A Robust Approach to Risk Aversion:

Disentangling risk aversion and elasticity of substitution without giving up preference monotonicity[*]

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Abstract

We investigate whether the set of Kreps and Porteus (1978) preferences include classes of preferences that are stationary, monotonic with respect to first order stochastic dominance, and well-ordered in terms of risk aversion. We prove that the class of preferences introduced by Hansen and Sargent (1995) in their robustness analysis is the only one that fulfills these properties. The paper therefore suggests a shift from the traditional approach to studying the role of risk aversion in recursive problems. We also provide applications, in which we discuss the impact of risk aversion on asset pricing and risk sharing.

Keywords: risk aversion, recursive utility, risk-sensitive preferences, ordinal dominance, risk free rate, risk sharing.

JEL codes: D90, D81.

1 Introduction

Since Koopmans (1960)'s early article, the assumption of preference stationarity plays a central role in the modeling of intertemporal choice under uncertainty. For many problems, it is indeed meaningful to assume that the agent's objective is independent of past events and from the calendar year. Preference stationarity is then required to generate time consistent planning.

The economic literature abounds in works focusing on stationary preferences. In decision theory, Epstein (1983), Epstein and Zin (1989), Klibanoff, Marinacci, and Mukerji (2009) made significant contributions extending Koopmans' initial contribution to more general settings. The assumption of preference stationarity is even more present in applied contributions, as it leads to recursive economic problems that can be tackled by using dynamic programming methods. A substantial part of the macroeconomic literature relies on models that assume stationary preferences.

However, imposing preference stationarity makes it difficult to discuss the role of risk aversion. In infinite horizon settings, two distinct stationary preference relations cannot fit simultaneously into the expected utility framework and be comparable in terms of risk aversion. Discussing the role of risk aversion – while maintaining preference stationarity – then involves either departing from the expected utility framework, as in Epstein and Zin (1989), or considering preferences defined on smaller domains, as in Bommier (2012).[1]

Epstein and Zin's (1989) preferences are considered as the standard procedure to study the role of risk aversion in intertemporal problems. These preferences extend the Kreps and Porteus (1978) framework to a stationary setting, focusing on homothetic and weakly separable preferences.[2] They are found to be extremely useful, as well as being appreciated for their tractability. However, such an approach has several shortcomings. First, as emphasized in Chew and Epstein (1990), Epstein-Zin preferences generally fail to fulfill a property of monotonicity with respect to first order stochastic dominance, also called ordinal dominance. With these preferences, an agent may end up choosing lotteries that are first-order stochastically dominated by other available lotteries. Such preference non-monotonicity may have unpleasant consequences. In particular, as shown in a two-period setting by Bommier, Chassagnon and Le Grand (2012), Epstein-Zin preferences are not

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[1] In Bommier (2012), consumption remains constant after a finite amount of time. It is then possible to find classes of stationary preferences that fit into the expected utility framework and are well-ordered in terms of risk aversion.

[2] The property of "weak separability" refers to preferences over deterministic consumption paths. It holds when the marginal rate of substitution between consumption at two different dates is independent of consumption at other dates.

well-ordered in terms of aversion for marginal increases in risk. This may generate misleading conclusions about the role of risk aversion in standard applied problems such as the analysis of precautionary savings.

Our paper aims at identifying classes of stationary preferences, which disentangle ordinal and risk preferences, without having the above-mentioned shortcomings. In fact, we explore the entire set of recursive preferences that are consistent with the Kreps-Porteus framework and look for classes of preferences that fulfill ordinal dominance and are properly suited to study risk aversion. Our representation results show that there is a single class of preferences that fulfill these requirements. These preferences are represented by utility functions V_t solving the following recursion ($t \geq 0$):

$$V_t = \begin{cases} (1-\beta)u(c_t) - \frac{\beta}{k} \log(E_m \left[e^{-kV_{t+1}} \right]) & \text{if } k \neq 0, \\ (1-\beta)u(c_t) + \beta E_m[V_{t+1}] & \text{if } k = 0, \end{cases} \tag{1}$$

where β and k are two real scalars, u is an increasing real function, and $E_m[\cdot]$ is the expectation with respect to a measure m. The parameter k determines the degree of risk aversion: a larger k is associated with a greater aversion for (marginal or non-marginal) increases in risk. This class of so-called risk-sensitive preferences corresponds to the one introduced by Hansen and Sargent (1995) in their approach to robustness analysis. As noted by Tallarini (2000) and Hansen and Sargent (2007), this class intersects the Epstein-Zin isoelastic preferences, when u is a log function (i.e., the intertemporal elasticity of substitution is one) or when $k = 0$ and u is isoelastic.[3] Otherwise, risk-sensitive and Epstein-Zin preferences are of different natures.

In order to illustrate the gains of using a well-ordered specification in terms of risk aversion, we develop two applications. In the first one, we investigate how risk aversion affects the risk-free rate and the market price of risk in a simple endowment economy. Relying on risk-sensitive preferences, defined by the recursion (1), we prove that, under fairly general conditions, a greater risk aversion means a lower risk-free rate and a larger market price of risk. Such a finding has clear-cut consequences when discussing key issues such as the choice of a proper social discount rate to evaluate public policy. In particular, the social discount rate should covariate negatively with the planner's risk aversion.

The second application focuses on risk sharing in a closed economy, in which two price-taker agents endowed with risk-sensitive preferences face an identical risk. The market equilibrium is characterized by a risk transfer from the less risk averse agent to the more risk averse agent, implementing therefore an intuitive risk reallocation.

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[3]The latter case also corresponds to the expected utility with time-separable preferences.

The remainder of the paper is organized as follows. In the next section, we expose the setting. Section 3 formalizes the notion of ordinal dominance and provides two representation results related to recursive preferences fulfilling ordinal dominance. These representation results can be viewed as the core contributions of the paper. Section 4 discusses the notion of comparative risk aversion and shows that preferences obtained in Section 3 are well-ordered in terms of risk aversion. Section 5 deals with applications and explores the role of risk aversion in an endowment economy and for risk sharing in a general equilibrium setup. Section 6 discusses further properties of risk-sensitive preferences and compares them with Epstein-Zin preferences. Section 7 concludes.

2 The setting

2.1 Preference domain

We consider preferences defined over the set of temporal lotteries in an infinite horizon setting. Time is discrete and indexed by $t = \{0, 1, \ldots\}$. For the sake of simplicity, we assume that per-period consumption is bounded. We note $C = [\underline{c}, \overline{c}] \subset \mathbb{R}^{++}$ the set of possible instantaneous consumptions at any date t, where $0 < \underline{c} < \overline{c}$. C is a compact Polish space. We denote by C^∞ the set of possible deterministic consumption paths, which is also a compact Polish space (by Tychonoff's theorem). We construct the set of temporal lotteries following Kreps and Porteus (1978) and Epstein and Zin (1989). Wakai (2007) also provides a detailed and precise construction of a similar consumption space.

More precisely, we define D_0 as the set of all singleton subsets of C^∞ (with a slight abuse of notation, $D_0 = C^\infty$). Then, for all $t \geq 1$, we define D_t by induction with:

$$D_t = C \times M(D_{t-1}),$$

where $M(D_{t-1})$ is the space of probability measures on D_{t-1} endowed with the Prohorov metric (metric of weak convergence).[4] It can be noticed that for all t, $D_t \subset D_{t+1}$. Moreover, we know that by induction all sets D_t are compact Polish spaces. The space of temporal lotteries D is then defined by:

$$D = \{(d_1, \ldots, d_t, \ldots) : \ \forall t \geq 1, \ d_t \in D_t \quad \text{and} \quad d_t = g_t(d_{t+1})\}, \tag{2}$$

where $g_t : D_{t+1} \to D_t$ is a projection on D_t, formally defined in Epstein and Zin (1989).

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[4]More generally, for any metric space X, $M(X)$ denotes the space of Borel probability measures on X endowed with the weak convergence topology.

To make it short, for all $d_{t+1} \in D_{t+1}$, the temporal lottery $g_t(d_{t+1}) \in D_t$ generates the same distribution of outcomes as d_{t+1} but the uncertainty vanishes in period t instead of period $t+1$. The space D defined by (2) can be shown to be a compact Polish space and homeomorphic to $C \times M(D)$. Moreover, the set $\bigcup_{t \geq 0} D_t$ is dense in D.

For practical purposes, we introduce two notations that will be useful in the remainder of the paper. First, since we often consider constant consumption paths, we denote by c_∞ the element of C^∞ providing the same consumption $c \in C$ in every period. Second, for any $t \geq 0$, we define the set Δ_t by:

$$\Delta_t = \bigcup_{\tau \geq 0} C^\tau \times D_t. \tag{3}$$

While the set D_t gathers lotteries for which the uncertainty is resolved prior to the beginning of period t, the set Δ_t contains lotteries for which the uncertainty is resolved within at most t successive periods (between period $\tau + 1$ and period $\tau + t$ for some $\tau \geq 0$). Lotteries in Δ_t may therefore include an arbitrary number of initial periods with a deterministic consumption before the uncertainty resolves. By construction, the set $\Delta_0 = D_0 = C^\infty$ is the set of deterministic consumption paths that we denote C^∞ throughout the paper for the sake of clarity. It is also straightforward to check that $\Delta_t \subset \Delta_{t+1}$ for all t and that $\bigcup_{t \geq 0} \Delta_t = \bigcup_{t \geq 0} D_t$. The set $\bigcup_{t \geq 0} \Delta_t$ is thus dense into D.

2.2 Recursive Kreps-Porteus preferences

Our paper explores the set of recursive preferences defined on D that fits into the framework introduced by Kreps and Porteus (1978). We restrict our attention to monotonic preferences. The elements $\underline{c}_\infty$ and $\overline{c}_\infty$ of D therefore provide respectively the lowest and highest levels of utility. By utility normalization, there is no loss of generality in assuming that a utility function U defined on D needs to fulfill $U(\underline{c}_\infty) = 0$ and $U(\overline{c}_\infty) = 1$. This leads to the following formal definition:

Definition 1 (Kreps-Porteus recursive preferences) *A utility function $U : D \rightarrow [0,1]$, such that $U(\underline{c}_\infty) = 0$ and $U(\overline{c}_\infty) = 1$ is said to be Kreps-Porteus recursive (henceforth KP-recursive) if and only if there exists a continuously differentiable function $W : (x,y) \in C \times [0,1] \mapsto W(x,y) \in [0,1]$ with strictly positive partial derivatives W_x and W_y, such that for all $c_0 \in C$ and $m \in M(D)$:*

$$U(c_0, m) = W(c_0, E_m[U]), \tag{4}$$

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where $E_m[\cdot]$ denotes the expectation with respect to the probability measure m.

The function W will be called an admissible aggregator for the KP-recursive utility function. Moreover, a preference relation on D will be said to be KP-recursive if and only if it can be represented by a KP-recursive utility function.

The most common example of KP-recursive utility function is the additive separable one:

$$U(c_0, m) = (1 - \beta)E_m\left[\sum_{i=0}^{+\infty} \beta^i u(c_i)\right],$$

where $0 < \beta < 1$ and $u : C \to [0, 1]$ is a continuously differentiable function such that $u(\underline{c}) = 0$, $u(\overline{c}) = 1$ and $u' > 0$. For this additively separable specification, equation (4) holds when:

$$W(x, y) = (1 - \beta)u(x) + \beta y. \tag{5}$$

Another famous example of KP-recursive preferences is the Epstein-Zin isoelastic preferences, usually represented by utility functions fulfilling the following recursion (see Epstein and Zin (2001) for example):[5]

$$V(c, m) = \begin{cases} \left[(1 - \beta)c^\rho + \beta(E_m[V^\alpha])^{\frac{\rho}{\alpha}}\right]^{\frac{1}{\rho}} & \text{if } 0 \neq \rho < 1, \alpha \neq 0, \\ \exp\left((1 - \beta)\log(c) + \frac{\beta}{\alpha}\log(E_m[V^\alpha])\right) & \text{if } \rho = 0, \alpha \neq 0, \\ [(1 - \beta)c^\rho + \beta\exp(\rho E_m[\log(V)])]^{\frac{1}{\rho}} & \text{if } 0 \neq \rho < 1, \alpha = 0, \\ \exp\left((1 - \beta)\log(c) + \beta E_m[\log(V)]\right) & \text{if } \rho = \alpha = 0, \end{cases} \tag{6}$$

with $0 < \beta < 1$. The utility function $V(c, m)$ is not KP-recursive in the sense of Definition 1, but an equivalent representation obtained by choosing:[6]

$$U(c, m) = \begin{cases} \dfrac{V(c, m)^\alpha - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha} & \text{if } \alpha \neq 0, \\ \dfrac{\log(V(c, m)) - \log(\underline{c})}{\log(\overline{c}) - \log(\underline{c})} & \text{if } \alpha = 0, \end{cases}$$

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[5]The cases where α or ρ are equal to zero can be obtained as limit cases of the general one ($\alpha \neq 0$ and $\rho \neq 0$). We provide their explicit formulations because this makes it easier to see the link with risk-sensitive preferences that are introduced later on.

[6]In the remainder of the paper, V refers to a non-normalized utility function, while U refers to a normalized one.

fulfills the requirements of Definition 1 with the following aggregator:

$$W(x,y) = \begin{cases} \dfrac{\left((1-\beta)x^\rho + \beta\left[y\left(\overline{c}^\alpha - \underline{c}^\alpha\right) + \underline{c}^\alpha\right]^{\frac{\rho}{\alpha}}\right)^{\frac{\alpha}{\rho}} - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha} & \text{if } 0 \neq \rho < 1, \alpha \neq 0, \\[2ex] \dfrac{\exp\left(\alpha(1-\beta)\log(x) + \beta\log(y\left(\overline{c}^\alpha - \underline{c}^\alpha\right) + \underline{c}^\alpha)\right) - \underline{c}^\alpha}{\overline{c}^\alpha - \underline{c}^\alpha} & \text{if } \rho = 0, \alpha \neq 0, \\[2ex] \dfrac{\frac{1}{\rho}\log\left((1-\beta)x^\rho + \beta\overline{c}^{\rho y}\underline{c}^{\rho(1-y)}\right) - \log(\underline{c})}{\log(\overline{c}) - \log(\underline{c})} & \text{if } 0 \neq \rho < 1, \alpha = 0, \\[2ex] (1-\beta)\dfrac{\log(x) - \log(\underline{c})}{\log(\overline{c}) - \log(\underline{c})} + \beta y & \text{if } \rho = \alpha = 0, \end{cases} \tag{7}$$

Even though the above examples (standard additive expected utility and Epstein-Zin preferences) are only two particular cases among a number of other possibilities, they are by far the most widely used in economics. However, they are inappropriate to study the role of risk aversion.[7]

Our purpose in the current paper is to look for classes of KP-recursive preferences that fulfill ordinal dominance and are suitable to discuss the role of risk aversion. We prove that this leads to preferences that can be represented by a utility function $V(c,m)$ fulfilling:

$$V(c,m) = \begin{cases} (1-\beta)u(c) - \frac{\beta}{k}\log(E_m\left[e^{-kV}\right]) & \text{if } k \neq 0, \\ (1-\beta)u(c) + \beta E_m[V] & \text{if } k = 0, \end{cases} \tag{8}$$

for some function u. This specification was first introduced in Hansen and Sargent (1995) as a tractable way to have a *risk-adjusted measure of cost* in a problem of optimal control. In Hansen, Sargent and Tallarini (1999), such a specification was used to represent the preferences of robust decision makers.[8] Due to the parallel between robustness and risk-sensitivity analysis, which is discussed in Hansen, Sargent and Tallarini (1999), these preferences might indifferently be called *robust preferences* or *risk-sensitive* preferences. Along the paper, we use the latter terminology (risk-sensitive preferences), which is more commonly used in the recent literature, as for example in the survey of Backus, Routledge and Zin (2005).[9] Here again, the utility representation in (8) does not fulfill the

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[7]See Bommier, Chassagnon and Le Grand (2012).

[8]"Robust agents" worry about possible model misspecifications, and account for them in their decisions. Hansen and Sargent (2007) provide a self-contained description of robustness applications in economics.

[9]The terminology *robust preferences* is now more often used to refer to max-min preferences.

requirements of Definition 1. But the utility function U defined by:

$$U(c,m) = \begin{cases} \dfrac{e^{-kV(c,m)} - e^{-ku(\underline{c})}}{e^{-ku(\overline{c})} - e^{-ku(\underline{c})}} & \text{if } k \neq 0, \\[2mm] \dfrac{V(c,m) - u(\underline{c})}{u(\overline{c}) - u(\underline{c})} & \text{if } k = 0, \end{cases}$$

represents the same preferences as V and fulfills Definition 1 requirements with the aggregator:

$$W(x,y) = \begin{cases} \dfrac{e^{-k(1-\beta)(u(x)-u(\underline{c}))} \left[1 + y(e^{-k(u(\overline{c})-u(\underline{c}))} - 1)\right]^{\beta} - 1}{e^{-k(u(\overline{c})-u(\underline{c}))} - 1} & \text{if } k \neq 0, \\[3mm] (1-\beta)\dfrac{u(x) - u(\underline{c})}{u(\overline{c}) - u(\underline{c})} + \beta y & \text{if } k = 0, \end{cases} \tag{9}$$

Choosing $u(c) = \log(c)$ in equation (9) yields exactly the same aggregator as in (7) with $\rho = 0$ and $\alpha = -k$. Thus, when the intertemporal elasticity of substitution is equal to one, Epstein-Zin preferences and risk-sensitive preferences coincide with each other, as was noticed by Tallarini (2000) for example. The class of risk-sensitive preferences also intersects with Epstein-Zin's one when $k = 0$ and u is isoelastic and concave. This corresponds to the standard additively separable model with a constant positive intertemporal elasticity of substitution. In all other cases, risk-sensitive preferences differ from Epstein-Zin's ones.

We now introduce the notion of ordinal dominance and show how it restricts our attention to risk-sensitive preferences.

3 Recursive preferences fulfilling ordinal dominance

3.1 Definition of ordinal dominance

The basic idea of ordinal dominance is that an individual should not prefer a lottery that is stochastically dominated at the first order by another one. When considering "atemporal uncertainty setting" – in the sense that uncertainty is always resolved with the same timing – the notion of ordinal dominance is relatively standard and can be found for example in Chew and Epstein (1990).

In order to define first order stochastic dominance over temporal lotteries, we proceed by induction on the sets Δ_t for $t = 0, 1, \ldots$ for which the uncertainty is resolved in t consecutive periods of time, between dates $\tau + 1$ and $t + \tau$ for some $\tau \geq 0$. For sake of simplicity, a typical element of Δ_t ($t \geq 1$) is still denoted (c,m), where $c \in C^{\tau+1}$ for some $\tau \geq 0$ and $m \in M(D_{t-1})$. The set $C^{\infty} = \Delta_0$ of deterministic consumption paths

is assumed to be endowed with a preference relation $\succeq_0$. We define a notion of ordinal dominance FSD_1 on Δ_1 as follows:

$$
\begin{cases}
\forall \left((c,m), (c',m') \right) \in \Delta_1 \times \Delta_1, \ (c,m) \ FSD_1 \ (c',m') \Leftrightarrow \\
c, c' \in C^\tau \text{ for some } \tau \geq 1 \text{ and} \\
\forall x \in C^\infty, \ m \left(\{ x' \in C^\infty | (c,x') \succeq_0 x \} \right) \geq m' \left(\{ x' \in C^\infty | (c',x') \succeq_0 x \} \right)
\end{cases}
\tag{10}
$$

By construction of FSD_1, comparisons are restricted to lotteries that have an identical timing of resolution of uncertainty. The definition of FSD_1 states that a temporal lottery in Δ_1 dominates at the first order another temporal lottery in Δ_1 with the same timing of uncertainty resolution if for any outcome $x \in C^\infty$, the probability that a realization of (c,m) is preferred to x is greater than the probability that a realization of (c',m') is preferred to x. This definition of first order stochastic dominance is the same as that of Chew and Epstein (1990).

To extend this first order dominance relationship to all Δ_t, we proceed as follows: Given a relation of first order stochastic dominance FSD_t on Δ_t, we define a relation of first order stochastic dominance FSD_{t+1} on Δ_{t+1} as follows ($t \geq 1$):

$$
\begin{cases}
\forall \left((c,m), (c',m') \right) \in \Delta_{t+1} \times \Delta_{t+1}, \ (c,m) \ FSD_{t+1} \ (c',m') \Leftrightarrow \\
c, c' \in C^\tau \text{ for some } \tau \geq 1 \text{ and} \\
\forall x \in \Delta_t, \ m \left(\{ x' \in D_t | (c,x') \ FSD_t \ x \} \right) \geq m' \left(\{ x' \in D_t | (c',x') \ FSD_t \ x \} \right)
\end{cases}
\tag{11}
$$

The interpretation of (11) is very similar to the definition of FSD_1 in (10), except that for FSD_{t+1}, we compare probabilities that lottery realizations of (c,m), which are elements of Δ_t, first order dominate a given $x \in \Delta_t$. Actually, the definition (10) can be seen as a particular case of the recursive definition (11), where FSD_0 is set equal to $\succeq_0$. Moreover, it is straightforward to check that if for some $t \geq 0$ we have $x, y \in \Delta_t$ such that $x FSD_t y$ then we also have $x FSD_{t+1} y$. The partial order FSD_{t+1} can therefore be considered as an extension of FSD_t making it possible to compare elements of Δ_{t+1}.

Our relation of stochastic dominance only depends on preferences over deterministic outcomes. As a consequence, two agents endowed with the same ordinal preferences always agree on whether a temporal lottery stochastically dominates another one, a property that turns out to be essential for comparing risk aversion. Nevertheless, since the notion of stochastic dominance may depend on ordinal preferences, we readily implicitly admit that risk aversion is only comparable among agents that have the same ranking over determin-

istic consumption paths. In that respect, we follow most of the literature on comparative risk (and ambiguity) aversion, including Kilhstrom and Mirman (1974), Epstein and Zin (1989), Chew and Epstein (1990) and Klibanoff, Marinacci, and Mukerji (2009).

By construction, our definition of first order stochastic dominance is only valid for lotteries that resolve in a finite amount of time, but as $\bigcup_{t \geq 0} \Delta_t = \bigcup_{t \geq 0} D_t$ is dense into D, it is sufficient to define a strong enough notion of ordinal dominance:

Definition 2 (Ordinal dominance) *Consider a relation of preferences $\succeq$ on D. Denote $\succeq_0$ its restriction to C^∞, from which the relations of first order stochastic dominance FSD_t are defined as in (10) and (11). The relation of preferences $\succeq$ is said to fulfill ordinal dominance, if for all $t \geq 0$ and all (c, m) and $(c', m') \in \Delta_t$, the following implication holds:*

$$(c, m) \ FSD_t \ (c', m') \Rightarrow (c, m) \succeq \ (c', m').$$

Ordinal dominance imposes some coherence between the relation of preferences $\succeq$ and its restriction $\succeq_0$ to the set of deterministic consumption space C^∞; or, put differently, between risk preferences and ordinal preferences. However, this remains quite a minimalist assumption and simply states that an individual prefers a lottery which stochastically dominates another one at the first order. In the expected utility framework, ordinal dominance is equivalent to assuming that the von-Neumann Morgenstern utility index (used to compute expected utility) is non-decreasing with respect to the order that is used to define first order stochastic dominance. In dimension one, with preferences over lotteries with outcomes in $\mathbb{R}^+$ (endowed with its natural order), ordinal dominance involves assuming that the utility index is a non-decreasing function.

The property of ordinal dominance seems to be no less desirable for temporal lotteries, even if its expression is less obvious. In particular, the monotonicity of the aggregator W, assumed in Definition 1, is not sufficient to ensure that preferences fulfill ordinal dominance. For example, Chew and Epstein (1990) explain that Epstein-Zin isoelastic preferences do not fulfill ordinal dominance, though the aggregator is monotonic. Bommier, Chassagnon and Le Grand (2012), as well as Section 6.2 of the current paper highlight some counterintuitive results derived when preferences do not fulfill ordinal dominance.

3.2   Representation results

The present section sets forth two representation results, affording the paper's central contribution. The first result shows that imposing recursivity and ordinal dominance readily leaves us with a small set of KP-recursive preferences.

Proposition 1 (Representation result) *A KP-recursive preference relation fulfills ordinal dominance if and only if it can be represented by a KP-recursive utility function that admits an aggregator $W : (x, y) \in C \times [0, 1] \mapsto W(x, y) \in [0, 1]$ taking one of the following forms:*

1.

$$W(x, y) = a(x) + b(x)y, \tag{12}$$

where $a, b : C \to [0, 1]$ are two continuously differentiable functions such that $a(\underline{c}) = 0$, $a(\overline{c}) + b(\overline{c}) = 1$ and for all $x \in C$, $a'(x) > 0$, $a'(x) + b'(x) > 0$ and $0 < b(x) < 1$.

2.

$$W(x, y) = \frac{1 - e^{-k(1-\beta)u(x)}\left(1 - (1 - e^{-k})y\right)^{\beta}}{1 - e^{-k}}, \tag{13}$$

where $0 < \beta < 1$, $k \neq 0$ and $u : C \to [0, 1]$ is a continuously differentiable function with a strictly positive derivative such that $u(\underline{c}) = 0$ and $u(\overline{c}) = 1$.

Moreover, for any such aggregator, there corresponds a unique KP-recursive preference relation.

Proof: See appendix.

Preferences obtained with (12) are in fact of the expected utility kind and correspond to those introduced in continuous time by Uzawa (1968), and discussed further in discrete time by Epstein (1983). Preferences obtained with the aggregator (13) correspond to risk-sensitive preferences. Moreover, in the limit case when k approaches zero, the aggregator (13) converges to the one in (12) with a constant function b and therefore leads to the class of additively separable utility functions.

Proposition 1 shows that imposing ordinal dominance leads us to restrict our attention to preferences *à la* Uzawa or to risk-sensitive preferences. However, preferences *à la* Uzawa turn out to be of little help in studying the role of risk aversion, as they do not make it possible to disentangle ordinal and risk preferences. Indeed, the following result shows that risk-sensitive preferences are the only potential candidates to study risk aversion.

Proposition 2 (Comparability of preferences) *Consider two KP-recursive preference relations $\succeq^A$ and $\succeq^B$ on D, which fulfill the ordinal dominance property and whose restrictions on C^{∞} are identical. Then:*

— *either both preference relations are identical: $\succeq^A = \succeq^B$,*

– or preferences relations $\succeq^A$ and $\succeq^B$ can be represented with KP-recursive utility functions, with admissible aggregators W^A and W^B such that for $i = A, B$ and $(x, y) \in C \times [0, 1]$:

$$W^i(x, y) = \begin{cases} \frac{1 - \exp(-k_i(1-\beta)u(x))(1 - y(1 - e^{-k_i}))^\beta}{1 - e^{-k_i}}, & k_i \neq 0, \\ (1 - \beta)u(x) + \beta y, & k_i = 0, \end{cases}$$

where $0 < \beta < 1$, $k_A, k_B \in \mathbb{R}$ and $u : C \to [0, 1]$ is a continuously differentiable function with a strictly positive derivative and $u(\underline{c}) = 0$, $u(\overline{c}) = 1$.

Proof: See appendix.

Proposition 2 shows that imposing ordinal dominance and non-trivial comparability of preferences reduces the sets of possible aggregators to those of risk-sensitive preferences. The class of risk-sensitive preferences is thus the only one – within the set of KP-recursive preferences – that may provide an appropriate basis to study the role of risk aversion.

One aspect of risk-sensitive preferences is that they are weakly separable, providing over C^∞ the same ranking as the standard additive utility function with constant discounting.[10] Interestingly enough, this property of weak separability was not introduced as an assumption – as it was for example in Epstein and Zin (1989), Chew and Epstein (1990) or Klibanoff, Marinacci, and Mukerji (2009). However, it is found to be a necessary condition to study the role of risk aversion while assuming preference stationarity. Further insight into why weak separability of preferences is a pre-requisite for comparing risk aversion of stationary preferences will be provided in Section 6.1.

The result of Proposition 2 is obtained without making precise statements regarding the meaning of comparative risk aversion. We now show that risk-sensitive preferences are indeed well-ordered in terms of risk aversion, even though we use strong notions of comparative risk aversion such as the one in Bommier, Chassagnon and Le Grand (2012).

4   Comparative risk aversion

As explained for instance in Chateauneuf, Cohen, Meijilson (2004), different notions of risk increases are associated with different notions of risk aversion, and may possibly yield different characterizations for utility functions. Similarly, different riskiness comparisons generate different notions of comparative risk aversion. We formalize the link between any risk relation and its corresponding notion of comparative risk aversion as follows:

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[10]The restriction of risk-sensitive references to C^∞ are represented by the utility function $U_0(c_0, c_1, ...) = (1 - \beta) \sum_{t=0}^{+\infty} \beta^t u(c_t)$.

Definition 3 (Comparative risk aversion) *Consider a binary relation R ("riskier than") defined over the set of temporal lotteries D. For any two preference relations $\succeq^A$ and $\succeq^B$ on D, the preference relation $\succeq^A$ on D is said to exhibit greater (R)–risk aversion than $\succeq^B$ if and only if for all (c, m) and (c', m') in D we have:*

$$\big((c,m) \ R \ (c',m') \ and \ (c,m) \ \succeq^A \ (c',m')\big) \Rightarrow \ (c,m) \ \succeq^B \ (c',m').$$

This procedure for defining comparative risk aversion from a riskiness relation can be attributed to Yaari (1969), even if Yaari only applied it to a very particular relation "riskier than" (the relation R_M introduced below). It was used in many theoretical contributions discussing the definition of risk (or inequality) aversion, including Grant and Quiggin (2005) or Bosmans (2007). Intuitively, if an agent A is more risk averse than B, any increase in risk perceived as worthwhile by A should also be perceived as worthwhile by B. One may apply this procedure to different notions of increase in risk, that is to different relations R. One possibility consists in focusing, as in Yaari (1969) and most of the subsequent literature, on comparisons between lotteries where at least one is degenerate and embeds no uncertainty (i.e., pays off a given outcome for sure).

Definition 4 (Minimalist risk comparison R_M) *We define the binary relation R_M as follows:*

$$\forall (c,m), (c',m') \in D, \quad (c,m) \ R_M \ (c',m') \ \Leftrightarrow \ (c',m') \in C^\infty.$$

This notion of comparative riskiness is the most minimalist one (from there the denomination R_M taken from Bosmans, 2007). It states that all degenerate lotteries, with no risk, are minimal elements in terms of the relation "riskier than". However, it excludes any comparison between two non-degenerate lotteries. The risk comparison R_M does not enable us to consider small variations in risk around a non-degenerate lottery. As a consequence, when using risk comparison R_M, the notion of marginal risk variations does not make sense.

Applying Definition 3 to the risk comparison R_M, which in fact involves focusing on certainty equivalents, is by far the most common way in the economic literature of defining comparative risk aversion. Kilhstrom and Mirman (1974), Epstein and Zin (1989), and Chew and Epstein (1990) for example, opt for this solution. We have the following result:

Proposition 3 (Weak comparative risk aversion) *Consider two KP-recursive utility functions U^A and U^B with aggregators W^A and W^B given by ($i = A, B$ and $(x, y) \in$*

$C \times [0, 1]$):

$$W^i(x, y) = \begin{cases} -\frac{\exp(-k_i(1-\beta)u(x))(1-y(1-e^{-k_i}))^\beta - 1}{1 - e^{-k_i}}, & k_i \neq 0, \\ (1-\beta)u(x) + \beta y, & k_i = 0, \end{cases} \quad (14)$$

with $k_A, k_B \in \mathbb{R}$, $0 < \beta < 1$, and $u : C \to [0, 1]$ a strictly increasing continuously derivable function.

If $k_A \geq k_B$, then the preferences represented by U^A exhibit more (R_M)-risk aversion than the preferences represented by U^B.

Proof: In appendix.

Proposition 3 tells us that the class of risk-sensitive preferences is well-ranked in terms of risk aversion, at least when using the less demanding (and also the most common) notion of comparative risk aversion. A similar property also holds with Epstein-Zin preferences. Paralleling the terminology of Chateauneuf, Cohen, Meijilson (2004), we say that risk-sensitive preferences, as well as Epstein-Zin preferences, are *weakly well-ordered* in terms of risk aversion. Focusing on the relation R_M, and the associated notion of comparative risk aversion, seems however minimal since R_M does not make it possible to consider marginal variations in risk and therefore aversion for marginal increases in risk. Such marginal risk variations may be important for applied problems, such as portfolio choices.

In order to be able to consider marginal risk variations, a risk relation that is less incomplete than R_M has to be considered. The literature on unidimensional lotteries suggests many notions of risk increases allowing for marginal comparisons. The most popular are those related to mean preserving spreads, second order stochastic dominance, or to Bickel-Lehman (1976) dispersions. A common drawback of these notions is that they are not invariant to a non-linear rescaling of the outcome space. Indeed, they assume that the set of outcomes is not only ordered but also admits a particular cardinalization. Using these notions for lotteries with outcomes in C^∞ would require C^∞ to be endowed with a given cardinalization, which would look arbitrary. Instead, in line with Jewitt (1989) or Bommier, Chassagnon and Le Grand (2012), we suggest using the single crossing of the cumulative distribution function as an indicator of an unambiguous increase or decrease in risk. Single crossing being unaffected by a monotonic rescaling, the associated risk relation depends only on the order of C^∞, that is on preferences under certainty. This leads to the following definition:

Definition 5 (Extended risk comparison R_{SC}) *Consider a relation of preferences $\succeq_0$ over C^∞ which is used to define the relation of first order stochastic dominance FSD_1 as*

in (10).

For (c, m) and $(c', m') \in D$, we state that $(c, m) R_{SC}(c', m')$ if $(c', m') \in C^\infty$ or (c, m) and $(c', m') \in \Delta_1$ and there exist $p \in [0, 1]$ and m_0, m_0', m_1, $m_1' \in M(C^\infty)$ such that:

$$(c, m) = p(c, m_0) \oplus (1 - p)(c, m_1), \tag{15}$$

$$(c', m') = p(c', m_0') \oplus (1 - p)(c', m_1'), \tag{16}$$

where $\oplus$ denotes the mixture operator, and:

$$\left(m_0' \times m_1' \right) \{ (x_0, x_1) \in C^\infty \times C^\infty | (c', x_1) \succeq_0 (c', x_0) \} = 1, \tag{17}$$

so that the probability that an outcome of m_1 first order dominates an outcome of m_0 equals 1, and the same holds for m_1' and m_0', and:

$$(c', m_0') \ FSD_1 \ (c, m_0),$$
$$(c, m_1) \ FSD_1 \ (c', m_1').$$

Intuitively, a lottery (c, m) is said to be riskier than a lottery (c', m'), either if this latter is with no risk, or if we can decompose each lottery into a two stage lottery, where in the first stage a random draw decides whether a "bad lottery" (i.e., (c, m_0) or (c', m_0')) or a "good lottery" (i.e., (c, m_1) or (c', m_1')) is played at the second stage. Moreover, these lotteries must have the following properties: (i) outcomes of bad lotteries are always worse than outcomes of good lotteries; (ii) conditional on playing a bad lottery, (c, m) is worse than (c', m') in terms of first order stochastic dominance, while conditional on playing a good lottery, the contrary holds; (iii) the outcomes of good and bad lotteries are riskless (i.e., elements of C^∞). In the end, (c, m) is worse than (c', m') in bad cases, but better than (c', m') in good cases, which makes it intuitive that (c, m) is riskier than (c', m').

By definition, the risk comparison R_{SC} depends on the preferences $\succeq_0$ over C^∞, the set of deterministic consumptions paths. People with different views on the ranking of deterministic consumption paths would therefore have diverging views on whether one lottery is riskier than another one. Indeed, agents need to be able to compare pay-offs of lotteries before assessing the risk of lotteries. When pay-offs are unidimensional, the comparison is trivial, but when pay-offs are consumption vectors, their comparison may be a matter of taste, embedded within the preference relation $\succeq_0$ (a similar discussion can be found in Kihlstrom and Mirman, 1974). However, the risk comparison R_{SC} is independent of any specific utility representation for $\succeq_0$, and in particular from any cardinalization. It

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therefore relies exclusively on aspects of preferences that can be revealed by choices under certainty.

The relation R_{SC} extends the relation R_M since $(c, m) \, R_M \, (c', m') \Rightarrow (c, m) \, R_{SC} \, (c', m')$. However, this risk relation R_{SC} is still relatively minimal, as it does not allow elements, which are neither in C^{∞} nor in Δ_1 to be compared. Focusing on this extension avoids controversial issues related to the comparison, in terms of riskiness, of temporal lotteries whose outcomes are themselves risky objects. Moreover, it is sufficient to highlight some crucial differences between our framework and more conventional ones.

Proposition 4 (Strong comparative risk aversion) *Consider two KP-recursive utility functions U^A and U^B with aggregators W^A and W^B as in Proposition 3 (i.e., risk-sensitive preferences with $k_A \geq k_B$). Then, the preferences represented by U^A exhibit more (R_{SC})-risk aversion than the preferences represented by U^B.*

Proof: See appendix.

This proposition establishes that risk-sensitive preferences obtained in Proposition 2 are well-ordered in terms of risk aversion, even if we use the stronger notion of comparative risk aversion based on the risk comparison R_{SC}. Consistent with the terminology suggested above, we state that risk-sensitive preferences are *strongly well-ordered* in terms of risk aversion, a property which is not shared with Epstein-Zin preferences (unless the intertemporal elasticity of substitution is equal to one).[11]

As risk-sensitive preferences are strongly well-ordered in terms of risk aversion, we may expect them to lead to intuitive conclusions as to the role of risk aversion when applied, for example, to asset pricing or risk sharing. Such applications are developed in the next section. We do indeed obtain meaningful results regarding the impact of risk aversion. In Section 6.2, we also report that addressing the same applications with preferences, which are weakly, but not strongly, well-ordered in terms of risk aversion, may lead to counter-intuitive conclusions.

5 Applications

In order to illustrate how risk-sensitive preferences yield intuitive conclusions regarding the role of risk aversion, we develop two simple applications. The first relates to the risk

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[11] A formal proof that Epstein-Zin preferences assuming an intertemporal elasticity of substitution different from one are not strongly well-ordered in terms of risk aversion can be found in Bommier, Chassagnon and Le Grand (2012).

free rate and the risk premium in a random endowment economy. The second concerns risk sharing in general equilibrium.

5.1 The risk free rate and the risk premium

We consider a random endowment economy in which c_t is random at each date (but typically not independently distributed). We focus on the risk free rate and the risk premium. More precisely, we compute the pricing kernel generated by risk-sensitive preferences and show that its mean (i.e., the inverse of the gross risk free rate) and the ratio of its standard deviation to its mean (i.e., one major driver of the risk premium as notably illustrated by the Hansen-Jagannathan (1991) bound) increases with risk aversion.

In the risk-sensitive approach, the utility at any date t can be expressed as follows:[12]

$$V_t = (1 - \beta)u(c_t) - \frac{\beta}{k} \log(E_t \left[e^{-kV_{t+1}} \right]), \tag{18}$$

where $E_t[\cdot]$ is the expectation conditional on the information available at date t (i.e., the σ-algebra generated by the process c_τ for $0 \le \tau \le t$).

The pricing kernel $M_{t,t+1}$, which is the intertemporal rate of substitution between dates t and $t + 1$, can be expressed as follows for risk-sensitive preferences (equation (18)):

$$M_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{\exp(-kV_{t+1})}{E_t[\exp(-kV_{t+1})]} \tag{19}$$

From the pricing kernel, we readily deduce the (gross) risk free rate $R_t^{-1} = E_t[M_{t,t+1}]$ and the market price of risk equal to $\frac{\sigma_t(M_{t,t+1})}{E_t[M_{t,t+1}]}$, where $\sigma_t(M_{t,t+1})$ denotes the standard deviation of $M_{t,t+1}$ conditional on the information available at date t. We have the following proposition regarding the impact of risk aversion on the risk free rate and market price of risk.

Proposition 5 (Endowment economy) *We assume that: (i) $k > 0$; (ii) at any date t, c_t, V_t, and $\frac{E_t[V_{t+1}e^{-kV_{t+1}}]}{E_t[e^{-kV_{t+1}}]}$ are (weakly) comonotonic;[13] (iii) at any date t, the utility V_{t+1} conditional on date t information admits a density defined on a compact set of $\mathbb{R}$. Then, a larger risk aversion through a larger k in (18) implies:*

  — a smaller risk free rate,

  — and a larger market price of risk.

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[12] For sake of clarity, we conduct applications using non-normalized utility functions V (equation (8)).

[13] Two random variables X and Y are (weakly) comonotonic if there exists a non-decreasing function f such that $X = f(Y)$ or $Y = f(X)$.

Proof: In appendix.

The assumption regarding the compact support of V_{t+1} is only a technicality ensuring the existence of all moments. We could also assume a more general distribution, while supposing that all considered moments do exist. Since the fraction $\frac{E_t[V_{t+1}e^{-kV_{t+1}}]}{E_t[e^{-kV_{t+1}}]}$ is a weighted expectation of future utility, the assumption of comonotonicity in Proposition 5 means that good news for consumption in period t cannot be bad news for the utility V_t in period t and for the (weighted) expected future utility. Such an assumption is for example always fulfilled when there is only one risky period or when the $(c_\tau)_{\tau \geq 0}$ are independently distributed, and also holds for many random growth models. However, it may fail if, for example, a high consumption in period t indicates that the period $t+1$ is likely to be bad. Out of these particular cases, risk aversion in the risk-sensitive preference framework has a "natural" effect on the risk free rate and the market price of risk. If the agent is more risk averse, he is willing to pay more to transfer resources from a certain state of the world (today) to an uncertain one (tomorrow), which raises the price of riskless savings and thus reduces the riskless interest rate. By the same token, a more risk averse agent requires a larger discount to hold a risky asset, which increases the market price of risk.

The results of Proposition 5 also have interesting consequences when discussing policy issues. For example, the on-going debate as to the cost of climate change is strongly influenced by the choice of the appropriate discount rate, and on how the risk and the planner's risk aversion affect this rate. Our result clearly states that the more risk averse the planner, the lower the discount rate.

As will be seen in Section 6.2, the results of Proposition 5 do not always hold for Epstein-Zin preferences.

5.2 Risk sharing

In this section, we consider a simple risk sharing problem.[14] The infinite-horizon economy is populated by two agents denoted A and B. They are endowed at date 0 with the certain income y_0. The income at all future dates is random but constant throughout the different dates. Formally, all revenues after date 1 are identical and equal to the realization of a single real random variable $\widetilde{y}$, which is unknown from the date 0 perspective. There is therefore no uncertainty after date 1 and only one aggregate income risk $\widetilde{y}$, which applies to both agents. We assume that $\widetilde{y}$ may take either the value y_h with probability $\eta \in (0, 1)$ or the value $y_l < y_h$ with probability $1 - \eta$.[15]

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[14] Anderson (2005) considers more complex risk sharing problems with risk-sensitive preferences.

[15] This simple risk structure is sufficient to deliver significant differences with Epstein-Zin preferences.

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At date 0, agents can share risk through a market of assets. All assets are in zero net supply. We do not need to further specify asset markets, which may be complete or incomplete or include redundant assets. Agents have a symmetric access to markets, which implies that any asset trade available for one agent is also available for the other agent. As usual, a market equilibrium is defined as a combination of asset prices and individual demands, such that (i) individual demands are optimal choices for price-taker agents (individual rationality); (ii) all assets are in zero aggregate demand (market clearing conditions).

Agents A and B are endowed with risk-sensitive preferences. They have the same ordinal preferences but differ with respect to the risk aversion parameter. The utility V_0^i of an agent $i = A, B$ at date 0, consuming the amount c_0^i at date 0 and having the continuation utility V_1^i can be expressed as follows: $V_0^i = (1-\beta)u(c_0^i) - \frac{\beta}{k_i}\log(E_0\left[e^{-k_i V_1^i}\right])$, where $0 < \beta < 1$ is the constant time discount factor, identical for both agents. The function u, common for both agents, is assumed to be strictly concave, which is consistent with a positive intertemporal elasticity of substitution.

At date 1, there is no uncertainty and agents maximize their intertemporal utility $(1 - \beta)\sum_{t\geq 1}\beta^{t-1}u(c_t^i)$, subject to per-period budget constraints. The resulting utility is the indirect utility denoted $\Omega(\widetilde{w}^i)$, which depends only upon the agent i's wealth $\widetilde{w}^i$ at date 1. The tilde highlights that the wealth is uncertain from a date 0 perspective. Since agents A and B have the same time discount factor β and the same instantaneous utility u, they have the same indirect utility function. In consequence, the utility V_0^i of an agent $i = A, B$ at date 0, consuming c_0^i at date 0 and having wealth $\widetilde{w}^i$ at date 1 can be expressed as follows:

$$V_0^i = (1-\beta)u(c_0^i) - \frac{\beta}{k_i}\log(E_0\left[e^{-k_i\Omega(\widetilde{w}^i)}\right]).$$

The ex-post lifetime utility V_s^i obtained in state $s = h, l$ by the agent i consuming c_0^i in period 0 and endowed with the wealth w_s^i at date 1 is:

$$V_s^i = (1-\beta)u(c_0^i) + \beta\Omega(w_s^i), \quad s = h, l.$$

The following result describes how asset trading in general equilibrium impacts agents' ex-post preferences.

Proposition 6 (Risk sharing with risk-sensitive preferences)   *We consider two agents A and B endowed with risk-sensitive preferences. A is assumed to be more risk averse than B, who is also more risk averse than the temporal risk neutral agent: $k_A \geq k_B \geq 0$. They are interacting in the economy described above.*

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We denote V_l^i and V_h^i the equilibrium ex-post utilities of agent i in states l and h, and V_l^{aut} and V_h^{aut} the ex-post utilities agents would get in autarky (i.e., in absence of trade).

Then, the market general equilibrium is such that:

$$V_l^B \leq V_l^{aut} \leq V_l^A \leq V_h^A \leq V_h^{aut} \leq V_h^B \tag{20}$$

Proof: In appendix.

Proposition 6 states that the more risk averse agent is better off than the less risk averse in the bad state of the world, while the contrary holds in the good state of the world. Moreover, compared to the no-trade allocation, the market equilibrium allocation is riskier for the less risk averse individual ($V_l^B \leq V_l^{aut} \leq V_h^{aut} \leq V_h^B$), and less risky for the more risk averse individual ($V_l^{aut} \leq V_l^A \leq V_h^A \leq V_h^{aut}$). The equilibrium asset trade therefore generates a risk transfer from the more risk averse agent to the less risk averse agent, which is consistent with economic intuition. We will again see in Section 6.2 that the results does not hold with Epstein-Zin preferences.

6 Discussion

In this section, we discuss certain aspects of risk-sensitive preferences, and also explain how they compare with Epstein-Zin preferences.

6.1 Preference for timing, homotheticity and utility independence

As is explained in Kreps and Porteus (1978), concavity (or convexity) with respect to the second argument of the aggregator in Definition 1 dictates: (i) preference for the timing of resolution of uncertainty and (ii) whether risk aversion is increasing or decreasing with time distance. A convex aggregator ($W_{yy} > 0$) is associated with preferences for early resolution of uncertainty and a greater risk aversion for lotteries that resolve in the distant future. A concave aggregator ($W_{yy} < 0$) generates preferences for an early resolution of uncertainty and a lower risk aversion for lotteries that resolve in the distant future. In the case of risk sensitive preferences, we derive from (9) that for all $(x, y) \in C \times [0, 1]$:

$$\frac{W_{yy}(x, y)}{W_y(x, y)} = \frac{(1 - \beta)(1 - e^{-k})}{1 - y + e^{-k}y}.$$

With $1 - \beta > 0$, the sign of W_{yy} is that of k. Agents, who are more risk averse than in the standard additive model (whenever $k > 0$) have preferences for an early resolution of uncertainty, while the reverse holds when $k < 0$. When $k = 0$, agents are indifferent

to the timing of lottery resolution. To gain a better insight, one may also notice that if preferences were defined on a smaller domain, so that preferences with zero or negative time preferences (i.e., $\beta \geq 1$) could be considered, we would obtain the opposite relation between risk aversion and preference for timing.[16] Time preference, risk aversion and preference for timing appear therefore to be intertwined.

The above interrelation can be interpreted as an intuitive consequence of the assumptions of stationarity and ordinal dominance. Let us consider an agent comparing temporal lotteries that provide the same consumption c during N periods, but may differ afterwards. On the one hand, with stationary preferences, the N periods of constant consumption c do not matter and the ranking is independent of c and N. On the other hand, ordinal dominance implies that risk aversion is considered with respect to lifetime utility, including the utility derived from the first N periods of life, and thus depends on c. For the ranking to be independent of c, preferences must exhibit a constant absolute risk aversion with respect to lifetime utility, such that the utility of the first N periods does not impact what happens afterwards. This explains the exponential functional form of risk-sensitive preferences, which, incidentally, makes them extremely tractable in dynamic problems.

Moreover, the larger the N, the smaller the utility risk the agent is facing, because of the discount factor $\beta < 1$ (the reverse would hold if $\beta > 1$). This generates a kind of non-stationarity of preferences, unless an "amplification" mechanism of risk attitudes regarding "utility risk" in the future is introduced. In consequence, an agent, who is risk averse with respect to lifetime utility ($k > 0$) and who has positive time preferences ($\beta < 1$) must exhibit greater risk aversion for lotteries resolving in the distant future in order to keep preferences stationary. This greater risk aversion should precisely compensate the discount of future risks due to the time preference parameter β. Similarly an agent, who is risk prone with respect to lifetime utility ($k < 0$) and has positive time preferences ($\beta < 1$) has to exhibit more risk loving (and thus less risk aversion) for lotteries resolving in the future. The symmetric arguments would hold in the case of negative time preferences ($\beta > 1$). Preferences for the timing (or, equivalently, a degree or risk aversion that depends on time distance) is a necessary ingredient to compensate for the existence of time preferences, as soon as temporal risk aversion is introduced and preferences are stationary.

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[16]The case where $\beta \geq 1$ can for example be considered when assuming that all consumption paths converge to an exogenous c^* within a finite amount of time. The normalization conditions imposed before Definition 1 are then no longer possible since preferences are defined on a domain that cannot include both $\underline{c}^\infty$ and $\bar{c}^\infty$. Risk-sensitive preferences can still be defined by the recursion $V(c, m) = u(c_t) - \frac{\beta}{k} \ln(E_m \left[e^{-kV} \right])$, leading to the same relation between risk aversion, time preferences and preferences for timing. The case $\beta = 1$ (zero time preferences) precisely corresponds to the multiplicative model of Bommier (2012), which fits into the expected utility framework and exhibits no preference for timing.

Another aspect of risk-sensitive preferences is that they are in general not homothetic, except in the particular cases when the intertemporal elasticity of substitution is equal to one or when u is isoelastic and $k = 0$. This feature may seem to be an unpleasant aspect, as it generates wealth effects in many problems, like those related to portfolio choices. However, in recent contributions, non-homotheticity of preferences has been found to contribute to explaining some empirical observed relationships, for example between trade flows and income per capita (Fieler, 2011) or between wealth and stock holdings (Wachter and Yogo, 2010).

Risk-sensitive preferences also have an interesting property that may significantly help for the resolution of other problems. Indeed, risk-sensitive preferences fulfill an assumption of (mutual) utility independence similar to that discussed by Keeney and Raiffa (1993). Preferences regarding what may happen in two periods of time t_0 and t_1, conditional on having consumption in another period of time ($t \neq t_0, t_1$) being equal to a given level c_t, is independent of c_t. The property also applies when c_t is random and independently distributed. In a dynamic setting, the assumption of preference stationarity forces preferences to be independent of the past history. The utility independence property in addition requires that preferences have to be independent of the (exogenous) future. This may simplify the analysis from both a theoretical and a numerical point of view in many intertemporal problems. For example, in the endowment economy of Section 5.1, in which the per period consumptions are independently distributed (which is a particular case of the assumptions in Proposition 5), we obtain that the pricing kernel is given by $M_{t,t+1} = \beta \frac{u'(c_{t+1})}{u'(c_t)} \frac{\exp(-k(1-\beta)u(c_{t+1}))}{E_t[\exp(-k(1-\beta)u(c_{t+1}))]}$. The risk free rate and the market price for risk are therefore independent of what may happen in period $t + 2$ and the subsequent periods of time, which is in fact a consequence of the property of utility independence.

Utility independence is not a direct consequence of the assumptions of stationarity and ordinal dominance. In fact, when the function b is non-constant, preferences associated with the aggregator (12) – i.e., Uzawa's preferences – fulfill the assumptions of stationarity and ordinal dominance but not that of utility independence. Utility independence only appears when restricting our attention to preferences that make it possible to disentangle ordinal and risk preferences. This comes from the – so far unnoticed – fact that, in the expected utility framework, history independence together with non-trivial risk aversion comparability imposes independence with respect to the future, and therefore mutual utility independence in the sense of Keeney and Raiffa (1993).

Mutual utility independence has in turn significant implications. Indeed, as known since Koopmans (1960), the combination of stationarity and independence with respect to

the future (called "period independence" in Koopmans' article) implies weak separability of preferences and constant time discounting. This explains why discussing risk aversion eventually requires us to consider preferences over deterministic consumption profiles that can be represented by an additive utility function with a constant time discounting. The property of weak separability, which is often introduced as a technical assumption, and sometimes justified by the long tradition of research that makes use of it, appears therefore to be a necessary condition to study risk aversion while assuming preference stationarity.

6.2   Comparison with Epstein-Zin preferences

Risk-sensitive preferences share many features with Epstein-Zin isoelastic preferences. They both rely on the Kreps-Porteus recursive framework, but use different specifications for the aggregator. However, this generates significant differences, such as the ability to conform with the ordinal dominance property. While risk-sensitive preferences always fulfill ordinal dominance, Epstein-Zin generally do not.[17] Moreover, Epstein-Zin preferences are only weakly (and not strongly) well-ordered in terms of risk aversion (unless the intertemporal elasticity of substitution is equal to one) and are therefore likely to generate odd conclusions on the role of risk aversion in settings where complete risk elimination is not possible or not optimal. In order to provide meaningful examples, we consider both applications developed in Section 5 that were shown to lead to intuitive conclusions when agents have risk-sensitive preferences, but now consider them while assuming that agents have Esptein-Zin preferences. Since the point is to emphasize that this may yield unappealing conclusions, we introduce further simplifications (for example, by focusing on some forms of risk) that are detailed along the text.

6.2.1   Risk free rate in a random economy

We consider a random endowment economy, as in Section 5.1. To further simplify matters, we assume a form of uncertainty similar to that of Section 5.2: consumption at date $t+1$ is random and this level of consumption is maintained for ever afterwards. Formally, the consumption in every period after time $t+1$ is equal to the realization of an unidimensional random variable $\widetilde{c}_{t+1}$.

In the risk-sensitive setting, if the function u is concave, the assumptions of Proposition 5 are fulfilled, and as a direct consequence, the risk free rate decreases with risk aversion. We now contrast these results with the ones derived with Epstein-Zin preferences.

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[17] The exceptions are when Epstein-Zin preferences are also risk-sensitive preferences (that is when $\rho = 0$ or $\alpha = \rho$ in (6)).

We consider the case where both α and ρ are different from zero. Epstein-Zin utility functions are then given by:

$$V_t = \left((1-\beta) (c_t)^\rho + \beta E_t \left[\left(V_{t+1}^\alpha \right) \right]^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}. \tag{21}$$

The parameter α is generally interpreted as an indicator of risk aversion, a smaller α indicating a greater risk aversion.[18] In the special case when the agent's consumption remains constant at the level $\widetilde{c}_{t+1}$ after period $t+1$, the continuation utility is $V_{t+1} = \widetilde{c}_{t+1}$. Thus, the utility at date t is: $V_t = \left((1-\beta)(c_t)^\rho + \beta \left(E_t \left[\widetilde{c}_{t+1}^\alpha \right] \right)^{\frac{\rho}{\alpha}} \right)^{\frac{1}{\rho}}$. We deduce that the gross risk free rate R_t is given by:

$$\frac{1}{R_t} = \frac{\beta}{1-\beta} \frac{E_t \left[\widetilde{c}_{t+1}^{\alpha-1} \right] E_t \left[\widetilde{c}_{t+1}^\alpha \right]^{\frac{\rho}{\alpha}-1}}{c_t^{\rho-1}}.$$

By derivation:

$$-\frac{1}{R_t} \frac{\partial R_t}{\partial \alpha} = \frac{E_t \left[\log(\widetilde{c}_{t+1}) \widetilde{c}_{t+1}^{\alpha-1} \right]}{E_t \left[\widetilde{c}_{t+1}^{\alpha-1} \right]} + \frac{\rho-\alpha}{\alpha} \frac{E_t \left[\log(\widetilde{c}_{t+1}) \widetilde{c}_{t+1}^\alpha \right]}{E_t \left[\widetilde{c}_{t+1}^\alpha \right]} - \frac{\rho}{\alpha^2} \log(E_t \left[\widetilde{c}_{t+1}^\alpha \right]),$$

which can be positive or negative. For example, when $\widetilde{c}_{t+1} = \hat{c} > 0$ with probability $p \ll 1$ and $\widetilde{c} = 1$ otherwise, the above expression is approximately equal to:

$$-\frac{1}{R_t} \frac{\partial R_t}{\partial \alpha} \simeq p \left[\log(\hat{c}) \hat{c}^{-(1-\alpha)} + \frac{\rho-\alpha}{\alpha} \log(\hat{c}) \hat{c}^\alpha - \frac{\rho}{\alpha^2} (\hat{c}^\alpha - 1) \right]$$

Assume that $0 < \alpha < 1$, and $\alpha < \rho$ (people are more risk averse than in the standard additive case). We observe that $\frac{\partial R_t}{\partial \alpha} > 0$ for $\hat{c}$ close to zero, and $\frac{\partial R_t}{\partial \alpha} < 0$ for $\hat{c}$ sufficiently large. The risk free rate varies non-monotonically with risk aversion. When the risk free rate increases with risk aversion, the willingness to save for precautionary motives decreases with risk aversion, which contradicts simple dominance arguments, as shown in Bommier, Chassagnon and Le Grand (2012).

6.2.2 Risk sharing

We now contrast the results derived in the risk sharing problem of Section 5.2 with those that could be obtained using Epstein-Zin preferences. The framework is very similar to the one with risk-sensitive preferences and in particular the risk structure is the same (cf. first

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[18]The interpretation is correct when focusing on weak comparative risk aversion, but does not extend to strong comparative risk aversion. In other words, decreasing α always increases the willingness to eliminate all risks, but does not necessarily yield a greater aversion for marginal increases in risk.

paragraph of Section 5.2). Epstein-Zin preferences are exactly the same as in equation (21) of the previous application. Both agents A and B have the same intertemporal elasticity of substitution $\frac{1}{1-\rho}$ and A is weakly more risk averse than B ($\alpha_A < \alpha_B$). We now have the following negative result.

Proposition 7 (Risk sharing with Esptein-Zin preferences) *We consider the risk sharing problem summarized above, in which agent A is weakly more risk averse than agent B (i.e., $\alpha_A < \alpha_B$). We denote $V_l^{EZ,i}$ and $V_h^{EZ,i}$ the ex-post utilities of agent $i = A, B$ in states l and h and $V_l^{EZ,aut}$ and $V_h^{EZ,aut}$ the ex-post utilities in autarky.*

If preference parameters are such that $\alpha_A < \alpha_B < 0 < \rho < 1$, there always exist an asset market and endowments such that the market equilibrium is characterized by the following properties:

1. *agent B saves while agent A borrows;*

2. *agent A is always ex-post better off than agent B or than in autarky (i.e., in the no trade allocation), while agent B is always ex-post worse off than in autarky:*

$$\begin{cases} V_l^{EZ,B} < V_l^{EZ,aut} < V_l^{EZ,A} \\ V_h^{EZ,B} < V_h^{EZ,aut} < V_h^{EZ,A} \end{cases} \tag{22}$$

Proof: In appendix.

The inequalities (22) show that whatever happens, agent B would be better off ex-post in absence of trade. While trading is not mandatory, agent B actually prefers to trade and gets an allocation, which is dominated at the first order by the no-trade allocation! Obviously, such a voluntary move to dominated allocations cannot arise with preferences fulfilling ordinal dominance. But with Epstein-Zin preferences, which fail to fulfill this property, it is "rational" and occurs in a simple market interaction. As a consequence, the ranking of ex-post utilities shown in (22) always appears to be favorable for the more risk averse agent and contrasts with the intuitive pattern of risk sharing that was obtained with risk-sensitive preferences.

7 Conclusion

When the horizon is infinite, the expected utility framework does not contain any class of stationary preferences that are well-ordered in terms of risk aversion. Today, the most popular approach to investigating the role of risk aversion, while assuming preference stationarity, involves using Epstein-Zin preferences, despite some shortcomings. These

preferences do not fulfill ordinal dominance (Chew and Epstein, 1990), are not well-ordered with respect to aversion for marginal risk variations, and may lead to counter-intuitive conclusions when used in applied problems.

In this paper, we explore whether Kreps-Porteus framework offers better alternatives. We first derive all Kreps-Porteus preferences that fulfill stationarity and ordinal dominance (Proposition 1): they are either of the expected utility kind (providing preferences *à la* Uzawa) or correspond to risk-sensitive preferences given by the equation (1), which were introduced by Hansen and Sargent (1995) in their works on robustness. Since preferences *à la* Uzawa are unable to achieve the separation between risk aversion and intertemporal substitutability (Proposition 2), the class of risk-sensitive preferences is the only one that provides an appropriate support to study risk aversion. In fact, Proposition 4 shows that varying parameter k that enters into the definition of risk-sensitive preferences, while leaving other parameters unchanged, generates a class of preferences that is well-ordered in terms of risk aversion, even when using a strong notion of comparative risk aversion.

An interesting aspect of our paper is that contrary to many well-known contributions, like Epstein and Zin (1989), or Klibanoff, Marinacci, and Mukerji (2009), we do not impose any restriction such as weak separability or constant time discounting for preferences over deterministic consumption paths. We prove that such properties are however necessary to study risk aversion while assuming stationarity. Hence, our paper provides an original argument for assuming weak separability of preferences.

In order to illustrate the interest of working with risk-sensitive preferences, we apply them to simple problems of asset pricing in a random endowment economy, and of risk sharing in a general equilibrium. They provide intuitive conclusions with regards to the role of risk aversion, contrary to what other models may generate.

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Appendix

A Proof of Proposition 1

A.1 Necessary conditions

We first prove that KP-recursive preferences fulfilling ordinal dominance admit a linear or a risk-sensitive aggregator (i.e., that (12) or (13) holds).

A preliminary lemma. The lemma provides a first set of restrictions on aggregators.

Lemma 1 *Consider a KP-recursive utility function U defined over D, whose admissible aggregator is denoted W. If the associated preference relation fulfills ordinal dominance, then the aggregator can be expressed as follows:*

$$\forall (x,y) \in C \times [0,1],\ W(x,y) = \phi\left(a(x) + yb(x)\right),$$

where $a, b : C \to [0,1]$, and $\phi : [0,1] \to [0,1]$ are continuously differentiable, with $a(\underline{c}) = \phi(0) = 0$, $a(\bar{c}) + b(\bar{c}) = \phi(1) = 1$, $a' > 0$, $(a' + b') > 0$ and $\phi' > 0$.

Proof.

For an aggregator W, we define preferences over $C \times M([0,1])$ by considering the utility function $\widetilde{U}$ defined by: $\forall (c,m) \in C \times M([0,1])$, $\widetilde{U}(c,m) = W(c, E_m[x])$, where $E_m[x] \in [0,1]$ denotes the expected payment associated with probability measure m. When c varies in C, $U(c_\infty)$ covers $[0,1]$ and the utility function U –applied to constant consumption paths– generates an isomorphism from C into $[0,1]$. As a consequence, if preferences over D fulfill ordinal dominance, preferences over $C \times M([0,1])$ represented by the utility function $\widetilde{U}$ also fulfill ordinal dominance.

First step. We consider $x_0 \in (\underline{c}, \bar{c})$ and $y_0 \in (0,1)$ (i.e., (x_0, y_0) lies in the interior of the definition domain of W). Since W is a continuously differentiable function with $W_y > 0$, the implicit function theorem states that there exist $\widetilde{B}_{x_0}$ and $\widetilde{B}_{y_0}$, respective neighborhoods of x_0 and y_0, and a continuously differentiable function η_{x_0, y_0} from $\widetilde{B}_{x_0}$ into $\widetilde{B}_{y_0}$ such that: $\forall x \in \widetilde{B}_{x_0}$, $W(x_0, y_0) = W(x, \eta_{x_0, y_0}(x))$. Let $y_1 \in \widetilde{B}_{y_0}$. By the same token, there exists a continuously differentiable function η_{x_0, y_1} from $\widehat{B}_{x_0}$ into $\widehat{B}_{y_1}$ such that, $\forall x \in \widehat{B}_{x_0}$, $W(x_0, y_1) = W(x, \eta_{x_0, y_1}(x))$. We define $B_{x_0} = \widetilde{B}_{x_0} \cap \widehat{B}_{x_0}$ and $B_{y_0, y_1} = \widetilde{B}_{y_0} \cap \widehat{B}_{y_1}$, which are non-empty and open sets. For all $x \in B_{x_0}$, we have:

$$W(x_0, y_0) = W(x, \eta_{x_0, y_0}(x)) \text{ and } W(x_0, y_1) = W(x, \eta_{x_0, y_1}(x)). \tag{23}$$

With the assumption of ordinal dominance, (23) implies that for all $p \in [0,1]$:

$$\forall x \in B_{x_0}, \; W(x, p\eta_{x_0,y_0}(x) + (1-p)\eta_{x_0,y_1}(x)) = W(x_0, py_0 + (1-p)y_1). \qquad (24)$$

Derivation with respect to x of equations (23)–(24) yields:

$$W_x(x, \eta_{x_0,y_0}(x)) + W_y(x, \eta_{x_0,y_0}(x))\frac{\partial \eta_{x_0,y_0}}{\partial x}(x) = 0,$$

$$W_x(x, \eta_{x_0,y_1}(x)) + W_y(x, \eta_{x_0,y_1}(x))\frac{\partial \eta_{x_0,y_1}}{\partial x}(x) = 0,$$

$$W_x(x, p\eta_{x_0,y_0}(x) + (1-p)\eta_{x_0,y_1}(x))$$

$$+W_y(x, p\eta_{x_0,y_0}(x) + (1-p)\eta_{x_0,y_1}(x)) \left(p\frac{\partial \eta_{x_0,y_0}}{\partial x_0}(x) + (1-p)\frac{\partial \eta_{x_0,y_1}}{\partial x_0}(x) \right) = 0.$$

By substitution of the two first equalities in the last one we deduce that:

$$\forall x \in B_{x_0}, \; \frac{W_x(x, p\eta_{x_0,y_0}(x) + (1-p)\eta_{x_0,y_1}(x))}{W_y(x, p\eta_{x_0,y_0}(x) + (1-p)\eta_{x_0,y_1}(x))} = p\frac{W_x(x, \eta_{x_0,y_0}(x))}{W_y(x, \eta_{x_0,y_0}(x))} + (1-p)\frac{W_x(x, \eta_{x_0,y_1}(x))}{W_y(x, \eta_{x_0,y_1}(x))},$$

which implies that the restriction of $\frac{W_x(x,y)}{W_y(x,y)}$ on $B_{x_0} \times B_{y_0,y_1}$ is linear in y.

Thus, for any $(x_0, y_0) \in (\underline{c}, \overline{c}) \times (0, 1)$, there exists a neighborhood B_{x_0,y_0} and two functions $\hat{a}_{x_0,y_0}$ and $\hat{b}_{x_0,y_0}$ such that for all $(x,y) \in B_{x_0,y_0}$ we have:

$$\frac{W_x(x,y)}{W_y(x,y)} = \hat{a}_{x_0,y_0}(x) + \hat{b}_{x_0,y_0}(x)y. \qquad (25)$$

Second step. Let $y_1 \in (0,1)$. For all $x \in (\underline{c}, \overline{c})$, we define $\hat{a}(x)$ and $\hat{b}(x)$ by:

$$((x, y_1) \in B_{x_0,y_0} \text{ for some } (x_0, y_0)) \Rightarrow \left(\hat{a}(x) = \hat{a}_{x_0,y_0}(x) \text{ and } \hat{b}(x) = \hat{b}_{x_0,y_0}(x) \right).$$

The functions $\hat{a}$ and $\hat{b}$ are well defined. Indeed, firstly, from (25), we know that for any $x \in (\underline{c}, \overline{c})$, there exists a pair (x_0, y_0) such that $(x, y_1) \in B_{(x_0,y_0)}$. Secondly, if for some $x_1 \in (\underline{c}, \overline{c})$ there are two pairs (x_0, y_0) and (x_0', y_0'), such that $(x_1, y_1) \in B_{(x_0,y_0)}$ and $(x_1, y_1) \in B_{(x_0',y_0')}$ then for all $(x, y) \in B_{(x_0,y_0)} \cap B_{(x_0',y_0')}$, we have:

$$\frac{W_x(x,y)}{W_y(x,y)} = \hat{a}_{x_0,y_0}(x) + \hat{b}_{x_0,y_0}(x)y = \hat{a}_{x_0',y_0'}(x) + \hat{b}_{x_0',y_0'}(x)y,$$

which implies $\hat{a}_{x_0,y_0}(x_1) = \hat{a}_{x_0',y_0'}(x_1)$ and $\hat{b}_{x_0,y_0}(x_1) = \hat{b}_{x_0',y_0'}(x_1)$.

The continuity of $\frac{W_x}{W_y}$ implies that $\hat{a}$ and $\hat{b}$ are continuous functions. Let us consider the set $\Gamma = \{(x,y) \in C \times [0,1] | \frac{W_x(x,y)}{W_y(x,y)} = \hat{a}(x) + \hat{b}(x)y\}$. This set is non-empty by construction and closed by continuity of $(x,y) \mapsto \frac{W_x(x,y)}{W_y(x,y)} - \hat{a}(x) - \hat{b}(x)y$.

Assume that there exists $(x_0, y_0) \in (\underline{c}, \overline{c}) \times (0, 1)$ which does not belong to Γ. The set $\Lambda = \{\lambda \in [0, 1] | (x_0, \lambda y_0 + (1 - \lambda)y_1) \in \Gamma\}$ is closed, and thus compact, contains 0 and does not contain 1. Let λ_1 be the supremum of Λ. By compactness, $\lambda_1 \in \Lambda$ and $\lambda_1 < 1$. We know from (25) that there exists $\varepsilon > 0$, such that for all λ with $|\lambda - \lambda_1| < \varepsilon$, we have:

$$\frac{W_x(x_0, \lambda y_0 + (1 - \lambda)y_1)}{W_y(x_0, \lambda y_0 + (1 - \lambda)y_1)} = \hat{a}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) + \hat{b}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) \left(\lambda y_0 + (1 - \lambda)y_1)\right).$$
(26)

Moreover since $\lambda_1 \in \Omega$, we have for all $\lambda \leq \lambda_1$:

$$\frac{W_x(x_0, \lambda y_0 + (1 - \lambda)y_1)}{W_y(x_0, \lambda y_0 + (1 - \lambda)y_1)} = \hat{a}(x_0) + \hat{b}(x_0) \left(\lambda y_0 + (1 - \lambda)y_1)\right).$$
(27)

We deduce that $\hat{a}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) = \hat{a}(x_0)$ and $\hat{b}_{x_0, \lambda_1 y_0 + (1 - \lambda_1)y_1}(x_0) = \hat{b}(x_0)$. Using (26), equation (27) extends to some $\lambda > \lambda_1$, contradicting λ_1 being the supremum of Λ. We conclude that $\Gamma = C \times [0, 1]$.

Third step.   Given x_0, we define $w_0 : y \mapsto w_0(y) = W(x_0, y)$, which is increasing continuously differentiable on $[0, 1]$. From equation (25), we know that W solves:

$$\forall (x, y) \in C \times [0, 1], \; W_x(x, y) = (\hat{a}(x) + \hat{b}(x)y)W_y(x, y),$$
$$\forall y \in [0, 1], \; W(x_0, y) = w_0(y).$$

The method of characteristics shows the existence and uniqueness of the solution given by:

$$W(x, y) = w_0 \left(\tilde{a}(x) + \tilde{b}(x)y\right),$$
(28)
$$\text{with: } \tilde{a}(x) = \int_{x_0}^{x} \left(\exp\left(\int_{x_0}^{s} \hat{b}(\tau)d\tau\right)\right) \hat{a}(s)ds \text{ and } \tilde{b}(x) = \exp\left(\int_{x_0}^{x} \hat{b}(s)ds\right).$$

$\tilde{a}$ and $\tilde{b}$ are continuously derivable on C. Using that $W(\overline{c}, 1) = 1$ and $W(\underline{c}, 0) = 0$, the representation of Lemma 1 is obtained for $a(x) = \frac{\tilde{a}(x) - \tilde{a}(\underline{c})}{\tilde{a}(\overline{c}) - \tilde{a}(\underline{c}) + \tilde{b}(\overline{c})}$, $b(x) = \frac{\tilde{b}(x)}{\tilde{a}(\overline{c}) - \tilde{a}(\underline{c}) + \tilde{b}(\overline{c})}$ and $\phi(z) = w_0 \left((\tilde{a}(\overline{c}) - \tilde{a}(\underline{c}) + \tilde{a}(\overline{c})) z + \tilde{a}(\underline{c})\right)$. Moreover, since w_0 and W are differentiable and strictly increasing, we have that a, $a + b$ and ϕ also are differentiable, increasing and with values in $[0, 1]$ (in $(0, 1)$ for the function b). □

Proof of the representation result in Proposition 1.   From Lemma 1, we know that KP-recursive preferences admit an aggregator $W(x, y) = \phi(a(x) + b(x)y)$ with $a(\underline{c}) = \phi(0) = 0$ and $a(\overline{c}) + b(\overline{c}) = \phi(1) = 1$. Let $x_1 \in C$ and let the function $\widetilde{W}$ be defined by:

$$\widetilde{W}(x, y) = W(x, W(x_1, y)) \text{ for all } (x, y) \in C \times [0, 1].$$

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With the same proof strategy as in Lemma 1, one may show that for all $(x, y) \in C \times [0, 1]$, $\widetilde{W}(x, y) = \phi\left(a(x) + b(x)\phi(a(x_1) + b(x_1)y)\right)$. By derivation:

$$\frac{\widetilde{W}_x(x, y)}{\widetilde{W}_y(x_0, y)} = \frac{a'(x) + b'(x)\phi(a(x_1) + b(x_1)y)}{b(x)b(x_1)\phi'(a(x_1) + b(x_1)y)},$$

which has to be linear in y. Since $a' > 0$, we deduce that $\frac{1 + \frac{b'(x)}{a'(x)}\phi(z)}{\phi'(z)}$ has to be linear in z.

First, assume that $\frac{b'(x)}{a'(x)}$ is not constant and takes at least two values $\lambda_1 \neq \lambda_2$. Since $\frac{1 + \lambda_1\phi(z)}{\phi'(z)}$ and $\frac{1 + \lambda_2\phi(z)}{\phi'(z)}$ are linear in z, $\frac{(\lambda_2 - \lambda_1)\phi(z)}{\phi'(z)}$ and $\frac{\phi(z)}{\phi'(z)}$ are also linear. Together with $\phi(0) = 0$ and $\phi(1) = 1$, we obtain that there exists $\nu \in \mathbb{R}$, such that $\forall y \in [0, 1]$, $\phi(y) = y^\nu$. The case where $\nu < 1$ contradicts the continuous differentiability of ϕ, while $\nu > 1$ contradicts $W_y(x, 0) > 0$. Therefore ϕ has to be linear providing $W(x, y) = a(x) + b(x)y$. The normalization, regularity and monotonicity conditions imposed in Definition 1 lead to $a(\underline{c}) = 0$, $a(\bar{c}) + b(\bar{c}) = 1$, $a'(x) > 0$ and $a'(x) + b'(x) > 0$. The condition $b(x) < 1$ comes from $a + b$ and b strictly increasing together with $a(\underline{c}) = 0$ and $a(\bar{c}) + b(\bar{c}) = 1$. We are thus left with the linear aggregators (12).

Second, we assume that $\frac{b'(x)}{a'(x)}$ is constant, necessarily larger than -1 (since $W_x > 0$) and different from zero (since $W_y > 0$). We define $k \neq 0$ with $\frac{b'(x)}{a'(x)} = e^{-k} - 1$. By integration:

$$b(x) = \left(e^{-k} - 1\right)a(x) + b_0, \quad \text{with } b_0 \in \mathbb{R}. \tag{29}$$

Let $h(z) = 1 + \left(e^{-k} - 1\right)\phi(z)$. Since $\frac{1 + \frac{b'(x)}{a'(x)}\phi(z)}{\phi'(z)}$ is linear in z, so is $\frac{h(z)}{h'(z)}$. $\phi(0) = 0$ and $\phi(1) = 1$ imply $h(0) = 1$ and $h(1) = e^{-k}$. By integration, there exists $\beta > 0$ such that:

$$\forall z \in [0, 1], \quad h(z) = \left(1 + \left(e^{-\frac{k}{\beta}} - 1\right)z\right)^\beta \quad \text{and} \quad \phi(z) = \frac{1 - \left(1 - (1 - e^{-\frac{k}{\beta}})z\right)^\beta}{1 - e^{-k}}. \tag{30}$$

We can remark that for all $z \in [0, 1]$:

$$\frac{1 + (e^{-k} - 1)\phi(z)}{\phi'(z)} = \frac{1}{\beta}\frac{1 - e^{-k}}{1 - e^{-\frac{k}{\beta}}}\left(1 - (1 - e^{-\frac{k}{\beta}})z\right). \tag{31}$$

Let $x_2 \in C$. Similarly to the definition of $\widetilde{W}$, we introduce a function $\hat{W}$ as follows:

$$\forall (x, y) \in C \times [0, 1], \ \hat{W}(x, y) = W(x, W(x_1, W(x_2, y))).$$

As in Lemma 1 and as with $\widetilde{W}$, one can show that ordinal dominance imposes that $\frac{\hat{W}_x}{\hat{W}_y}$ has to be linear in y. Writing $\hat{W}(x, y) = \phi\left(a(x) + b(x)\phi(a(x_1) + b(x_1)\phi(a(x_2) + b(x_2)y))\right)$ to

compute $\frac{\hat{W}_x}{\hat{W}_y}$ and using (31), we obtain that: $\frac{1-(1-e^{-\frac{k}{\beta}})(a(x_1)+b(x_1)\phi(a(x_2)+b(x_2)y))}{\phi'(a(x_2)+b(x_2)y)}$ is linear in y. However we know that $\frac{1-(1-e^{-k})\phi(a(x_2)+b(x_2)y))}{\phi'(a(x_2)+b(x_2)y)}$ is linear in y. So, after substitution in the above expression, we have $\left(1-(1-e^{-\frac{k}{\beta}})a(x_1)-\frac{(1-e^{-\frac{k}{\beta}})b(x_1)}{(1-e^{-k})}\right)\frac{\phi(a(x_2)+b(x_2)y))}{\phi'(a(x_2)+b(x_2)y)}$ is linear in y. Thus, either ϕ is linear (case already considered) or $1-(1-e^{-\frac{k}{\beta}})a(x_1)=\frac{(1-e^{-\frac{k}{\beta}})b(x_1)}{(1-e^{-k})}$ for any $x_1 \in C$. Equation (29) implies that $b_0 = \frac{1-e^{-k}}{1-e^{-\frac{k}{\beta}}}$ and

$$\forall x \in C, \ b(x) = \left(1-e^{-k}\right)\left(\frac{1}{1-e^{-\frac{k}{\beta}}} - a(x)\right).$$

Then (30) implies that for all x and y:

$$\phi(a(x)+b(x)y) = \frac{1-\left[\left(1-(1-e^{-\frac{k}{\beta}})a(x)\right)\left(1-(1-e^{-k})y\right)\right]^\beta}{1-e^{-k}}.$$

We define $u:[0,1] \to \mathbb{R}$ as follows:

$$u(x) = -\frac{\beta}{k(1-\beta)}\log\left(1-(1-e^{-\frac{k}{\beta}})a(x)\right), \ \text{or} \ 1-(1-e^{-\frac{k}{\beta}})a(x) = e^{-\frac{k(1-\beta)u(x)}{\beta}}, \quad (32)$$

which leads to the specification (13). Eq. (32) implies $u(\underline{c}) = 0$. Moreover $a(\bar{c}) + b(\bar{c}) = 1$ imposes that $a(\bar{c}) = \frac{1-e^{-k\frac{1-\beta}{\beta}}}{1-e^{-\frac{k}{\beta}}}$ and $u(\bar{c}) = 1$. The condition $\beta < 1$ results from the fact that the restriction of U to C^∞ has to be monotonic. Indeed from (13) and $u(\underline{c}) = 0$, the utility associated with the consumption of the same $c \in C$ for n periods and $\underline{c}$ afterward is $\frac{1-e^{-k(1-\beta)\sum_{i=0}^{n-1}\beta^i u(c)}}{1-e^{-k}}$, which is monotonic in c if and only if $\beta < 1$.

A.2 Existence and uniqueness

We prove that the aggregator (13) defines a unique utility function.[19] We wish to use the Banach fixed point theorem to show that there exists a function V such that:

$$\forall(x,m) \in C \times D, \ V(c,m) = (1-\beta)u(c) - \frac{\beta}{k}\ln E_m e^{-kV}. \quad (33)$$

We define $\mathcal{C}(D,[0,1])$ the set of continuous functions from D into $[0,1]$. We know that C^∞ endowed with the product norm is a compact Polish space and that D endowed with the Prohorov metric is also a compact Polish space. The metric space $(\mathcal{C}(D,[0,1]),\|\cdot\|_\infty)$ is thus a Banach space.[20]

---

[19]The proof in the case of the aggregator (12) is not provided as it would follow the same arguments.
[20]See for example Theorem 9.3 in Aliprantis and Burkinshaw (1998).

Denote by $\mathcal{F}(D, \mathbb{R})$ the set of all functions from D to $\mathbb{R}$. We consider the mapping T from $\mathcal{C}(D, [0, 1])$ into $\mathcal{F}(D, \mathbb{R})$ such that for $F \in \mathcal{C}(D, [0, 1])$:

$$\forall (c, m) \in D, \ TF(c, m) = (1 - \beta)u(c) - \frac{\beta}{k} \ln E_m e^{-kF}. \tag{34}$$

In order to use the fixed point theorem, we prove that T is a contraction. More specifically, we check that T verifies Blackwell's (1965) sufficient conditions: (1) T is a mapping from $\mathcal{C}(D, [0, 1])$ into itself; (2) T is increasing; (3) there exists a constant $\theta \in (0, 1)$, such that for all $F \in \mathcal{C}(D, [0, 1])$, for all $A \in [0, 1]$, we have: $\forall x \in D, T(F + A)(x) \le TF(x) + \theta A$.[21]

1. Let $F \in \mathcal{C}(D, [0, 1])$ and $(c, m) \in D$. Since $0 \le u(c) \le 1$ and $0 \le -\frac{1}{k} \ln \left(E_m \left[e^{-kF} \right] \right) \le 1$, we also have $0 \le TF(c, m) \le 1$. Let $(c_n, m_n)_{n \ge 0} \in D^\infty$ converging towards $(c, m) \in D$. Since F is bounded and continuous, e^{-kF} is also bounded (below by e^{-k}) and continuous, we have: $\ln E_{m_n} e^{-kF} \to \ln E_m e^{-kF}$. Since u is continuous, $TF(c_n, m_n) \to TF(c, m)$ and $TF \in \mathcal{C}(D, [0, 1])$.

2. Let $F_1, F_2 \in \mathcal{C}(D, [0, 1])$, such that $\forall x \in D, F_2(x) \ge F_1(x) \ge 0$. We have for all k and for all $m \in M(D)$, $E_m \left[e^{-kF_2} \right]^{-\frac{1}{k}} \ge E_m \left[e^{-kF_1} \right]^{-\frac{1}{k}} > 0$ and $-\frac{1}{k} \ln E_m \left[e^{-kF_2} \right] \ge -\frac{1}{k} \ln E_m \left[e^{-kF_1} \right]$. Since $\beta > 0$, we deduce that T is increasing.

3. Let $0 \le A \le 1$, $F \in \mathcal{C}(D, [0, 1])$ and $(c, m) \in D$. Noticing that $-\frac{\beta}{k} \ln \left(E_m \left[e^{-k(F+A)} \right] \right) = -\frac{\beta}{k} \ln \left(E_m \left[e^{-kF} \right] \right) + \beta A$, it is straightforward that $T(F + A)(c, m) - TF(c, m) = \beta A$.

The map T is a contraction of modulus $\beta \in (0, 1)$ on $(\mathcal{C}(D, [0, 1]), \|\cdot\|_\infty)$. The Banach fixed point theorem implies that (33) admits a unique solution $V \in \mathcal{C}(D, [0, 1])$. We now define U as follows: $\forall (c, m) \in D, U(c, m) = \frac{1 - e^{-kV(c, m)}}{1 - e^{-k}}$. It is straightforward to check that $U \in \mathcal{C}(D, [0, 1])$ and that U verifies the following relationship:

$$\forall (c, m) \in D, \ U(c, m) = \frac{1 - e^{-k(1-\beta)u(c)} \left(E_m \left[1 - (1 - e^{-k})U \right] \right)^\beta}{1 - e^{-k}}$$

Therefore U is the unique fixed-point of the aggregator $W(x, y) = \frac{1 - e^{-k(1-\beta)u(x)}(1 - (1 - e^{-k})y)^\beta}{1 - e^{-k}}$.

A.3 Sufficient conditions

We conduct the proof for the aggregator $W(x, y) = \frac{1 - e^{-k(1-\beta)u(x)}(1 - \widetilde{k}y)^\beta}{1 - e^{-k}}$. The other case is analogous. Let U be the corresponding utility function. We show by induction that the associated preference is monotonic with respect to the partial orders FSD_t for all t. We assume that $k > 0$ (the proof when $k < 0$ is similar) and denote $\widetilde{k} = 1 - e^{-k}$, which is also positive.

---

[21]With an obvious and standard abuse of notation, the letter A can mean either the constant $A \in [0, 1]$ or the constant function of $\mathcal{C}(D, [0, 1])$, whose value is always equal to A .

First step: monotonicity with respect to FSD_1. We consider (c,m) and (c',m') in Δ_1 with $(c,m)\ FSD_1\ (c',m')$, and $c=(c_0,\ldots,c_\tau)\in C^{\tau+1}$ and $c'=(c'_0,\ldots,c'_\tau)\in C^{\tau+1}$ for some $\tau\geq 0$. By definition, since the function U fulfills FSD_1, we have:

$$
\forall x\in C^\infty,\ m\left(\left\{x'\in C^\infty\Big|\frac{1-e^{-k(1-\beta)\sum_{j=0}^\tau \beta^j u(c_j)}(1-\widetilde{k}U(x'))^{\beta^\tau}}{\widetilde{k}}\geq U(x)\right\}\right)\geq \quad (35)
$$
$$
m'\left(\left\{x'\in C^\infty\Big|\frac{1-e^{-k(1-\beta)\sum_{j=0}^\tau \beta^j u(c'_j)}(1-\widetilde{k}U(x'))^{\beta^\tau}}{\widetilde{k}}\geq U(x)\right\}\right).
$$

Since $k,\ \widetilde{k}>0$, (35) becomes:

$$
\forall x\in C^\infty,\ m\left(\left\{x'\in C^\infty\big|e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c_j)}(1-\widetilde{k}U(x'))\leq (1-\widetilde{k}U(x))^{\frac{1}{\beta^\tau}}\right\}\right)\geq \quad (36)
$$
$$
m'\left(\left\{x'\in C^\infty\big|e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c'_j)}(1-\widetilde{k}U(x'))\leq (1-\widetilde{k}U(x))^{\frac{1}{\beta^\tau}}\right\}\right).
$$

We wish to prove that (c,m) is preferred to (c',m'), i.e. $U(c,m)\geq U(c',m')$ or that:
$\frac{1-e^{-k(1-\beta)\sum_{j=0}^\tau \beta^j u(c_j)}\left(1-\widetilde{k}\int_{C^\infty}U(x)m(dx)\right)^{\beta^\tau}}{\widetilde{k}}\geq \frac{1-e^{-k(1-\beta)\sum_{j=0}^\tau \beta^j u(c'_j)}\left(1-\widetilde{k}\int_{C^\infty}U(x)m'(dx)\right)^{\beta^\tau}}{\widetilde{k}}$. Since $k,\widetilde{k}>0$, this simplifies into:[22]

$$
\int_{C^\infty}e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c_j)}(1-\widetilde{k}U(x))m(dx)\leq \int_{C^\infty}e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c'_j)}(1-\widetilde{k}U(x))m'(dx).
$$
$$(37)$$

For any (c,m) of Δ_1 with $c=(c_0,\ldots,c_\tau)\in C^{\tau+1}$, we define the function $F_{c,m,\tau}(\cdot)$ by:

$$
t\in[0,1]\mapsto F_{c,m,\tau}(t)=m\left(\left\{x\in C^\infty\big|e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c_j)}\left(1-\widetilde{k}U(x)\right)\leq t\right\}\right).
$$

$F_{c,m,\tau}$ is well-defined, non-decreasing with $F_{c,m,\tau}(0)=0$ and $F_{c,m,\tau}(1)=1$. To show that $F_{c,m,\tau}$ is right-continuous, we consider $t\in[0,1]$ and $(t_n)_{n\geq 0}\in[0,1]^\infty\searrow t$. Measure additivity implies that $F_{c,m,\tau}(t_n)-F_{c,m,\tau}(t)=m\left(\left\{x\in C^\infty\big|t<e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^\tau \beta^j u(c_j)}\left(1-\widetilde{k}U(x)\right)\leq t_n\right\}\right)$, which can be made arbitrarily small by taking n large enough. We deduce that $F_{c,m,\tau}$ is the cdf of the utility associated to (c,m). (36) becomes:

$$
\forall t\in[0,1],\ F_{c,m,\tau}(t)\geq F_{c',m',\tau}(t). \quad (38)
$$

Using the image measure theorem (e.g., Theorem 4.1.11 in Dudley, 2002), (37) simplifies to $\int_{[0,1]}t\,dF_{c,m,\tau}(t)\leq \int_{[0,1]}t\,dF_{c',m',\tau}(t)$, or after integration by parts $1-\int_{[0,1]}F_{c,m,\tau}(t)dt\leq 1-\int_{[0,1]}F_{c',m',\tau}(t)$, which holds because of (38). So (c,m) is preferred to (c',m').

---

[22]When $k<0$, inequalities (36) and (37) are reversed, but the rest of the proof is similar.

Second step: monotonicity with respect to FSD_t. We consider (c, m) and (c', m') in Δ_t with $(c, m) \, FSD_t \, (c', m')$ and $c = (c_0, \ldots, c_\tau) \in C^{\tau+1}$ and $c' = (c'_0, \ldots, c'_\tau) \in C^{\tau+1}$ for some $\tau \geq 0$. By definition, since the function U fulfills FSD_{t-1}, we have:

$$\forall x \in \Delta_{t-1}, \; m\left(\left\{x' \in D_{t-1} | \frac{1 - e^{-k(1-\beta)\sum_{j=0}^{\tau} \beta^j u(c_j)}(1 - \widetilde{k}U(x'))^{\beta^\tau}}{\widetilde{k}} \geq U(x)\right\}\right) \qquad (39)$$
$$\geq m'\left(\left\{x' \in D_{t-1} | \frac{1 - e^{-k(1-\beta)\sum_{j=0}^{\tau} \beta^j u(c'_j)}(1 - \widetilde{k}U(x'))^{\beta^\tau}}{\widetilde{k}} \geq U(x)\right\}\right).$$

We wish to prove that $U(c, m) \geq U(c', m')$ or since $k, \widetilde{k} > 0$:

$$\int_{D_{t-1}} e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^{\tau} \beta^j u(c_j)}(1 - \widetilde{k}U(x))m(dx) \leq \int_{D_{t-1}} e^{-k\frac{(1-\beta)}{\beta^\tau}\sum_{j=0}^{\tau} \beta^j u(c'_j)}(1 - \widetilde{k}U(x))m'(dx).$$
$$(40)$$

Equations (39) and (40) are totally similar to (35) and (37). It is then possible to follow exactly the same path as in the first step to terminate the proof.

B Proof of proposition 2

Consider two KP-recursive utility functions U^A and U^B which fulfill ordinal dominance and represent the same preferences over C^∞. We assume that one of them (e.g., U^A) is not a risk-sensitive utility function (i.e. its aggregator W^A is not as in (14)). Proposition 1 implies that $W^A(x, y) = a_1(x) + b_1(x)y$ for some non-constant function b_1. Such preferences do not fulfill the assumption of weak separability and therefore cannot have the same restriction over C^∞ as U^A (risk-sensitive preferences are weakly separable). Then, U^B must also correspond to an aggregator W^B such that $W^B(x, y) = a_2(x) + b_2(x)y$ for some non-constant function b_2. In order to prove that if U^A and U^B represent the same preferences over C^∞, then $W^A = W^B$, we first state the following simple result:

Lemma 2 (Aggregators with identical ordinal preferences) *We assume that $W^A(x, y)$ and $W^B(x, y)$ are two aggregators whose corresponding KP-recursive utility functions represent the same preferences over deterministic consumption paths. Then there exists an increasing function ψ such that $\psi(0) = 0$, $\psi(1) = 1$ and:*

$$\forall (x, y) \in C \times [0, 1], \; \psi\left(W^A(x, y)\right) = W^B(x, \psi(y)).$$

Proof. Consider U^A and U^B the utility functions. Since they represent the same ordinal preferences, there exists an increasing function ψ, such that for all $x \in C^\infty$ we have $U^B(x) = \psi\left(U^A(x)\right)$. From $U^A(\underline{c}_\infty) = U^B(\underline{c}_\infty) = 0$ and $U^A(\overline{c}_\infty) = U^B(\overline{c}_\infty) = 1$,

we have $\psi(0) = 0$, $\psi(1) = 1$. For any $(c_0, c_1, c_2 \ldots) \in C^\infty$, we have: $U^B(c_0, c_1, \ldots) = W^B(c_0, U^B(c_1, \ldots)) = W^B(c_0, \psi(U^A((c_1, \ldots)))$. We also have: $U^B(c_0, c_1, \ldots) = \psi(U^A(c_0, c_1, \ldots)) = \psi(W^A(c_0, U^A(c_1, \ldots)))$. Noting $y = U^A(c_1, c_2, \ldots)$ which covers $[0, 1]$, we deduce that $\psi\left(W^A(c_0, y)\right) = W^B(c_0, \psi(y))$ for all $(c_0, y) \in C \times [0, 1]$. ∎

We now terminate the proof. Lemma 2 implies that there exists an increasing function ψ with $\psi(0) = 0$ and $\psi(1) = 1$, such that for all $(x, y) \in C \times [0, 1]$:

$$\psi(a_1(x) + b_1(x)y) = a_2(x) + b_2(x)\psi(y). \tag{41}$$

Since W^A and W^B are continuously derivable on $C \times [0, 1]$, a_i and b_i are (on C) for $i = 1, 2$.

As a preliminary remark, it would be straightforward to show that $x \mapsto a_i(x) + b_i(x)y$ ($i = 1, 2$) defines a strictly increasing bijection from C into $[b_i(\underline{c})y, a_i(\overline{c}) + b_i(\overline{c})y]$, with $a_i(\overline{c}) + b_i(\overline{c})y < 1$ if and only if $y < 1$ and $b_i(\underline{c})y > 0$ if and only if $y > 0$.

Let $Y = \{z \in [0, 1] | \psi \text{ is continuously derivable on } [0, z]\}$. From (41) in $y = 0$ and our preliminary remark for $y = 0$, we deduce that ψ is continuously derivable on $[0, a_1(\overline{c})]$. So, $Y \neq \emptyset$ and Y is also bounded by 1. Let $\overline{y} = \sup Y$ with $0 < a_1(\overline{c}) \leq \overline{y} \leq 1$. If $\overline{y} < 1$, our preliminary remark in $\overline{y}$ and (41) contradicts the definition of $\overline{y}$. So $\overline{y} = 1$. As for ψ, after deriving (41) with respect to y, ψ' can be shown to be continuously derivable, as well as strictly positive. Since b_1 and b_2 also are, we obtain, after deriving (41) twice, that $b_1(x)\frac{\psi''}{\psi'}(a_1(x) + b_1(x)y) = \frac{\psi''}{\psi'}(y)$. The preliminary remark implies that for any $y \in [0, 1]$, there exists $x_y \in C$ such that $a_1(x_y) + b_1(x_y)y = y$, and $(1 - b_1(x_y))\frac{\psi''}{\psi'}(y) = 0$. Since $b_1(x_y) < 1$, we have $\psi'' = 0$ which, with $\psi(0) = 0$ and $\psi(1) = 1$, imply that $\psi(x) = x$. From (41), it follows that $a_1(x) = a_2(x)$ and $b_1(x) = b_2(x)$ and finally $W^A = W^B$.

C Proof of Proposition 3

We assume that $k_A > k_B$ and $k_A, k_B \neq 0$ (the case where either k_A or k_B equals zero can be treated similarly). For $i = A, B$ and all $x = (c_0, c_1, \ldots) \in C^\infty$ we have:

$$\forall x \in C^\infty, \ U^i(x) = \frac{1 - \exp(-k_i U_0(x))}{1 - e^{-k_i}}, \tag{42}$$

where $U_0(x) = (1 - \beta)\sum_{t=0}^{+\infty} \beta^t u(c_t)$ is the utility over deterministic consumption paths and is the same for both agents. We define $\psi_{A,B} : [0, 1] \to [0, 1]$:

$$\forall y \in [0, 1], \ \psi_{A,B}(y) = \frac{1 - (1 - (1 - e^{-k_B})y)^{k_A/k_B}}{1 - e^{-k_A}}. \tag{43}$$

$\psi_{A,B}$ is increasing and concave. We have from (42): $U^A(c) = \psi_{A,B}(U^B(c))$ for all $c \in C^\infty$ and $\psi_{A,B}(W^B(c, y)) = W^A(c, \psi_{A,B}(y))$ for all $c \in C$ and $y \in [0, 1]$. We show by induction

that for all $(c, m) \in D_t$, $\psi_{A,B}(U^B(c, m)) \geq U^A(c, m)$. It holds for $t = 0$ since $U^A(c) = \psi_{A,B}(U^B(c))$. We assume that it holds up to $t - 1$ and let $(c, m) \in D_t$:

$$\psi_{A,B}(U^B(c, m)) = \psi_{A,B}(W^B(c, E_m[U^B])) = W^A(c, \psi_{A,B}(E_m[U^B]))$$
$$\geq W^A(c, E_m[\psi_{A,B}(U^B)]) \geq W^A(c, E_m[U^A])) = U^A(c, m),$$

where the first inequality is a consequence of Jensen inequality and of W^B being increasing with respect to its second argument, while the second comes from the induction hypothesis.

Since $\cup_{t \geq 0} D_t$ is dense in D, we deduce by continuity that:

$$\psi_{A,B}(U^B(c, m)) \geq U^A(c, m) \text{ for all } (c, m) \in D. \tag{44}$$

We deduce that for all $x \in C^\infty$ and $(c, m) \in D$, $U^A(c, m) \geq U^A(x) \Rightarrow U^B(c, m) \geq U^B(x)$, which terminates the proof.

D   Proof of Proposition 4

We assume that $k_A \neq 0$ and $k_B \neq 0$. We prove that if (c, m) and $(c', m') \in D$ are such that $(c, m) R_{SC}(c', m')$ and $(c, m) \succeq_A (c', m')$, then $(c, m) \succeq_B (c', m')$. The case when $(c', m') \in C^\infty = \Delta_0$ has been treated in Section C. We assume that (c, m) and (c', m') are in Δ_1, where $c = (c_0, \ldots, c_\tau) \in C^{\tau+1}$ and $c' = (c'_0, \ldots, c'_\tau) \in C^{\tau+1}$ for $\tau \geq 0$. By definition, there exist $p \in [0, 1]$ and m_0, m'_0, m_1, $m'_1 \in M(C^\infty)$ such that (15) and (16) hold.

The utility of agent A can be expressed as follows:

$$U_A(c, m) = \kappa(p \int_{C^\infty} f_{k_A}(U_0(c, x_0)) m_0(dx_0) + (1 - p) \int_{C^\infty} f_{k_A}(U_0(c, x_1)) m_1(dx_1)),$$

where: $f_{k_i} : t \mapsto \frac{1 - e^{-\frac{k_i}{\beta} t}}{1 - e^{-\frac{k_i}{\beta}}}$ $(i = A, B)$, $U_0(c, x_0) = (1 - \beta) \sum_{t=0}^\tau \beta^j u(c_t) + \beta^\tau U^0(x_0)$ slightly extends (42) and $\kappa : x \mapsto \frac{1 - (1 - (1 - e^{-\frac{k_A}{\beta}}) x)^\beta}{1 - e^{-k_A}}$. So, $(c, m) \succeq_A (c', m')$ implies:

$$p \int_{C^\infty} f_{k_A}(U_0(c, x_0)) m_0(dx_0) + (1 - p) \int_{C^\infty} f_{k_A}(U_0(c, x_1)) m_1(dx_1) \geq \tag{45}$$
$$p \int_{C^\infty} f_{k_A}(U_0(c', x_0)) m'_0(dx_0) + (1 - p) \int_{C^\infty} f_{k_A}(U_0(c', x_1)) m'_1(dx_1),$$

To reexpress (45), we define $F_{c, m_0, A} : [0, 1] \ni t \mapsto m_0 (\{x' \in C^\infty | f_{k_A}(U_0(c, x')) \geq t\}) \in [0, 1]$. The image measure theorem gives $\int_{C^\infty} f_{k_A}(U_0(c, x_0)) m_0(dx_0) = - \int_0^1 t dF_{c, m_0, A}(t)$. Since f_{k_A} is an increasing bijection from $[0, 1]$ onto itself, a change of variable yields:

$$\int_{C^\infty} f_{k_A}(U_0(c, x_0)) m_0(dx_0) = - \int_0^1 f_{k_A}(t) d (F_{c, m_0, A} \circ f_{k_A}) (t),$$

where $\circ$ is the composition operator. Moreover, since f_{k_A} is bijective and increasing, we have $F_{c,m_0,A} \circ f_{k_A}(t) = m_0(\{x' \in C^\infty | U_0(c,x') \geq t\})$. So, $G_{c,m_0} = F_{c,m_0,A} \circ f_{k_A}$ depends only on ordinal preferences and is the same for both agents. Integrating by part, we obtain:

$$\int_{C^\infty} f_{k_A}(U_0(c,x_0))m_0(dx_0) = -\int_0^1 f_{k_A}(t)dG_{c,m_0}(t) = \int_0^1 f'_{k_A}(t)G_{c,m_0}(t)dt. \qquad (46)$$

Using obvious notations to extend (46), we deduce that (45) can be re-expressed as follows:

$$(1-p)\int_0^1 f'_{k_A}(t)\left(G_{c,m_1}(t) - G_{c',m_1'}(t)\right)dt \geq p\int_0^1 f'_{k_A}(t)\left(G_{c',m_0'}(t) - G_{c,m_0}(t)\right)dt. \qquad (47)$$

We also have (c',m_0') FSD_1 (c,m_0) and (c,m_1) FSD_1 (c',m_1'). These relations imply that for all $t \in [0,1]$, $G_{c',m_0'}(t) - G_{c,m_0}(t) \geq 0$ and $G_{c,m_1}(t) - G_{c',m_1'}(t) \geq 0$. Since f_{k_A} is increasing and bijective, (c',m_0') FSD_1 (c,m_0) simplifies into $m_0'(\{x' \in C^\infty | U_0(c',x') \geq t\}) \geq m_0(\{x' \in C^\infty | U_0(c,x') \geq t\})$ for $t \in [0,1]$, which means $G_{c',m_0'}(t) - G_{c,m_0}(t) \geq 0$.

Moreover, m_0, m_0', m_1, m_1' verify (17), which implies that there exists $t_0 \in (0,1)$ such that $G_{c',m_1'}(t) = 1$ for $0 \leq t \leq t_0$ and $G_{c',m_0'}(t) = 0$ for $t_0 \leq t \leq 1$. Since for all $t \in [0,1]$, $G_{c',m_0'}(t) \geq G_{c,m_0}(t)$ and $G_{c,m_1}(t) \geq G_{c',m_1'}(t)$, we have $G_{c',m_1'}(t) = G_{c,m_1}(t) = 1$ for $0 \leq t \leq t_0$ and $G_{c',m_0'}(t) = G_{c,m_0}(t) = 0$ for $t_0 \leq t \leq 1$. We deduce that (47) becomes:

$$(1-p)\int_{t_0}^1 f'_{k_A}(t)\left(G_{c,m_1}(t) - G_{c',m_1'}(t)\right)dt \geq p\int_0^{t_0} f'_{k_A}(t)\left(G_{c',m_0'}(t) - G_{c,m_0}(t)\right)dt. \qquad (48)$$

We have $f_{k_A} = \psi_{A,B} \circ f_{k_B}$, where $\psi_{A,B}$ defined in (43) is derivable. Since $\psi_{A,B}$ is concave and f_{k_B} increasing, $\psi'_{A,B}(f_{k_B}(t)) \geq \psi'_{A,B}(f_{k_B}(t_0))$ for $t \in (0,t_0)$ and $\psi'_{A,B}(f_{k_B}(t)) \leq \psi'_{A,B}(f_{k_B}(t_0))$ for $t \in (t_0,1)$. We deduce from (48): $\psi'_{A,B}(f_{k_B}(t_0))(1-p)\int_0^{t_0} f'_{k_B}(t)(G_{c,m_1}(t) - G_{c',m_1'}(t))dt \geq \psi'_{A,B}(f_{k_B}(t_0))p\int_{t_0}^1 f'_{k_B}(t)(G_{c',m_0'}(t) - G_{c,m_0}(t))dt$, which implies $(c,m) \succeq_B (c',m')$ using $\psi'_{A,B}(f_{k_B}(t_0)) \geq 0$ and (48).

E   Proof of Proposition 5

E.1   A preliminary lemma

Lemma 3 *We consider a continuous decreasing function g, a continuous positive function h and a random variable $\widetilde{x}$ admitting a density f with a compact support in $\mathbb{R}$.[23] Then:*

$$E[h(\widetilde{x})e^{-k\widetilde{x}}]E[g(\widetilde{x})e^{-k\widetilde{x}}] - E[g(\widetilde{x})h(\widetilde{x})e^{-k\widetilde{x}}]E[e^{-k\widetilde{x}}] \geq 0.$$

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[23]This technical assumption is not central to our argument but simply ensures the existence of all moments considered below. Alternatively, at the cost of additional technicalities, we could have assumed that all moments of $\widetilde{x}$ exist.

Moreover if $h(x)e^{-kx}$ is decreasing:

$$E[g(\widetilde{x})h(\widetilde{x})e^{-k\widetilde{x}}]E[h^2(\widetilde{x})e^{-2k\widetilde{x}}] - E[g(\widetilde{x})h^2(\widetilde{x})e^{-2k\widetilde{x}}]E[h(\widetilde{x})e^{-k\widetilde{x}}] \geq 0.$$

Proof.

Let $f(x)$ be the pdf of $\widetilde{x}$ with the support $[\underline{x}, \overline{x}]$. We define $f_1(x) = e^{-kx}f(x)$ and:

$$\Delta_1(a) = \int_{\underline{x}}^{a} h(x)f_1(x)dx \int_{\underline{x}}^{a} g(x)f_1(x)dx - \int_{\underline{x}}^{a} g(x)h(x)f_1(x)dx \int_{\underline{x}}^{a} f_1(x)dx.$$

Since $\Delta_1'(a) = f_1(a)\int_{\underline{x}}^{a} f_1(x)(h(x)-h(a))(g(a)-g(x))dx \geq 0$ and $\Delta_1(\underline{x}) = 0$, we obtain $\Delta_1(\overline{x}) \geq 0$, which proves the first point.

For the second point, we define $f_2(x) = e^{-kx}f(x)h(x)$ and:

$$\Delta_2(a) = \int_{\underline{x}}^{a} h(x)e^{-kx}f_2(x)dx \int_{\underline{x}}^{a} g(x)f_2(x)dx - \int_{\underline{x}}^{a} g(x)e^{-kx}h(x)f_2(x)dx \int_{\underline{x}}^{a} f_2(x)dx.$$

Since $\Delta_2'(a) = f_2(a)\int_{-\infty}^{a}(h(a)e^{-ka} - h(x)e^{-kx})(g(x)-g(a))f_2(x)dx \geq 0$ and $\Delta_2(\underline{x}) = 0$, we obtain $\Delta_2(\overline{x}) \geq 0$. ∎

E.2 Proof of Proposition 5

Since $\frac{1}{R_t} = \frac{\beta}{u'(c_t)}\frac{E_t[u'(c_{t+1})\exp(-kV_{t+1})]}{E_t[\exp(-kV_{t+1})]}$, deriving with respect to k yields:

$$\frac{1}{R_t}\frac{\partial R_t}{\partial k} = \frac{E_t[\frac{\partial(kV_{t+1})}{\partial k}u'(c_{t+1})e^{-kV_{t+1}}]E_t[e^{-kV_{t+1}}] - E_t[u'(c_{t+1})e^{-kV_{t+1}}]E_t[\frac{\partial(kV_{t+1})}{\partial k}e^{-kV_{t+1}}]}{E_t[e^{-kV_{t+1}}]E_t[u'(c_{t+1})e^{-kV_{t+1}}]}.$$

However, $\frac{\partial(kV_{t+1})}{\partial k} = (1-\beta)u(c_{t+1}) + \beta\frac{E_t[V_{t+2}e^{-kV_{t+2}}]}{E_t[e^{-kV_{t+2}}]}$. By assumption, $\frac{\partial(kV_{t+1})}{\partial k}$ and V_{t+1} are comonotonic, while V_{t+1} and $u'(c_{t+1})$ are anticomonotonic. We thus have $\frac{\partial(kV_{t+1})}{\partial k} = h(V_{t+1})$ for some increasing function h and $u'(c_{t+1}) = g(V_{t+1})$ for some decreasing function g. Moreover, V_{t+1} has a compact support. Lemma 3 implies thus that $\frac{\partial R_t}{\partial k} < 0$.

For the market price of risk, we have: $\left(\frac{\sigma_t(M_{t,t+1})}{E_t[M_{t,t+1}]}\right)^2 = \frac{E_t[(u'(c_{t+1}))^2\exp(-2kV_{t+1})]}{E_t[(u'(c_{t+1}))\exp(-kV_{t+1})]^2} - 1$. After derivation, we obtain:

$$\frac{\sigma_t(M_{t,t+1})}{E_t[M_{t,t+1}]}\frac{\partial}{\partial k}\left(\frac{\sigma_t(M_{t,t+1})}{E_t[M_{t,t+1}]}\right) = \frac{E_t\left[(u'(c_{t+1}))\frac{\partial(kU_{t+1})}{\partial k}e^{-kU_{t+1}}\right]E_t\left[(u'(c_{t+1}))^2 e^{-2kU_{t+1}}\right]}{E_t\left[(u'(c_{t+1}))e^{-kU_{t+1}}\right]^3}$$

$$- \frac{E_t\left[\frac{\partial(kU_{t+1})}{\partial k}(u'(c_{t+1}))^2 e^{-2kU_{t+1}}\right]E_t\left[(u'(c_{t+1}))e^{-kU_{t+1}}\right]}{E_t\left[(u'(c_{t+1}))e^{-kU_{t+1}}\right]^3}.$$

Again, Lemma 3 implies that $\frac{\partial}{\partial k}\left(\frac{\sigma_t(M_{t,t+1})}{E_t[M_{t,t+1}]}\right) > 0$.

F Proof of Proposition 6

Since u is strictly increasing and concave, the indirect utility function Ω also is. The concavity of u and Ω implies that $\frac{1}{2}\left(u(c_0^A) + u(c_0^B)\right) \leq u(\frac{c_0^A+c_0^B}{2})$ and $\frac{1}{2}\left(\Omega(w^A) + \Omega(w^B)\right) \leq \Omega(\frac{w^A+w^B}{2})$. Market equilibrium then implies that $\frac{1}{2}\left(V_s^B + V_s^A\right) \leq V_s^{aut}$ for $j = h, l$ (where V_s^j is the ex-post utility of agent $j = A, B$ in state $s = h, l$ and V_s^{aut} the ex-post utility in autarky in state s). We consider the three following consumption plans: the one of A, the one of B, and the autarkic one, which are available to both agents by assumption. The plan of A cannot strictly first order stochastically dominate the plan of B, otherwise it would contradict the choice of B (and vice-versa). Moreover, since $\frac{1}{2}\left(V_j^B + V_j^A\right) \leq V_j^{aut}$, the autarkic allocation cannot be dominated by the plan of A; otherwise the plan of B would be dominated by the autarkic one, which would contradict the choice of B.

We now prove that $V_h^A \geq V_l^A$ and $V_h^B \geq V_l^B$. Let c^{aut} be the consumption in autarky, and c^A the consumption of A at the equilibrium. If $c^{aut} = c^A$, our result holds. Let us assume that $c^{aut} \neq c^A$. For any $\lambda \in [-1, 1]$, we consider the allocation $c(\lambda) = c^{aut} + \lambda(c^A - c^{aut})$. We have $c^A = c(1)$ and $c^B = c(-1)$. Since $c(-1)$ and $c(1)$ are available, the consumption plan $c(\lambda)$ is available for agents A and B. We denote $c_l(\lambda)$ (resp. $c_h(\lambda)$) the (deterministic) consumption vectors obtained in state l (resp. h). Since agents A and B preferences are identical for deterministic outcomes, $V_s(\lambda)$ ($s = h, l$) is the utility of agents (A or B) endowed with $c_s(\lambda)$. We have $V_s^A = V_s(1)$ and $V_s^B = V_s(-1)$.

Let us assume that $V_l(1) > V_h(1)$ and show that it yields a contradiction. In autarky, we have by definition: $V_l(0) < V_h(0)$. Since u is strictly concave, the functions V_l and V_h are also strictly concave. Because of ordinal dominance, $V_l'(1)$ and $V_h'(1)$ cannot have the same signs (otherwise, A would like to pick up a different $c(\lambda)$). Similarly, considering that $c(-1)$ is the optimal behavior of B, $V_l'(-1)$ and $V_h'(-1)$ cannot have the same sign.

Assume that $V_l'(\lambda_0) = 0$ for some $\lambda_0 \in (-1, 1)$. The concavity of $V_l(\lambda)$ implies that $V_l'(-1) \geq 0$ and $V_l'(1) \leq 0$. The above remarks imply that $V_h'(-1) \leq 0$ and $V_h'(1) \geq 0$, which contradicts the strict concavity of V_h. Thus $V_l'(\lambda_0) = 0$ is impossible and V_l' and V_h' keep constant but opposite signs over $(-1, 1)$. Since $V_l(1) > V_h(1)$ and $V_l(0) < V_h(0)$, we have $V_l'(\lambda) \geq 0$ and $V_h'(\lambda) \leq 0$ for all $\lambda \in (-1, 1)$. Consider now a small $\varepsilon > 0$. We have $V_h(1) \leq V_h(1 - \varepsilon) < V_l(1 - \varepsilon) \leq V_l(1)$ and $V_l(-1) \leq V_l(-1 + \varepsilon) < V_h(-1 + \varepsilon) \leq V_h(-1)$.

Since $c(1)$ is riskier than $c(1 - \varepsilon)$, choosing $c(1)$ instead of $c(1 - \varepsilon)$ is a risk increase perceived as worthwhile by A. Since A is more risk averse than B, B must also prefer $c(1)$ to $c(1 - \varepsilon)$, or said differently $\frac{dV^B(c(\lambda))}{d\lambda}|_{\lambda=1} \geq 0$. But since B prefers $c(-1)$ to $c(-1 + \varepsilon)$ by definition, we also have $\frac{dV^B(c(\lambda))}{d\lambda}|_{\lambda=-1} \leq 0$. This contradicts the strict concavity of $\lambda \mapsto V^B(c(\lambda))$, which can be proved to hold if u is strictly concave and $k_B \geq 0$. We deduce then that $V_h(1) \geq V_l(1)$. By the same token, we can prove that $V_h(-1) \geq V_l(-1)$.

We have two possibilities: Either (20) holds, or its symmetric where A is replaced by B holds. This latter possibility however contradicts that A is more risk averse than B.

G Proof of Proposition 7

We assume that agents can only trade a riskless bond of price p_t at date t, which pays 1 unit of consumption at all future dates (starting at $t+1$). Given the shock structure, all asset trades will take place at date 0 and consumption from date 1 and onward is constant. Using the functional form (21) for Epstein-Zin preferences and denoting by a^i the riskless asset quantity purchased by agent i, the program of $i = A, B$ can be expressed as follows:

$$\max_{a^i} \left((1-\beta) \left(y_0 - p_0 a^i \right)^\rho + \beta \left(\eta \left(y_h + a^i \right)^{\alpha_i} + (1-\eta) \left(y_l + a^i \right)^{\alpha_i} \right)^{\frac{\rho}{\alpha_i}} \right)^{\frac{1}{\rho}}, \qquad (49)$$

The asset price p_0 adjusts, such that the aggregate demand is null: $a^A + a^B = 0$.

We allow ourselves to choose a value $y_h \gg 1$ as large as we like, and then, for this given y_h, a probability $0 < \eta \ll 1$ as close to zero as we wish. For a given y_h, making η converging to zero involves getting closer to the no risk situation. The market equilibrium is therefore close to the no-trade equilibrium: $|a^i| \ll 1$.

From (49), we deduce the following Euler equations ($i = A, B$):

$$p_0 \left(y_0 - p_0 a^i \right)^{\rho-1} = \frac{\beta}{1-\beta} \left((1-\eta) \left(y_l + a^i \right)^{\alpha_i-1} + \eta \left(y_h + a^i \right)^{\alpha_i-1} \right) \qquad (50)$$
$$\times \left((1-\eta) \left(y_l + a^i \right)^{\alpha_i} + \eta \left(y_h + a^i \right)^{\alpha_i} \right)^{\frac{\rho}{\alpha_i}-1}.$$

We note $a = a^A = -a^B$. Taking the ratio of both Euler equations (50) $i = A, B$, we obtain using $y_h \gg 1$, $0 < \eta \ll 1$, $|a| \ll 1$ and $\alpha_A, \alpha_B < 0 < \rho < 1$ that:

$$2(1-\rho) \left(\frac{p_0}{y_0} + \frac{1}{y_l} \right) a \approx -\frac{\rho(\alpha_B - \alpha_A)}{\alpha_A \alpha_B} \eta < 0. \qquad (51)$$

Regarding the price, we obtain using (50) for $i = A$ and (51):

$$p_0 \simeq \frac{\beta}{1-\beta} \left(\frac{y_l}{y_0} \right)^{\rho-1} \left(1 - \rho \frac{\alpha_A + \alpha_B}{2\alpha_A \alpha_B} \eta \right) > \frac{\beta}{1-\beta} \left(\frac{y_l}{y_0} \right)^{\rho-1}. \qquad (52)$$

From (21) and (49), ex-post utility of agent $i = A, B$ in state $s = h, l$ can be expressed as:

$$V_s^{EZ,i} = \left((1-\beta) \left(y_0 - p_0 a^i \right)^\rho + \beta \left(y_s + a^i \right)^\rho \right)^{\frac{1}{\rho}},$$
$$\approx V_s^{EZ,aut} + \left(V_s^{EZ,aut} \right)^{1-\rho} (\beta \left(\frac{y_s}{y_0} \right)^{\rho-1} - (1-\beta) p_0 y_0^{\rho-1}) a^i,$$

since the autarky utility in state $s = h, l$ is $V_s^{EZ,aut} = ((1-\beta)(y_0)^\rho + \beta(y_s)^\rho)^{\frac{1}{\rho}}$. Equation (51) finally implies that $V_s^{EZ,A} > V_s^{EZ,aut} > V_s^{EZ,B}$ for $s = h, l$.