e^{\frac{(\nu-\omega)+F}{\eta}x}\Big|_{x=0}^{X}\\
&=\omega\ t\ \ \left[e^{\frac{(\nu-\omega)+F}{\eta}X}-1\right].
\end{aligned}$$

We can rewrite this equality as

$$\begin{aligned}\frac{\nu}{\omega}-1&=\left[e^{\frac{(\nu-\omega)+F}{\eta}X}-1\right]\\ \frac{\nu}{\omega}&=e^{\frac{(\nu-\omega)+F}{\eta}X}\\ \log(\nu)-\log(\omega)&=(\nu-\omega+F)\frac{X}{\eta}\end{aligned}$$

(b). Fix any ν and let ω^{**} be the output rate in a constant growth path with forgetful worker. Then ω^{**} solves equation (11), which can be written as

$$h(\omega)=h(\nu)+f\cdot\frac{\nu-\omega}{\omega}\frac{X}{\eta}. \tag{27}$$

Suppose $f>0$, and by contradiction, that the output rate in a constant growth path with non-forgetful worker, call it ω^*, is smaller than ω^{**}. Since obviously, $\omega^{**}<\nu$, we have $\omega^*\leq\omega^{**}<\nu$. By definition of ω^* we have $h(\omega^*)=h(\nu)$, and since the function h is convex, it follows that $h(\omega^{**})\leq h(\nu)$. But then since $f>0$, the right-hand side in (27) must exceed the left-hand side, and so the equation cannot be satisfied. We have reached a contradiction. ■

A.5 Material for Section 6.2

The first goal must be to specify the (private or social) objective function. To this end, some preliminaries need to be introduced.

Definition 6 ***(rate of clearing intermediate goals)*** *Denote by $\omega_t(x)$ the rate at which, at each instant t, projects clear threshold $x\in[0,X]$.*

For example, $\omega_t(X/2)$ denotes the rate at which projects clear the $X/2$ mark, that is, the rate at which projects become half done. In this notation, the output rate ω_t corresponds to $\omega_t(0)$. We allow for weight to be placed on any number of these "intermediate rates" by postulating the following objective function:

$$\int_0^\infty e^{-\rho t}\left[\int_0^X u(\omega_t(x))\ P(dx)\right]\ dt. \tag{P}$$

Here ρ represents a (social or private) discount factor. The function u is increasing and its curvature measures the degree to which low clearing rates are penalized in the objective function. $P(x)$ is a probability measure that specifies the weight placed on intermediate step x, so for example if one third of the value is created when projects clear the $X/2$ mark then $P(X/2) = 1/3$. A slightly different interpretation of $P(X/2) = 1/3$ is that for half the projects two thirds of the value is created by clearing the $X/2$ mark, and for the other half no value is created. In this latter interpretation projects are heterogeneous with respect to when value is created.

Lemma 4 *Along a constant growth path, $\omega_t(x) = \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}$. Thus, along a constant growth path $\omega_t(x)$ is stationary and is denoted by $\overline{\omega}(x)$.*

Proof. The mass of projects that clear x between t and $t+\Delta$ is approximately equal to

$$\int_x^{x+\frac{\eta\Delta}{A_t}} \varphi_t(s)\ ds \approx \varphi_t(x)\frac{\eta\Delta}{A_t},$$

and the rate $\omega_t(x)$ obtains dividing by Δ and letting $\Delta \to 0$. Formally,

$$\omega_t(x) = \frac{\eta}{A_t}\varphi_t(x).$$

We have

$$\omega_t(x) = \frac{\eta\varphi_t(x)}{A_t} = \frac{\eta\varphi_t(0)}{A_t}e^{\frac{(\nu-\omega)}{\eta}x} = \omega e^{\frac{(\nu-\omega)}{\eta}x}, \tag{28}$$

where the second equality holds along a constant growth path. Now,

$$\frac{(\nu-\omega)}{\eta}x = \frac{x}{X}\frac{(\nu-\omega)}{\eta}X = \frac{x}{X}\log\frac{\nu}{\omega},$$

where the last equality follows from equation (5) which must hold along a constant growth path. Taking exponentials on both sides and substituting into (28) yields

$$\omega_t(x) = \omega\left(\frac{\nu}{\omega}\right)^{\frac{x}{X}},$$

which is equivalent to the expression in the lemma. ∎

Considerable simplification in (**P**) is achieved because Lemma 4 shows that $\omega_t(x)$ is stationary, i.e., time-independent, when ω is generated by ν and η. , and in addition it offers a convenient expression for $\omega_t(x)$ as a geometric mean of ν and ω with weight x/X. In light of this lemma, the inner integral in problem (**P**) is stationary and so, along a constant growth path, (**P**) simplifies to

$$\frac{1}{\rho}\int_0^X u(\overline{\omega}(x))\ P(dx).$$

If, moreover, we assume $u(\cdot) = \log(\cdot)$, then (**P**) equals

$$\begin{aligned} & \frac{1}{\rho} \int_0^X \log\left(\nu^{\frac{x}{X}}\ \omega^{\left(1-\frac{x}{X}\right)}\right)\ P(dx) \\ & = \frac{1}{\rho}\left[\log(\omega) + \frac{\mathbb{E}(P)}{X}\log\left(\frac{\nu}{\omega}\right)\right] \\ & = \frac{1}{\rho}\log\left(\overline{\omega}\left(\mathbb{E}(P)\right)\right). \end{aligned} \tag{P'}$$

where $\mathbb{E}(P) = \int_0^X x\ P(dx)$ denotes the expected value of the c.d.f. P. Expression (**P'**) shows that, despite the fact that the measure P may place weight on clearing a great number of intermediate goals, when $u(\cdot) = \log(\cdot)$ the worker's problem can always be reduced to caring about clearing a single "average" goal $\mathbb{E}(P)$. This is a considerable analytical advantage.

The next proposition shows that, the more we care about clearing early intermediate goals, the higher the optimal input rate ν. Thus we have the basis for a theory optimally chosen input rates that might account for rates exceeding η/X.

Proposition 11 *Suppose the pair (ν, ω) is part of a constant growth path given η/X. If ν is chosen to maximize (**P'**) then ν is strictly increasing in $\mathbb{E}(P)$, and the associated ω is therefore strictly decreasing in $\mathbb{E}(P)$.*

Proof. Along a constant growth path (**P'**) is proportional to

$$U\left(\nu; \widetilde{P}\right) = \log\left(\Omega\left(\nu; \eta/X\right)\right) + \frac{\mathbb{E}(P)}{X}\log\left(\frac{\nu}{\Omega\left(\nu; \eta/X\right)}\right)$$

We know from Proposition 1 (a) that $\log\left(\frac{\nu}{\Omega(\nu;\eta/X)}\right)$ is strictly increasing in ν. It follows that, if $\mathbb{E}\left(\widetilde{P}\right) > \mathbb{E}(P)$, then the expression

$$U\left(\nu; \widetilde{P}\right) - U(\nu; P)$$

is an increasing function of ν. Then for any pair $\nu < \nu'$ we have

$$U\left(\nu; \widetilde{P}\right) - U(\nu; P) < U\left(\nu'; \widetilde{P}\right) - U(\nu'; P).$$

Rearranging yields

$$U\left(\nu'; \widetilde{P}\right) - U\left(\nu; \widetilde{P}\right) > U(\nu'; P) - U(\nu; P).$$

Now set $\nu' = \nu^* = \sup \{\arg\max U(\nu; P)\}$. Then the right-hand side is no smaller than zero, which implies that $U\left(\nu^*; \widetilde{P}\right) > U\left(\nu; \widetilde{P}\right)$ for any $\nu < \nu^*$. This shows that the maximizer(s) of $U\left(\cdot; \widetilde{P}\right)$ must be at least as large as ν^*. To finish the proof we need to show that the maximizer(s) of $U\left(\cdot; \widetilde{P}\right)$ are in fact strictly larger than ν^*. The fact that the function $\Omega(\nu)$ is differentiable in ν guarantees that $\partial U(\nu; P) / \partial \nu$ is zero at ν^*. But then $\partial U\left(\nu; \widetilde{P}\right) / \partial \nu$ cannot be zero at ν^* (recall that $U\left(\nu; \widetilde{P}\right) - U(\nu; P)$ is a strictly increasing function of ν). Therefore the maximizer(s) of $U\left(\cdot; \widetilde{P}\right)$ cannot include ν^* and thus they must be strictly larger than ν^*. ■

This result makes intuitive sense if we think about polar cases. If $\mathbb{E}(P)$ assumes the largest possible value, namely X, then the worker only cares about the rate at which projects are opened. In this case, it makes sense for the worker to chose the largest possible ν because the negative effect on the completion rate is irrelevant. Conversely, if $\mathbb{E}(P)$ assumes the smallest possible value, namely 0, then the worker only cares about the rate at which projects are completed. In this case, it makes sense for the worker to chose the smallest ν compatible with no idleness, that is, η/X.

Until now we have assumed equal treatment of all open projects. The equal treatment assumption makes sense in some circumstances. If, however, it is possible to treat projects disparately then it may be optimal to do so. We now define a strategy which allows effort to be tailored according to a project's level of completion, and partially-completed projects to be kept in a queue.

Definition 7 *Consider a partition of $[0, X]$ with generic element $I_i = (x_{i-1}, x_i)$, where we posit $x_0 = 0, x_i < x_{i+1}$, and $x_N = X$. A **variable-speed work strategy** is a vector of pairs $(\eta_i, \nu_i)_{i=1}^N$ such that*

(a) $\sum_i \eta_i = \eta$, and

(b) $\nu_i \leq \Omega\left(\nu_{i+1}; \eta_{i+1} \frac{1}{x_{i+1} - x_i}\right)$.

The interpretation is as follows. I_i is an interval of completion levels, η_i the effort devoted to projects whose level of completion at any point in time belongs to I_i, and ν_i is the rate at which projects are allowed to transit into I_i. The worker is allowed to distribute his total effort η in any way she chooses across these intervals, which means that she is allowed to focus on projects with different completion levels. Moreover, the worker is allowed to tailor the input rate ν_i for I_i, that is, the worker chooses how fast to feed into I_i projects coming out of I_{i+1}. Once η_{i+1} and ν_{i+1} are set, the output rate out of interval I_{i+1} is given by $\Omega\left(\nu_{i+1};\eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$. Obviously, the worker cannot feed projects into I_i any faster than they come out of I_{i+1}; this accounts for the inequality in part (b) of Definition 7. When strict inequality holds, the worker is slowing down projects coming out of interval I_{i+1} and putting them in a queue of projects waiting to enter interval I_i.

Definition 8 *An* ***equal treatment*** *work strategy is a special variable-speed strategy for which* $\nu_i = \Omega\left(\nu_{i+1};\eta_{i+1}\frac{1}{x_{i+1}-x_i}\right)$ *for all* i.

The strategies considered elsewhere in the paper are in fact equal treatment strategies. To see this, observe that in a strategy where all projects move to the right at the same rate $\eta\frac{1}{A_t}$ we can fix any x and think of projects that cross x (moving from right to the left) as projects that have just outputed "completion level below x" and are just being inputed into "completion rate higher than x." Obviously, these two artificial output and input rates are the same.

Since equal treatment strategies are a special case of variable-speed strategies, obviously the latter are going to be at least as good along whatever dimension we choose to measure. When is the converse true? That is, when is the "best" variable-speed strategy actually an equal treatment one? The next proposition addresses this question.

Proposition 12 *(a) For any variable-speed work strategy, there is an equal treatment strategy that yields the same output rate and requires (weakly) less effort.*

(b) Consider the profile of intermediate output rates $\overline{\omega}(x)$ generated by an equal treatment strategy with effort η. The sequence of variable-speed work strategies which in the limit yields the same profile requires the same amount of effort η in the limit.

*(c) Fix η and ν, and let ω be the output rate along the associated constant growth path. Suppose we want to maximize (**P**) with $u(\cdot) = \log(\cdot)$ and intermediate outputs being valued according to $\overline{P}(dx) = \overline{\omega}(x)\ dx$. Then equal treatment strategies do just as well as variable-speed work strategies.*

Proof. (a) Suppose there is an $j > 0$ such that

$$\nu_j < \Omega\left(\nu_{j+1};\eta_{j+1}\frac{1}{x_{j+1}-x_j}\right). \tag{29}$$

Then we can decrease η_{j+1} slightly without changing any other element of $(\eta_i,\nu_i)_{i=1}^N$ and obtain a new completion-dependent strategy which requires less effort than the original one, and which has the same output since ν_j and all the other variables with index j or lower are unaffected. Moreover, the new strategy satisfies part (b) of Definition 7 for all j if the decrease in η_{j+1} is small enough. Continue this process until (29) binds for all j, and an equal treatment strategy is obtained with the desired property.

(b) Fix η and ν in an equal treatment strategy, and let ω be the associated completion rate along the constant growth path. These ν and ω generate an entire profile $\overline{\omega}(x)$ according to Lemma 4. We turn now to the variable-speed strategy which is going to approximate the profile $\overline{\omega}(x)$. For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N}..., X\right)$. The cheapest way to generate a given output rate ω out of an interval of size $\frac{X}{N}$ is to have input rate ω and effort $\frac{X}{N}\omega$. So the cheapest way to generate intermediate output rates $\left(\omega, \overline{\omega}\left(\frac{X}{N}\right), \overline{\omega}\left(\frac{2X}{N}\right)..., \nu\right)$ is with a variable-speed strategy $(\eta_i, \nu_i)_{i=0}^N = \left(\frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right), \overline{\omega}\left(\frac{i}{N}X\right)\right)_{i=0}^N$. The effort required by this strategy is

$$\eta = \sum_{i=0}^{N} \eta_i = \sum_{i=0}^{N} \frac{X}{N}\overline{\omega}\left(\frac{i}{N}X\right).$$

Taking the limit as $N \to \infty$, this effort converges to

$$\begin{aligned}\int_0^X \overline{\omega}(x)\ dx &= \int_0^X \nu^{\frac{x}{X}} \cdot \omega^{1-\frac{x}{X}}\ dx \\ &= \omega \int_0^X \left(\frac{\nu}{\omega}\right)^{\frac{x}{X}}\ dx\end{aligned}$$

where the first equality follows from Lemma 4. Performing the change of variable $y = x/X$ yields

$$\begin{aligned}\eta &= \omega \int_0^1 \left(\frac{\nu}{\omega}\right)^y X\ dy \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega}\right)^y \Bigg|_{y=0}^1 \\ &= \frac{\omega X}{\log\left(\frac{\nu}{\omega}\right)} \left(\frac{\nu}{\omega} - 1\right).\end{aligned}$$

Rearranging yields $(\nu - \omega)\frac{X}{\eta} = \log(\nu) - \log(\omega)$, which is exactly equation (5). So the effort required by the limit of the sequence of cheapest variable-speed strategies is the same effort that generates ω in the equal-treatment strategy.

c) For each N consider the partition $\left(0, \frac{X}{N}, \frac{2X}{N} ..., X\right)$ and the associated variable-speed strategy $(\eta_i, \nu_i)_{i=0}^N$. Given this strategy space, the maximization of problem (**P**) reads

$$\begin{aligned}
&\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\nu_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx) \\
&\text{s.t. } \nu_i \leq \Omega\left(\nu_{i+1}, N\eta_{i+1}\right) \text{ for all } i, \\
&\sum_i \eta_i = \eta.
\end{aligned}$$

Since $\frac{N}{X}\eta_i \leq \nu_i$, the objective function is smaller than

$$\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx).$$

and also the constraint is more restrictive than

$$\nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i.$$

We therefore define the relaxed problem as

$$\begin{aligned}
&\max_{(\eta_i,\nu_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left([\nu_i]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx). \\
&\text{s.t. } \nu_i \leq \Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right) \text{ for all } i. \\
&\sum_i \eta_i = \eta.
\end{aligned}$$

In the solution to the relaxed problem the constraints on each ν_i bind and so the relaxed problem reads

$$\begin{aligned}
&\max_{(\eta_i)_{i=0}^N} \sum_i \int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}} \log\left(\left[\Omega\left(\frac{N}{X}\eta_{i+1}, \frac{N}{X}\eta_{i+1}\right)\right]^{\frac{x}{X}} \cdot \left[\Omega\left(\frac{N}{X}\eta_i, \frac{N}{X}\eta_i\right)\right]^{1-\frac{x}{X}} \right) \overline{P}(dx) \\
&\text{s.t. } \sum_i \eta_i = \eta.
\end{aligned}$$

Substituting for $\Omega(y,y)=y$, the relaxed problem reads

$$\max_{(\eta_i)_{i=0}^N}\sum_i\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}}\left[\frac{x}{X}\log\left(\frac{N}{X}\eta_{i+1}\right)+\left(1-\frac{x}{X}\right)\log\left(\frac{N}{X}\eta_i\right)\right]\overline{P}(dx)$$
$$=\max_{(\eta_i)_{i=0}^N}\sum_i\left[\log\left(\frac{N}{X}\eta_{i+1}\right)\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}}\frac{x}{X}\overline{P}(dx)+\log\left(\frac{N}{X}\eta_i\right)\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}}\left(1-\frac{x}{X}\right)\overline{P}(dx)\right],$$

subject to the effort constraint. Define

$$\overline{P}_i=\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}}\frac{x}{X}\overline{P}(dx)+\int_{i\frac{X}{N}}^{(i+1)\frac{X}{N}}\left(1-\frac{x}{X}\right)\overline{P}(dx)$$

We can rewrite the relaxed problem as

$$\max_{(\eta_i)_{i=0}^N}\sum_i\overline{P}_i\log\left(\frac{N}{X}\eta_i\right)$$
$$\text{s.t. }\sum_i\eta_i=\eta.$$

This is a concave problem so the solution is identified from the first order conditions of the associated Lagrangean. At the optimum these conditions imply that $\overline{P}_i\frac{1}{\eta_i^*}$ is the same for all i. Since η_i^* converges to zero as $N\to\infty$, it is convenient to write the first order conditions as

$$\overline{P}_i\frac{1}{\frac{N}{X}\eta_i^*}=\text{const for all } i.$$

As $N\to\infty,\overline{P}_i$ converges to $\overline{P}(x_i)$. Moreover, since $\frac{N}{X}\eta_i^*=\Omega\left(\frac{N}{X}\eta_i^*,\frac{N}{X}\eta_i^*\right)$, $\frac{N}{X}\eta_i^*$ is also equal to an intermediate output rate, which as $N\to\infty$ converges to the intermediate output rate $\omega^*(x_i)$ in the relaxed problem. Therefore, in the solution to the relaxed problem the intermediate output rate the first order conditions imply

$$\omega^*(x_i)=\text{const}\cdot\overline{P}(x_i)=\text{const}\cdot\overline{\omega}(x_i),$$

where the last equality follows by definition of $\overline{P}(\cdot)$. This means that in the solution to the relaxed problem, the profile of intermediate output rates is proportional to that associated by an equal treatment strategy with effort η. The budget constraint then ensures that the constant is equal to 1, and thus that the solution to the relaxed problem is in fact an equal treatment strategy with effort η. Since this strategy is obviously feasible in the original problem, the proof is done. ■

Part (a) of this proposition shows that any output rate that can be achieved by a variable-speed strategy can also be achieved, more cheaply, by an equal treatment strategy. In this sense, there is no better strategy than an equal treatment strategy.[16] This result justifies the focus on equal treatment strategies *if we only care about output rates*. Because this result holds only if we care exclusively about output rates, it does not apply to the setup of Section 6.2. That is to say, part (a) of this proposition applies in a world in which equal treatment functions with $\nu > \omega$ are necessarily suboptimal.

Parts (b) and (c) rehabilitate equal treatment strategies when $\nu > \omega$. In part (b), for any pair $(\nu, \eta/X)$ these strategies are shown to be the most efficient way to get the profile of intermediate outputs they generate. So, to the extent that we deem these strategies "inefficient" when $\nu > \eta/X$, it is only because we are evaluating their profile of intermediate outputs according to a criterion that they do not meet (perhaps because we only value final output). But if we wish to generate the intermediate output profile they generate, there is no other strategy that attains it for cheaper. In part (c) we show that equal treatment strategies with $\nu > \omega$ can be optimal within the class of variable-speed strategies, by reverse-engineering the parameters that ensure that the variable-speed strategy that maximizes $(\mathbf{P})$ is, in fact, an equal treatment strategy.

A.6 Proofs for Section 6.3

Proof of Proposition 7

Proof. Think of the worker as grouping projects by type, and working on each group of projects separately. Accordingly, we denote by η_{it} the (still to be computed) amount of effort allocated to A_{it}, the mass of projects of type i at time t. By definition, $\sum_i \eta_{it} = \eta$ and $\sum_i A_{it} = A_t$. In order for this representation to be valid, the η_{it}'s must be such that all groups of projects move at the same speed, so for all i, j we must have

$$\frac{\eta_{it}}{A_{it}}\Delta = \frac{\eta_{jt}}{A_{jt}}\Delta. \tag{30}$$

We conjecture, and later verify, that there exists a unique set of time-invariant $\{\eta_{it}\}_i = \{\eta_i\}_i$ that solves this equation. In this case each group of projects follows a constant growth path, and so from Proposition 2 we have $A_{it} = (\nu_i - \omega_i)\, t$. Substituting into equation (30) yields

$$\frac{\eta_i}{(\nu_i - \omega_i)} = \frac{\eta_j}{(\nu_j - \omega_j)}. \tag{31}$$

[16] However, equal treatment strategies are not necessarily optimal if the objective is not only to increase the output rate, but also to clear intermediate goals. This point will be addressed at the end of Section 6.2.

For all i, ν_i and ω_i are linked by expression (5) and so we may replace ω_i with $\Omega(\nu_i; \eta_i/X_i)$. Since $\Omega(\nu; \eta/X)$ is increasing in η, the left- and right-hand sides of equation (31) are increasing in η_i and η_j respectively. This fact implies that there exists a unique set $\{\eta_i\}_{i=1}^N$ which solves equation (31) and simultaneously meets the constraint $\sum_i \eta_{it} = \eta$. This verifies that our conjecture was correct.

Now, recall that expression (5) reads

$$(\nu_i - \omega_i)\frac{X_i}{\eta_i} = \log\left(\frac{\nu_i}{\omega_i}\right).$$

Substituting into (31) yields

$$\frac{\log\left(\frac{\nu_i}{\omega_i}\right)}{X_i} = \frac{\log\left(\frac{\nu_j}{\omega_j}\right)}{X_j},$$

or equivalently

$$\left(\frac{\nu_i}{\omega_i}\right)^{X_j} = \left(\frac{\nu_j}{\omega_j}\right)^{X_i}. \tag{32}$$

Thus, if $X_i < X_j$ then $\frac{\nu_i}{\omega_i} < \frac{\nu_j}{\omega_j}$. Moreover, equation (32) is verified we replace ω_i with $\nu_i \cdot K^{X_i}$, and do the same for ω_j. This means that, given η and a constellation of $\{\nu_i, X_i\}_i$, there exists a constant K such that for all i

$$\omega_i = \nu_i \cdot K^{X_i}.$$

The constant K cannot exceed 1 for otherwise $\omega_i > \nu_i$ for all i. The constant equals 1 only if completion time is zero, which requires $\eta \geq \sum_{i=1}^N \nu_i X_i$. Otherwise, $K < 1$. ■

A.7 Proofs for Section 6.5

Proof of Proposition 8

Proof. Define

$$\dot{\mathfrak{T}}(t) = \nu_t/\nu_0,$$

so that

$$\nu_t = \nu_0 \cdot \dot{\mathfrak{T}}(t)$$

and

$$\eta_t = \frac{\eta_t}{\nu_t} \cdot \nu_t = \frac{\eta_0}{\nu_0} \cdot \dot{\mathfrak{T}}(t) \cdot \nu_0 = \eta_0 \cdot \dot{\mathfrak{T}}(t).$$

Now use the function $\dot{\mathfrak{T}}(t)$ to define a synthetic "worker time scale" where

$$\mathfrak{T}(t) = \int_0^t \dot{\mathfrak{T}}(y)\ dy$$

represents how much worker time has elapsed between calendar times zero and t. Note that since $\mathfrak{T}(0) = 0$ both clocks, the worker clock and the calendar clock, start at the same time.We want to show that ν_t, η_t are stationary in worker time (even though they are not stationary in calendar time). To show this we compute that the mass of input which accrues between worker times $\mathfrak{t}$ and $\mathfrak{t} + \mathfrak{D}$ (here $\mathfrak{D}$ represents a unit of worker time). This mass is given by the mass of input which accrues between calendar times $\mathfrak{T}^{-1}(\mathfrak{t})$ and $\mathfrak{T}^{-1}(\mathfrak{t} + \mathfrak{D})$, which is

$$\begin{aligned}\int_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \nu_t\ dt &= \int_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \nu_0 \cdot \dot{\mathfrak{T}}(t)\ dt \\ &= \nu_0 \cdot \left[\mathfrak{T}(t)\Big|_{\mathfrak{T}^{-1}(\mathfrak{t})}^{\mathfrak{T}^{-1}(\mathfrak{t}+\mathfrak{D})} \right] = \nu_0 \cdot \mathfrak{D}.\end{aligned}$$

Dividing by $\mathfrak{D}$ gives the input rate in worker time, which is constant and equal to ν_0. The same argument shows that the effort rate is constant in worker time. Now, if $\nu_{\mathfrak{t}}$ and $\eta_{\mathfrak{t}}$ are constant in worker time $\mathfrak{t}$, then the theory developed in the previous sections applies with respect to worker time and guarantees that there is an output rate $\omega_{\mathfrak{t}}$ which is also constant in worker time $\mathfrak{t}$. These constant input, effort, and output rate are given by ν_0, η_0 and $\omega_0 = \Omega(\nu_0, \eta_0/X)$. Since the output rate is constant in worker time, it evolves through calendar time as $\omega_0 \cdot \dot{\mathfrak{T}}(t)$. This concludes the proof. ■